What is form?

Draft

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Abstract

What is the form of a set? Though there are many vague descriptions of the form of a set, there remains one inherent property: the form does not change by rotations and translations.

So there are on one side the objects and on the other side the transactions which can be carried out on an object without changing its form.

It is of significance, that the transactions themselves may be considered as objects. The leading thought for the following is to create a kind of basis of the transaction objects in order to describe all other objects. We will focus on plane, closed, rectifiable curves building the border of a simply connected domain. It turns out, that there exists a simple uniform relationship of four fundamental entities up to normalization for orthogonal trajectories:

1. The sum of the changes of geodesic curvature is zero.
2. The difference of the changes of geodesic curvature is the real part of the Schwarzian derivative.
3. The difference of the changes of geodesic acceleration is the imaginary part of the Schwarzian derivative.
4. The sum of the changes of geodesic acceleration is the Gaussian curvature.

1 Plane rotations and translations

The plane rotations are a commutative group, any representation is equivalent to a representation consisting of representations of first order, Smirnov [8]. The complex circular functions are the striking examples if the respective angle is an integer multiple of the circumference. By that way we get a representation for any integer and thus an infinite number of representations.

The theorem of Weierstrass on trigonometric approximation proves that any closed continuous function can be approximated by a trigonometric polynom. Thus the continuous closed curves can be approximated arbitrarily precise, so that we can focus on Fourier approximation. A basis function of the Fourier approximation is at least nothing but a curve which rotates n times around a circle transformed by a (2,2) matrix, that is a curve floating n-times around an ellipse. However, due to the Riemannian mapping theorem, a simply connected domain can be mapped injectively and conform on a circle. It is therefore already sufficient to use Fourier approximations consisting of similarities matrices as coefficients matrices.

For the plane translations, which also do not alter the form of an object, there are no (2,2) matrices, which could be used for a representation. However, there are (3,3) matrices, considered as dual projective transformations which can be used for a representation. But we only get the translations themselves and pure projective transformations. The effect of the projective self mappings on a circle border can be considered as a reparameterization that is equivalent to a Moebius transformation restricted to a circle border. But the geometric effect of a translation and of a projective reparameterization is already incorporated in the Fourier basis approximation, so that the basis needs not to be extended. The representation theory for translations is carried out by Vilenkin [9] emphasizing the connection to Bessel series expansion. Due to Vilenkin the group of translations and rotations can be considered as the limiting case of the group of three dimensional sphere rotations when the sphere radius tends to infinity. However, we will ignore the basis of cylindrical harmonics for simplicity and restrict to the Fourier basis as a guide.

The preferred coordinate nets of a form are the conformal net restricted to a basis with positive exponents for the circular functions or the harmonic net if both positive and negative exponents are allowed.

The question is, what is the striking characteristic of such a coordinate net. From a differential viewpoint the characterization at a point, where two lines (trajectories) cross, is in the conformal case, that the sum of the changes of curvature vanishes. In case of the general harmonic net, also considered as the projection of a minimal surface, that characteristic is, that the sum of the changes of geodesic curvature of the associated orthogonal trajectories on the corresponding surface vanishes. So generally speaking, harmonic resolution of two crossing lines comes from the orthogonal crossing lines in space projected orthogonally in the plane, with the sum of the changes of the geodesic curvature vanishing.

It turns out, that the notion form is characterized by the duality of curvature and acceleration, imaginable in the way, that one curve is curved stronger, but passed slower while the other curve is less curved but passed accelerated, as two parts of a custom vehicle travel gently driving along a circular curve and accelerating and the end of the curve facing the nearly straight line, up two the difference, that these two types of parameterized curves are crossing in out context.

This Yin and Yang of curvature and acceleration can be described in another way. If you want to describe the duality based on just one of the terms, it turns out, that the correspondence of curvature and acceleration is also expressed by the fact, that the differentials of the curvatures of the orthogonal curves are equivalent. It is pivotal, that we are concerned with the differentials of curvature: on the one hand it is obvious, that if the curvatures themselves were forced to be equal, then the two crossing curves would have to be congruent, i.e. the one curve would be transformed into the other by a rotation and translation, a phenomenon, analogue to self similarity, characterising fractals. But this is, as we are concerned with the differentials of curvature, not mandatory. One the other hand, it is obvious, that on differentiation, the constant parts vanish, $(f(x) + b)' = f'(x)$, so from this viewpoint, the curvatures may deviate by nonzero constants. Characteristic for transformations, with curvatures of crossing curves deviating by constants, are Moebius transformations. These transformations are also characterized by maps mapping circles on circles. As a circle has constant curvature, with the constant just depending on the radius of the circle, it is evident again, that for two crossing circle segments the curvature differential will be zero, but i.e., the differentials of curvature will be equal and also the sum of the curvature differential will be zero. So from a simplified perspective, imagine the harmony of crossing lines as a locally Moebius fractal attribute.

2 Preface form essentials

2.1 Motivation to derive curvature

The angle ϕ at parameter t is determined by

$$\tan\phi(t) = \frac{\dot{x}(t)}{\dot{y}(t)}$$

$$\arctan \tan \phi(t) = \phi(t) = \arctan \frac{\dot{x}(t)}{\dot{y}(t)}$$

The change of the tangential angle, its derivative is due to:

$$\frac{\dot{x}(t)\ddot{y}(t) - \ddot{x}(t)\dot{y}(t)}{\dot{x}(t)^2} \frac{1}{1 + \left(\frac{\dot{x}(t)}{\dot{y}(t)}\right)^2} = \frac{\dot{x}(t)\ddot{y}(t) - \ddot{x}(t)\dot{y}(t)}{\dot{x}(t)^2 + \dot{y}(t)^2}$$

Curvature is derived by dividing with arclength.

2.2 Conformal viewpoints

viewpoint	f(t)	viewpoint	f'(t)
inverse velocity	$\frac{1}{\sqrt{x'(t)^2 + y('t)^2}}$	inverse acceleration	$(-1)\frac{x'(t)x''(t) + y'(t)y''(t)}{\sqrt{x'(t)^2 + y('t)^2}^3}$
logarithm of velocity square	$\ln(x'(t)^2 + y('t)^2)$	logarithmic acceleration	$2\frac{x'(t)x''(t) + y'(t)y''(t)}{x'(t)^2 + y('t)^2}$
angle	$\arctan \frac{x'(t)}{y'(t)}$	change of angle	$\frac{x'(t)y''(t) - y'(t)x''(t)}{x'(t)^2 + y('t)^2}$
		curvature	$\frac{x'(t)y''(t) - y'(t)x''(t)}{\sqrt{x'(t)^2 + y('t)^2}^3}$

Conformal mappings are characterized by the fact that the sums of the changes of curvature of the orthogonal trajectories vanish, see Needham [5] referencing Bivens. But the fact can be proven without refering to arc length parameterization and instead considering the inherent parameterization.

Let K^x be the curvature of the trajectory in argument x and K^y be the orthogonal trajectory in argument y of a conformal mapping $(u(x,y), v(x,y))$, with

$$K^x(x,y) = \frac{\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial x^2} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial x^2} \end{vmatrix}}{\left(\sqrt[2]{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2}\right)^3} = -\frac{\left(\frac{\partial v}{\partial y}\frac{\partial v}{\partial y^2} + \frac{\partial u}{\partial y^2}\frac{\partial u}{\partial y}\right)}{\left(\sqrt[2]{\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2}\right)^3} = -B^y(x,y)$$

$$K^y(x,y) = \frac{\begin{vmatrix} \frac{\partial u}{\partial y} & \frac{\partial u}{\partial y^2} \\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial y^2} \end{vmatrix}}{\left(\sqrt[2]{\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2}\right)^3} = \frac{\left(\frac{\partial v}{\partial x}\frac{\partial v}{\partial x^2} + \frac{\partial u}{\partial x^2}\frac{\partial u}{\partial x}\right)}{\left(\sqrt[2]{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2}\right)^3} = B^x(x,y)$$

and analogue

$$\tilde{K}^x(x,y) = \frac{\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial x^2} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial x^2} \end{vmatrix}}{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2} = -\frac{\left(\frac{\partial v}{\partial y}\frac{\partial v}{\partial y^2} + \frac{\partial u}{\partial y^2}\frac{\partial u}{\partial y^2}\right)}{\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2} = \tilde{B}^y(x,y)$$

then

$$\frac{\partial K^x}{\partial x} + \frac{\partial K^y}{\partial y} = 0$$

The classical Schwarzian derivative may be considered as the differences of derivatives:

based on curvature and inverse acceleration:

$$\begin{aligned} S &= \frac{1}{\sqrt{\left(\frac{\partial u}{\partial x}\right)^2 \left(\frac{\partial v}{\partial x}\right)^2}} \left(\frac{\partial K^x}{\partial x} - \frac{\partial K^y}{\partial y}, \frac{\partial B^x}{\partial x} - \frac{\partial B^y}{\partial y}\right) \\ &= \frac{1}{\sqrt{\left(\frac{\partial u}{\partial x}\right)^2 \left(\frac{\partial v}{\partial x}\right)^2}} \left(2\frac{\partial K^x}{\partial x}, \frac{\partial B^x}{\partial x} - \frac{\partial B^y}{\partial y}\right) \end{aligned}$$

or a modified Schwarzian derivative based on change of angle and logarithmic acceleration:

$$\tilde{S} = \left(\frac{\partial \tilde{K}^x}{\partial x} - \frac{\partial \tilde{K}^y}{\partial y}, \frac{\partial \tilde{B}^x}{\partial x} - \frac{\partial \tilde{B}^y}{\partial y}\right) = \left(2\frac{\partial \tilde{K}^x}{\partial x}, \frac{\partial \tilde{B}^x}{\partial x} - \frac{\partial \tilde{B}^y}{\partial y}\right)$$

The generalized Schwarzian derivative may be based on the changes of geodesic curvature (the curvature of the orthogonal on the tangent plane projected curve) and the changes of the geodesic acceleration (the inverse acceleration of the orthogonal on the tangent plane projected curve).

All in all we will find four basic entity relations:

Sum of the changes of geodesic curvature	$=$	0
Sum of the changes of geodesic acceleration	$\approx$	Gaussian curvature
Difference of the changes of geodesic curvature	$\approx$	Imaginary part of the Schwarzian derivative
Difference of the changes of the geodesic acceleration	$\approx$	Real part of the Schwarzian derivative

for orthogonal trajectories.

3 Sum of the changes of curvature for orthogonal trajectories of conformal mappings

Theorem 3.1 *The changes of the Euclidean curvatures of the component function of a conformal mapping are identical up to the sign.*

Let

$$\begin{aligned} z &= x + iy \\ f(x+iy) &= u(x,y) + iv(x,y) \end{aligned}$$

The parameterization invariant representations of the Euclidean curvature of the component functions u and v are:

$$\frac{\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial x^2} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial x^2} \end{vmatrix}}{\left(\sqrt{\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)^{\mathbf{2}} + \left(\frac{\partial \mathbf{v}}{\partial \mathbf{x}}\right)^{\mathbf{2}}}\right)^{\mathbf{3}}} \qquad \frac{\begin{vmatrix} \frac{\partial u}{\partial y} & \frac{\partial u}{\partial y^2} \\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial y^2} \end{vmatrix}}{\left(\sqrt{\left(\frac{\partial \mathbf{u}}{\partial \mathbf{y}}\right)^{\mathbf{2}} + \left(\frac{\partial \mathbf{v}}{\partial \mathbf{y}}\right)^{\mathbf{2}}}\right)^{\mathbf{3}}}$$

The following is true for the Euclidean changes of curvature:

$$\frac{\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial x^3} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial x^3} \end{vmatrix} \left(\sqrt{\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)^{\mathbf{2}} + \left(\frac{\partial \mathbf{v}}{\partial \mathbf{x}}\right)^{\mathbf{2}}}\right)^{\mathbf{3}} - \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial x^2} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial x^2} \end{vmatrix} \mathbf{3}\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\frac{\partial \mathbf{u}}{\partial \mathbf{x^2}} + \frac{\partial \mathbf{v}}{\partial \mathbf{x}}\frac{\partial \mathbf{v}}{\partial \mathbf{x^2}}\right)\sqrt{\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)^{\mathbf{2}} + \left(\frac{\partial \mathbf{v}}{\partial \mathbf{x}}\right)^{\mathbf{2}}}}{\left(\sqrt[2]{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2}\right)^{\mathbf{6}}} =$$

$$(-\mathbf{1})\frac{\begin{vmatrix}\frac{\partial u}{\partial y} & \frac{\partial u}{\partial y^3}\\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial y^3}\end{vmatrix}\left(\sqrt{\left(\frac{\partial \mathbf{u}}{\partial \mathbf{y}}\right)^{\mathbf{2}}+\left(\frac{\partial \mathbf{v}}{\partial \mathbf{y}}\right)^{\mathbf{2}}}\right)^{\mathbf{3}}-\begin{vmatrix}\frac{\partial u}{\partial y} & \frac{\partial u}{\partial y^2}\\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial y^2}\end{vmatrix}\mathbf{3}\left(\frac{\partial \mathbf{u}}{\partial \mathbf{y}}\frac{\partial \mathbf{u}}{\partial \mathbf{y^2}}+\frac{\partial \mathbf{v}}{\partial \mathbf{y}}\frac{\partial \mathbf{v}}{\partial \mathbf{y^2}}\right)\sqrt{\left(\frac{\partial \mathbf{u}}{\partial \mathbf{y}}\right)^{\mathbf{2}}+\left(\frac{\partial \mathbf{v}}{\partial \mathbf{y}}\right)^{\mathbf{2}}}}{\left(\sqrt[2]{\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2}\right)^{\mathbf{6}}}$$

Proof

CR-equations:

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

Laplace-equations

$$\frac{\partial u}{\partial x^2}+\frac{\partial u}{\partial y^2}=0 \qquad \frac{\partial v}{\partial x^2}+\frac{\partial v}{\partial y^2}=0$$

1. Derivative of the Laplace-equations

$$\frac{\partial u}{\partial x^3}+\frac{\partial u}{\partial x\partial y^2}=0 \qquad \frac{\partial v}{\partial x^3}+\frac{\partial v}{\partial x\partial y^2}=0$$

Therefore

$$\frac{\partial u}{\partial x^3}=-\frac{\partial u}{\partial x\partial y^2} \qquad \frac{\partial v}{\partial x^3}=-\frac{\partial v}{\partial x\partial y^2}$$

so further

$$\frac{\partial u}{\partial x^3}=-\frac{\partial u}{\partial y^2\partial x} \qquad \frac{\partial v}{\partial x^3}=-\frac{\partial v}{\partial y^2\partial x}v$$

and because of the CR-equations follows:

$$\frac{\partial u}{\partial x^3}=-\frac{\partial v}{\partial y^2\partial y} \qquad \frac{\partial v}{\partial x^3}=\frac{\partial u}{\partial y^2\partial y}$$

and thus altogether:

$$\frac{\partial u}{\partial x^3}=-\frac{\partial v}{\partial y^3} \qquad \frac{\partial v}{\partial x^3}=\frac{\partial u}{\partial y^3}$$

$$\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial x^3} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial x^3} \end{vmatrix} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & -\dfrac{\partial v}{\partial y^3} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial u}{\partial y^3} \end{vmatrix} =$$

$$\begin{vmatrix} \dfrac{\partial v}{\partial y} & -\dfrac{\partial v}{\partial y^3} \\ -\dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial y^3} \end{vmatrix} = -\begin{vmatrix} \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial y^3} \\ \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial y^3} \end{vmatrix}$$

2. Due to the CR-equations, the absolute values of the derivatives in x and y, i.e. the velocities in x and y are identical.

$$\sqrt[2]{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2} = \sqrt[2]{\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2}$$

3. The Euclidean acceleration raised to the second power and the affine velocity of the orthogonal trajectories are anti-dual.

$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial x^2} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial x^2} \end{vmatrix} = -\left(\frac{\partial u}{\partial y}\frac{\partial u}{\partial y^2} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial y^2}\right)$	$\begin{vmatrix} \frac{\partial u}{\partial y} & \frac{\partial u}{\partial y^2} \\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial y^2} \end{vmatrix} = -\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x^2} + \frac{\partial v}{\partial x}\frac{\partial x}{\partial x^2}\right)$

So altogether there is the duality of curvature and inverse acceleration.

$$\frac{\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial x^2} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial x^2} \end{vmatrix}}{\left(\sqrt[2]{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2}\right)^3} = (-1)\frac{\frac{\partial u}{\partial y}\frac{\partial u}{\partial y^2} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial y^2}}{\left(\sqrt[2]{\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2}\right)^3}$$

∎

4 Schwarzian derivative

The classical complex form of the Schwarzian derivative, see Duren, Osgood, Chuaqui [2], [4] or Ovsienko, Tabachnikov [7], is:

$$
\begin{aligned}
\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 &= \frac{f'''f'}{(f')^2} - \frac{3}{2}\frac{(f'')^2}{(f')^2} \\
&= \frac{2f'''f' - 3(f'')^2}{2(f')^2} \\
&= \frac{(2f'''f' - 3(f'')^2)\overline{(f')^2)}}{2(f')^2\overline{(f')^2)}} \\
&= \frac{2f'''\|f'\|^2\overline{f'} - 3(f''\overline{f'})^2}{2\|f'\|^4}
\end{aligned}
$$

However, to get more geometrical insight into the Schwarzian derivative, is it useful, to derive the explicit real formulation $f(x,y) = (u(x,y), v(x,y))$, for the real and imaginary part we have:

$$
\frac{2\left(\frac{\partial u}{\partial x^3}+i\frac{\partial v}{\partial x^3}\right)\left(\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)\right)^2\left(\frac{\partial u}{\partial x}-i\frac{\partial v}{\partial x}\right)-3\left(\left(\frac{\partial u}{\partial x^2}+i\frac{\partial v}{\partial x^2}\right)^2\left(\frac{\partial u}{\partial x}-i\frac{\partial v}{\partial x}\right)^2\right)}{2\left(\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right)^2}
$$

4.1 Real part of the Schwarzian derivative

$$
\frac{2\left(\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)\right)^2\left(\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x^3}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial x^3}\right)\right)-3\left(\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x^2}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial x^2}\right)^2-\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x^2}-\frac{\partial v}{\partial x}\frac{\partial u}{\partial x^2}\right)^2\right)}{2\left(\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right)^2}
$$

4.2 Changes of 'inverse acceleration'

Theorem 4.1 *The differences of the changes of the inverse acceleration of the orthogonal trajectories are equal to the real part of the Schwarzian derivative of a conformal mapping up to normalization.*

Proof For the inverse acceleration in direction x we have:

$$
\frac{\partial B^x}{\partial x} =
$$

$$
\frac{\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x^3}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial x^3}+\left(\frac{\partial u}{\partial x^2}\right)^2+\left(\frac{\partial v}{\partial x^2}\right)^2\right)\left(\sqrt{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2}\right)^3-\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x^2}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial x^2}\right)3\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x^2}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial x^2}\right)\sqrt{\left(\frac{\partial u}{\partial x}\right)^2\left(\frac{\partial v}{\partial x}\right)^2}}{\left(\sqrt[2]{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2}\right)^6} =
$$

$$\frac{\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x^3}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial x^3}+\left(\frac{\partial u}{\partial x^2}\right)^2+\left(\frac{\partial v}{\partial x^2}\right)^2\right)\left(\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right)-3\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x^2}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial x^2}\right)^2}{\left(\sqrt[2]{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2}\right)^5}$$

$$\frac{\partial B^x}{\partial y} =$$

$$\frac{\left(\frac{\partial u}{\partial y}\frac{\partial u}{\partial y^3}+\frac{\partial v}{\partial y}\frac{\partial v}{\partial y^3}+\left(\frac{\partial u}{\partial y^2}\right)^2+\left(\frac{\partial v}{\partial y^2}\right)^2\right)\left(\sqrt{\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2}\right)^3-\left(\frac{\partial u}{\partial y}\frac{\partial u}{\partial y^2}+\frac{\partial v}{\partial y}\frac{\partial v}{\partial y^2}\right)3\left(\frac{\partial u}{\partial y}\frac{\partial u}{\partial x^2}+\frac{\partial v}{\partial y}\frac{\partial v}{\partial x^2}\right)\sqrt{\left(\frac{\partial u}{\partial y}\right)^2\left(\frac{\partial v}{\partial y}\right)^2}}{\left(\sqrt[2]{\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2}\right)^6} =$$

$$\frac{\left(\frac{\partial -v}{\partial x}\frac{\partial v}{\partial x^3}+\frac{\partial -u}{\partial x}\frac{\partial u}{\partial x^3}+\left(\frac{\partial u}{\partial x^2}\right)^2+\left(\frac{\partial v}{\partial x^2}\right)^2\right)\left(\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right)-3\left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial x^2}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial x^2}\right)^2}{\left(\sqrt[2]{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2}\right)^5} =$$

$$\left(\frac{\partial B^x}{\partial x}-\frac{\partial B^x}{\partial y}\right)\frac{2}{\sqrt[2]{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2}} =$$

$$\frac{2\left(\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)\right)^2\left(\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x^3}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial x^3}\right)\right)-3\left(\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x^2}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial x^2}\right)^2-\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x^2}-\frac{\partial v}{\partial x}\frac{\partial u}{\partial x^2}\right)^2\right)}{2\left(\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right)^2}$$

$\blacksquare$

4.3 Imaginary part of the Schwarzian derivative

$$\frac{2\left(\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)\right)^2\left(\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x^3}-\frac{\partial v}{\partial x}\frac{\partial u}{\partial x^3}\right)\right)-3\left(2\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x^2}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial x^2}\right)\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x^2}-\frac{\partial v}{\partial x}\frac{\partial u}{\partial x^2}\right)\right)}{2\left(\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right)^2}$$

4.4 Changes of curvature

Theorem 4.2 *The differences of the changes of the curvature of the orthogonal trajectories are equal to the real part of the Schwarzian derivative of a conformal mapping up to normalization.*

Proof As already achieved in section 3 the following equation holds:

$$\frac{\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial x^3} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial x^3} \end{vmatrix} \left(\sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2} \right)^3 - \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial x^2} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial x^2} \end{vmatrix} 3 \left(\frac{\partial u}{\partial x} \frac{\partial u}{\partial x^2} + \frac{\partial v}{\partial x} \frac{\partial v}{\partial x^2} \right) \sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2}}{\left(\sqrt[2]{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2} \right)^6} =$$

$$(-1) \frac{\begin{vmatrix} \frac{\partial u}{\partial y} & \frac{\partial u}{\partial y^3} \\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial y^3} \end{vmatrix} \left(\sqrt{\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2} \right)^3 - \begin{vmatrix} \frac{\partial u}{\partial y} & \frac{\partial u}{\partial y^2} \\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial y^2} \end{vmatrix} 3 \left(\frac{\partial u}{\partial y} \frac{\partial u}{\partial y^2} + \frac{\partial v}{\partial y} \frac{\partial v}{\partial y^2} \right) \sqrt{\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2}}{\left(\sqrt[2]{\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2} \right)^6}$$

The differences of the changes of the curvature form the imaginary part of the Schwarzian derivative, i.e. two times the changes of curvature up to normalization. $\blacksquare$

The relation of the Schwarzian derivative to the changes of curvature was mentioned by Chuaqui, Duren, Osgood [4] in connection with their generalized Schwarzian derivative and it was emphasized by Osgood, [6], from an advanced differential geometric perspective, that the Schwarzian derivative might be considered as the difference of connections.

The rank of the curvature in Euclidean geometry (invarince to rotations and translations) is transferred to the Schwarzian derivative in conformal geoemtry (invariance to Moebius transformations). Here the "connection" is based on geodesic curvature and geodesic acceleration and the Schwarzian derivative is the difference of two such orthogonal connections.

5 Geodesic Curvature

The geodesic curvature of a curve on the surface is the curvature of that curve, that is derived by orthogonally projecting the curve on the surface on the tangent plane. Due to the formula from Eisenhart [3] for geodesic curvature $\frac{1}{\rho_g}$ in case that the parametric lines form an orthogonal system with respect to the parameter u and v:

$$\frac{1}{\rho_{gu}} = \frac{-1}{\sqrt{EG}} \frac{\partial \sqrt{E}}{\partial v}$$

$$\frac{1}{\rho_{gv}} = \frac{1}{\sqrt{EG}} \frac{\partial \sqrt{E}}{\partial u}$$

For isothermal parameterization we have

$$\frac{1}{\rho_{gu}} = \frac{-1}{E} \frac{\partial \sqrt{E}}{\partial v} = \frac{-1}{E} \frac{\partial E}{\partial v} \frac{1}{2} \frac{1}{\sqrt{E}} = \frac{-1}{\sqrt{E}^3} \frac{\partial E}{\partial v}$$

$$\frac{1}{\rho_{gv}} = \frac{1}{E} \frac{\partial \sqrt{E}}{\partial u} = \frac{1}{E} \frac{\partial E}{\partial u} \frac{1}{2} \frac{1}{\sqrt{E}} = \frac{1}{\sqrt{E}^3} \frac{\partial E}{\partial u}$$

Change of geodesic curvature:

$$\frac{\partial}{\partial u} \left(\frac{1}{\rho_{gu}} \right) = -1 \left(\frac{\frac{\partial E}{\partial u \partial v} \sqrt{E}^3 + \frac{\partial E}{\partial v} \frac{3}{2} \frac{\partial E}{\partial u} \sqrt{E}}{\sqrt{E}^6} \right) = -1 \left(\frac{\frac{\partial E}{\partial u \partial v} \sqrt{E}^2 + \frac{\partial E}{\partial v} \frac{3}{2} \frac{\partial E}{\partial u}}{\sqrt{E}^5} \right)$$

$$\frac{\partial}{\partial v} \left(\frac{1}{\rho_{gv}} \right) = \frac{\frac{\partial E}{\partial u \partial v} \sqrt{E}^3 - \frac{\partial E}{\partial u} \frac{3}{2} \frac{\partial E}{\partial v} \sqrt{E}}{\sqrt{E}^6} = \frac{\frac{\partial E}{\partial u \partial v} E - \frac{\partial E}{\partial u} \frac{3}{2} \frac{\partial E}{\partial v}}{\sqrt{E}^5}$$

Therefore the sum of the changes of geodesic curvature of the orthogonal trajectories is zero:

$$\frac{\partial}{\partial u} \frac{1}{\rho_{gu}} + \frac{\partial}{\partial v} \frac{1}{\rho_{gv}} = 0$$

Furthermore, if the sums of the changes of geodesic curvature vanish, the projected curve is locally conformal.

6 Geodesic Acceleration

Instead of calculating the changes of geodesic curvature in x with respect to x and the geodesic curvature in y with respect to y, we change the arguments for taking the derivative, regard the sign and get:

$$\frac{\partial}{\partial v}\left(\frac{1}{\rho_{gu}}\right) = -1\left(\frac{\frac{\partial E}{\partial v^2}\sqrt{E}^3 - \frac{3}{2}\frac{\partial E}{\partial v}^2}{\sqrt{E}^5}\right)$$

$$\frac{\partial}{\partial u}\left(\frac{1}{\rho_{gv}}\right) = \frac{\frac{\partial E}{\partial u^2}\sqrt{E}^3 - \frac{3}{2}\frac{\partial E}{\partial u}^2}{\sqrt{E}^5}$$

So for the differences of geodesic acceleration we find:

$$-\frac{\partial}{\partial v}\frac{1}{\rho_{gu}} - \frac{\partial}{\partial u}\frac{1}{\rho_{gv}} =$$

$$\frac{E\left(\frac{\partial E}{\partial u^2} - \frac{\partial E}{\partial v^2}\right) - \frac{3}{2}\left(\frac{\partial E}{\partial u}^2 - \frac{\partial E}{\partial v}^2\right)}{\sqrt{E}^5}$$

7 Gaussian curvature

For an orthogonal parameterization, Gaussian curvature is

$$K = \frac{1}{2\sqrt{EG}}\left(\frac{\partial}{\partial u}\frac{G_u}{\sqrt{EG}} + \frac{\partial}{\partial v}\frac{E_v}{\sqrt{EG}}\right)$$

for an isothermal parameterization:

$$K = \frac{1}{2E}\left(\frac{\partial}{\partial u}\frac{E_u}{E} + \frac{\partial}{\partial v}\frac{E_v}{E}\right)$$

as the appropriate representation for an interpretation in terms of logarithmic acceleration.

The sum of the changes of geodesic acceleration with respect to logarithmic acceleration is evidently zero for a conformal plane mapping:

$$\frac{\partial}{\partial x}\left(\frac{\left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial x^2}+\frac{\partial v}{\partial x^2}\frac{\partial u}{\partial x^2}\right)}{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2}\right)+\frac{\partial}{\partial y}\left(\frac{\left(\frac{\partial v}{\partial y}\frac{\partial u}{\partial y^2}+\frac{\partial v}{\partial y^2}\frac{\partial u}{\partial y^2}\right)}{\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2}\right)=0$$

8 Harmonic Schwarzian derivative

The classical Schwarzian derivative based on the differences of the changes of geodesic curvature and the differences of the changes of geodesic acceleration is equivalent up to normalization to the definition for the harmonic Schwarzian derivative derived by Duren [2] with respect to conformal metrics of minimal surfaces. (Duren [2] - let f be harmonic, locally univalent $f = g+\overline{h}$ with dilation $\omega = \frac{g'}{h'}$ of the form $\omega = q^2$ for some analytic function q, which can be lifted to a minimal surface with metric $ds = \lambda|dz|$ where $\lambda = |h'|+|g'|$. Then $S_z(f) = 2((\log\lambda)_{zz} - ((\log\lambda)_z)^2)))$

In case of an isothermal lines of curvature parameterization there is $\lambda^2 = E$.

Let $\frac{\partial E}{\partial z} = \frac{1}{2}\left(\frac{\partial E}{\partial x} - i\frac{\partial E}{\partial y}\right)$

$$\frac{\partial E}{\partial z^2} = \frac{1}{4}\left(\frac{\partial E}{\partial x^2} - i\frac{\partial E}{\partial x\partial y}\right) + \frac{1}{4}\left(-i\frac{\partial E}{\partial x\partial y} - \frac{\partial E}{\partial y^2}\right)$$

$$2((\log\sqrt{E})_{zz} - ((\log\sqrt{E})_z)^2) =$$

$$2\left(\frac{\frac{1}{2}\frac{\partial E}{\partial z}}{\partial\sqrt{E}}\frac{1}{\sqrt{E}}\right)_z - \left(\frac{\frac{1}{2}\frac{\partial E}{\partial z}}{E}\right)^2 =$$

$$2\left(\frac{\frac{1}{2}\frac{\partial E}{\partial z}}{\partial E}\right)_z - \left(\frac{\frac{1}{2}\frac{\partial E}{\partial z}}{E}\right)^2 =$$

$$2\left(\frac{1}{2}\frac{\frac{\partial E}{\partial z^2}E - \left(\frac{\partial E}{\partial z}\right)^2}{E^2} - \left(\frac{1}{4}\frac{\frac{\partial E}{\partial z}}{E^2}\right)^2\right) =$$

$$2\left(\frac{\frac{1}{2}\frac{\partial E}{\partial z^2}E - \frac{3}{4}\left(\frac{\partial E}{\partial z}\right)^2}{E^2}\right) =$$

$$\left(\frac{\frac{\partial E}{\partial z^2} E - \frac{3}{2} \left(\frac{\partial E}{\partial z} \right)^2}{E^2} \right) =$$

$$\frac{1}{4} \left(\frac{\left(\frac{\partial E}{\partial x^2} - 2i \frac{\partial E}{\partial x \partial y} - \frac{\partial E}{\partial y^2} \right) E - \frac{3}{2} \left(\left(\frac{\partial E}{\partial x} \right)^2 - 2i \frac{\partial E}{\partial x} \frac{\partial E}{\partial y} - \left(\frac{\partial E}{\partial y} \right)^2 \right)}{E^2} \right) =$$

$$\frac{1}{4} \left(\frac{\left(\frac{\partial E}{\partial x^2} - \frac{\partial E}{\partial y^2} \right) E - \frac{3}{2} \left(\left(\frac{\partial E}{\partial x} \right)^2 - \left(\frac{\partial E}{\partial y} \right)^2 \right) - 2i \left(\frac{\partial E}{\partial x \partial y} E - \frac{3}{2} \frac{\partial E}{\partial x} \frac{\partial E}{\partial y} \right)}{E^2} \right)$$

So the real part is evidently the difference of the changes of geodesic acceleration up to normalization, see section 6 and the imaginary part is evidently the difference of the changes of geodesic curvature up to normalization, see section 5.

Which surfaces do have the same harmonic Schwarzian derivative? It is obvious that a threedimensional rotation does neither modify the changes of geodesic curvature nor the changes of geodesic acceleration. However, besides the rotations we have to take associate surfaces (isometries) into consideration, see Blaschke [1] (Bonnet) and Duren [2] and furthermore the remarks of R.M. Kiehn about Cartan Spinors generating two minimal surfaces.

9 Lines of curvature, minimal geodesic torsion, miminal surfaces

Conformality enforces that the sum of the changes of geodesic curvature vanishes and the orthogonality of the trajectories. Such pairwise requirements for three surfaces intersecting leads to the fact, that such surfaces intersect in lines of curvature, (theorem of Dupin), (based already on orthogonality) and proven for example by Blaschke [1] based on the fact, that the sum of the geodesic torsions of the intersecting surfaces is pairwise zero, so that each geodesic torsion is zero and thus the lines of intersection are lines of curvature. (However, there is no rich structure of triply orthogonal minimal surfaces, last exercise in the book of Eisenhart citeeisenhart.) With regard to this it is intuitive to have in mind the isothermic surfaces, surfaces with conformally parameterized lines of curvatures - like surfaces with constant mean curvature - as the metastructure of surfaces, where the sums of the changes of geodesic curvature vanishes. That said, we want to characterize the lines in the plane, which are the images of projecting lines of curvature from minimal surfaces onto the plane.

Due to Eisenhart, [3]: the parameters of lines of curvature of a minimal surface (isothermal system with $D = -D'' = 1, D' = 0$ may be chosen so, that the linear elements of the surface has the respective form:

$$ds^2 = \rho(du^2 + dv^2)$$

where ρ is the absolute value of each principal radius. (Any other value of the constant $D = -D'' = 1$ leads to homothetic surfaces.)

So the following must hold for the Gaussian curvature K of the associated minimal surfaces in case of the lines of curvature parameterization due to Duren [2], referring to standard harmonic mapping decomposition as $f = h + \overline{g}$:

$$ds^2 = \lambda^2 |dz|^2 = (|h'| +' g'|)^2 |dz|^2$$

$$\begin{aligned}
K = \frac{|\left(\frac{g'}{h'}\right)'|^2}{|h'g'|\left(1 + |\frac{g'}{h'}|\right)^4} &= \frac{1}{(|h'|+|g'|)^2 \, (|h'|+|g'|)^2} \\
\left|\frac{g''h' - g'h''}{(h')^2}\right|^2 &= \frac{1}{(|h'|+|g'|)^4 \, |h'g'|} \frac{(|g'|+|h'|)^4}{|h'|^4} \\
|g''h' - g'h''|^2 &= |h'g'|
\end{aligned}$$

Let again $u = \Re f(z)$ and $v = \Im f(z)$

$$
\begin{aligned}
\begin{vmatrix} \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial z^2} & \frac{\partial v}{\partial z^2} \end{vmatrix} &= \\
\begin{vmatrix} \frac{1}{2}(h' + g') & \frac{1}{2i}(h' - g') \\ \frac{1}{2}(h'' + g'') & \frac{1}{2i}(h'' - g'') \end{vmatrix} &= \\
\frac{1}{4i}\left(h'h'' - h'g'' + g'h'' - g'g''\right) - \frac{1}{4i}\left(h''h' - h''g' + g''h' - g'g''\right) &= \\
\frac{2}{4i}\left(-h'g'' + h''g'\right) &= \\
\frac{1}{2i}\left(g'h'' - h'g''\right)
\end{aligned}
$$

Due to the equation Duren [2],

$$
|h'g'| = |\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2|
$$

the following holds, in analogy to the ordinary curvature representation of plane real curves:

$$
4|\begin{vmatrix} \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial z^2} & \frac{\partial v}{\partial z^2} \end{vmatrix}|^{\mathbf{2}} = |\left(\frac{\partial \mathbf{u}}{\partial \mathbf{z}}\right)^{\mathbf{2}} + \left(\frac{\partial \mathbf{v}}{\partial \mathbf{z}}\right)^{\mathbf{2}}|
$$

or

$$
\boxed{\frac{|\begin{vmatrix} \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial z^2} & \frac{\partial v}{\partial z^2} \end{vmatrix}|}{\sqrt{|\left(\frac{\partial \mathbf{u}}{\partial \mathbf{z}}\right)^{\mathbf{2}} + \left(\frac{\partial \mathbf{v}}{\partial \mathbf{z}}\right)^{\mathbf{2}}|}} = \frac{\mathbf{1}}{\mathbf{4}}}
$$

that is, the 'complex curvature' is constant for the coordinate net representing the lines of curvature pararameterization, the parameterization with minimal geodesic torsion.

10 Conclusion

In a simplified world, where there exists nothing else but geometries with objects in coordinate net series expansion based on the canonical group representation in its inherent parameterization and projections and transitions of geometries, it was tried to describe the notion "form" as an entity of the

Euclidean group in similarity group representation leading to an inherent conformal coordinate parameterization: the geometry's intrinsic representation parameterizes and curves the coordinate net and in the present case, dynamics and curvature are duals.

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