A New Look at the Position Operator in Quantum Theory

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Abstract:

The postulate that coordinate and momentum representations are related to each other by the Fourier transform has been accepted from the beginning of quantum theory by analogy with classical electrodynamics. As a consequence, an inevitable effect in standard theory is the wave packet spreading (WPS) of the photon coordinate wave function in directions perpendicular to the photon momentum. This leads to several paradoxes. The most striking of them is that coordinate wave functions of photons emitted by stars have cosmic sizes and strong arguments indicate that this contradicts observational data. We argue that the above postulate is based neither on strong theoretical arguments nor on experimental data and propose a new consistent definition of the position operator. Then WPS in directions perpendicular to the particle momentum is absent and the paradoxes are resolved. Different components of the new position operator do not commute with each other and, as a consequence, there is no wave function in coordinate representation. Implications of the results for entanglement, quantum locality and the problem of time in quantum theory are discussed.

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1 Status of the position operator in quantum theory

1.1 Historical reasons for choosing standard form of position operator

It has been postulated from the beginning of quantum theory that the coordinate and momentum representations of wave functions are related to each other by the Fourier transform. One of the historical reasons was that in classical electrodynamics the coordinate and wave vector $\mathbf{k}$ representations are related analogously and we postulate that $\mathbf{p} = \hbar\mathbf{k}$ where $\mathbf{p}$ is the particle momentum. Then, although the interpretations of classical fields on one hand and wave functions on the other are fully different,

from mathematical point of view classical electrodynamics and quantum mechanics have much in common (and such a situation does not seem to be natural).

Similarity of classical electrodynamics and quantum theory is reflected even in the terminology of the latter. The terms "wave function", "particle-wave duality" and "de Broglie wave length" have arisen at the beginning of quantum era in efforts to explain quantum behavior in terms of classical waves but now it is clear that no such explanation exists. The notion of wave is purely classical; it has a physical meaning only as a way of describing systems of many particles by their mean characteristics. In particular, such notions as frequency and wave length can be applied only to classical waves, i.e. to systems consisting of many particles. If a particle state vector contains $exp[i(\mathbf{pr} - Et)/\hbar]$, where E is the energy, then by analogy with the theory of classical waves one might say that the particle is a wave with the frequency $\omega = E/\hbar$ and the (de Broglie) wave length $\lambda = 2\pi\hbar/|\mathbf{p}|$. However, such defined quantities ω and λ are not real frequencies and wave lengths measured on macroscopic level. A striking example showing that on quantum level λ does not have the usual meaning is that from the point of view of classical theory an electron having the size of the order of the Bohr radius cannot emit a wave with $\lambda = 21cm$ (this observation has been pointed out to me by Volodya Netchitailo).

In quantum theory the photon and other particles are characterized by their energies, momenta and other quantities for which there exist well defined operators while the notion of coordinates on quantum level is a problem which is investigated in the present paper. The term "wave function" might be misleading since in quantum theory it defines not amplitudes of waves but only amplitudes of probabilities. So, although in our opinion the term "state vector" is more pertinent than "wave function" we will use the latter in accordance with the usual terminology, and the phrase that a photon has a frequency ω and the wave length λ will be understood only such that $\omega = E/\hbar$ and $\lambda = 2\pi\hbar/|\mathbf{p}|$.

One of the examples of the above similarity follows. Consider a wave function of the form $\psi(\mathbf{r},t) = a(\mathbf{r},t)exp[iS(\mathbf{r},t)/\hbar]$, where $S(\mathbf{r},t)$ is the classical action as a function of coordinates and time. Then

$$\frac{\partial\psi(\mathbf{r},t)}{\partial\mathbf{r}} = [\frac{i}{\hbar}\frac{\partial S(\mathbf{r},t)}{\partial\mathbf{r}} + \frac{1}{a(\mathbf{r},t)}\frac{\partial a(\mathbf{r},t)}{\partial\mathbf{r}}]\psi(\mathbf{r},t) \tag{1}$$

and analogously for $\partial\psi(\mathbf{r},t)/\partial t$. In the formal limit $\hbar \to 0$ the second term in the square brackets can be neglected and, as explained in textbooks on quantum mechanics (see e.g. Ref. [1]) the Schrödinger equation becomes the Hamilton-Jacoby equation. This situation is analogous to the approximation of geometrical optics in classical electrodynamics (see e.g. Ref. [2]) when fields contain a rapidly oscillating factor $exp[i\varphi(\mathbf{r},t)]$ where the function $\varphi(\mathbf{r},t)$ is called eikonal. It satisfies the eikonal equation which coincides with the relativistic Hamilton-Jacobi equation for a particle with zero mass. This is reasonable in view of the fact that electromagnetic waves consist of photons.

Another example follows. In classical electrodynamics a wave packet moving even in empty space inevitably spreads out and this fact has been known for a long time. For example, as pointed out by Schrödinger (see pp. 41-44 in Ref. [3]), in standard quantum mechanics a packet does not spread out if a particle is moving in a harmonic oscillator potential in contrast to "a wave packet in classical optics, which is dissipated in the course of time". However, as a consequence of the similarity, a free quantum mechanical wave packet inevitably spreads out too. This effect is called wave packet spreading (WPS) and it is described in textbooks and many papers (see e.g. Refs. [4, 5] and references therein). In the present paper this effect is discussed in detail and we argue that it plays a crucial role in drawing a conclusion on whether standard position operator is consistently defined.

The requirement that the momentum and position operators are related to each other by the Fourier transform is equivalent to standard commutation relations between these operators and to the Heisenberg uncertainty principle (see Sec. 2).

A reason for choosing standard form of the position operator is described, for example, in the Dirac textbook [4]. Here Dirac argues that the momentum and position operators should be such that their commutator should be proportional to the corresponding classical Poisson bracket with the coefficient $i\hbar$. However, this argument is not convincing because only in very special cases the commutator of two physical operators is a c-number. One can check, for example, a case of momentum and position operators squared.

In Ref. [6] Heisenberg argues in favor of his principle by considering *Gedankenexperiment* with Heisenberg's microscope. Since that time the problem has been investigated in many publications. A discussion of the current status of the problem can be found e.g. in Ref. [7] and references therein. A general opinion based on those investigations is that Heisenberg's arguments are problematic but the uncertainty principle is valid, although several authors argue whether standard mathematical notion of uncertainty (see Sec. 2) is relevant for describing a real process of measurement. However, a common assumption in those investigations is that one can consider uncertainty relations for all the components of the position and momentum operators independently. Below we argue that this assumption is not based on solid physical arguments.

1.2 Problem of consistency of standard position operator

Usual arguments in favor of choosing standard position and momentum operators are that these operators have correct properties in semiclassical approximation (see e.g. Ref. [1]). However, this requirement does not define the operator unambiguously. Indeed, if the operator B becomes zero in semiclassical limit then the operators A and $A + B$ have the same semiclassical limit.

One of the principles of physics is the correspondence one according to which any new theory should reproduce results of the old well tested theory at some

conditions. As noted above, in the main approximation in $1/\hbar$ the Schrödinger equation becomes the Hamilton-Jacoby equation if the coordinate wave function $\psi(\mathbf{r},t)$ contains a factor $exp[iS(\mathbf{r},t)/\hbar]$. In textbooks this is usually treated as the correspondence principle between quantum and classical theories. However, the following question arises.

As follows from Eq. (1), the Hamilton-Jacoby equation is a good approximation for the Schrödinger equation if the index of the exponent changes much faster than the amplitude $a(\mathbf{r},t)$. Is this correct to define semiclassical approximation by this condition? Quantum theory fully reproduces the results of classical one when not only this condition is satisfied but, in addition, the amplitude has a sharp maximum along the classical trajectory. If the latter is true at some moment of time then, in view of the WPS effect, one cannot gurantee that this will be true always.

At the beginning of quantum theory the WPS effect has been investigated by de Broglie, Darwin and Schrödinger. The fact that WPS is inevitable has been treated by several authors as unacceptable and as an indication that standard quantum theory should be modified. For example, de Broglie has proposed to describe a free particle not by the Schrödinger equation but by a wavelet which satisfies a nonlinear equation and does not spread out (a detailed description of de Broglie's wavelets can be found e.g. in Ref. [8]). Sapogin writes (see Ref. [9] and references therein) that "Darwin showed that such packet quickly and steadily dissipates and disappears" and proposes an alternative to standard theory which he calls unitary unified quantum field theory.

At the same time, it has not been explicitly shown that numerical results on WPS are incompatible with experimental data. For example, it is known (see Sec. 3) that for macroscopic bodies the effect of WPS is extremely small. Probably it is also believed that in experiments on the Earth with atoms and elementary particles spreading does not have enough time to manifest itself although we have not found an explicit statement on this problem in the literature. According to our observations, different physicists have different opinions on the role of WPS in different phenomena but in any case the absolute majority of physicists do not treat WPS as a drawback of the theory.

A natural problem arises what happens to photons which can travel from distant objects to Earth even for billions of years. As shown in Sec. 10, standard theory predicts that, as a consequence of WPS, wave functions of such photons will have the size of the order of millions or billions kilometers or even more. Does this contradict observations? We argue that it does and the reason of the paradox is that standard position operator is not consistently defined. Hence the inconsistent definition of the position operator is not only an academic problem but leads to the above paradox.

In view of the fact that the coordinate and momentum representations are related to each other by the Fourier transform, one might think that the position and momentum operators are on equal footing. However, this is not the case for the

following reasons. In quantum theory each elementary particle is described by an irreducible representation (IR) of the symmetry algebra. For example, in Poincare invariant theory the set of momentum operators represents three of ten linearly independent representation operators of the Poincare algebra and hence those operators are consistently defined. On the other hand, among the representation operators there is no position operator. So the assumption that the position operator in momentum representation is $i\hbar\partial/\partial\mathbf{p}$ should be substantiated.

Consider first a one-dimensional case. As argued in textbooks (see e.g. Ref. [1]), if the mean value of the x component of the momentum p_x is rather large, the definition of the coordinate operator $i\hbar\partial/\partial p_x$ can be justified but this definition does not have a physical meaning in situations when p_x is small. This is clear even from the fact that if p_x is small then $exp(ip_x x/\hbar)$ is not a rapidly oscillating function of x.

Consider now the three-dimensional case. If all the components p_j ($j = 1, 2, 3$) are rather large then all the operators $i\hbar\partial/\partial p_j$ can have a physical meaning. A semiclassical wave function $\chi(\mathbf{p})$ in momentum space should describe a narrow distribution around the mean value $\mathbf{p}_0$. Suppose now that coordinate axes are chosen such $\mathbf{p}_0$ is directed along the z axis. Then the mean values of the x and y components of the momentum operator equal zero and the operators $i\hbar\partial/\partial p_j$ cannot be physical for $j = 1, 2$, i.e. in directions perpendicular to the particle momentum. The situation when a definition of an operator is physical or not depending on the choice of coordinate axes is not acceptable. Hence standard definition of the position operator is not physical.

1.3 When do we need position operator in quantum theory?

The position operator is used in many standard problems of quantum theory. For example, one of the arguments in favor of its validity is that the nonrelativistic Schrödinger equation correctly describes the hydrogen energy levels, the Dirac equation correctly describes fine structure corrections to these levels etc. Historically these equations have been first written in coordinate space and in textbooks they are still discussed in this form. However, from the point of view of the present knowledge those equations should be treated as follows.

A fundamental theory describing electromagnetic interactions on quantum level is quantum electrodynamics (QED). This theory proceeds from quantizing classical Lagrangian which is only an auxiliary tool for constructing S-matrix. The argument $\mathbf{x}$ in the Lagrangian density $L(t, \mathbf{x})$ cannot be treated as a position operator because $L(t, \mathbf{x})$ is constructed from field functions which do not have a probabilistic interpretation. When quantization is accomplished, the results of QED are formulated exclusively in momentum space and the theory does not contain space-time at all.

In particular, as follows from Feynman diagrams for the one-photon ex-

change, in the approximation $(v/c)^2$ the electron in the hydrogen atom can be described in the potential formalism where the potential acts on the wave function in momentum space. So for calculating energy levels one should solve the eigenvalue problem for the Hamiltonian with this potential. This is an integral equation which can be solved by different methods. One of the convenient methods is to apply the Fourier transform and get standard Schrödinger or Dirac equation in coordinate representation with the Coulomb potential. Hence the fact that the results for energy levels are in good agreement with experiment shows only that QED defines the potential correctly and *standard coordinate Schrödinger and Dirac equations are only convenient mathematical ways of solving the eigenvalue problem*. For this problem the physical meaning of the position operator is not important at all. One can consider other transformations of the original integral equation and define other position operators. The fact that for non-standard choices one might obtain something different from the Coulomb potential is not important on quantum level. On classical level the interaction between two charges can be described by the Coulomb potential but this does not imply that on quantum level the potential in coordinate representation should be necessarily Coulomb.

Let us also note the following. In the literature the statement that the Coulomb law works with a high accuracy is often substantiated from the point of view that predictions of QED have been experimentally confirmed with a high accuracy. However, as follows from the above remarks, the meaning of distance on quantum level is not clear and in QED the law $1/r^2$ can be tested only if we assume additionally that the coordinate and momentum representations are related to each other by the Fourier transform. So a conclusion about the validity of the law can be made only on the basis of macroscopic experiments. A conclusion made from the results of classical Cavendish and Maxwell experiments is that if the exponent in Coulomb's law is not 2 but $2 \pm q$ then $q < 1/21600$. The accuracy of those experiments have been considerably improved in the experiment [10] the result of which is $q < 2 \cdot 10^{-9}$. However, the Cavendish-Maxwell experiments and the experiment [10] do not involve pointlike electric charges. Cavendish and Maxwell used a spherical air condenser consisting of two insulated spherical shells while the authors of Ref. [10] developed a technique where the difficulties due to spontaneous ionization and contact potentials were avoided. Therefore the conclusion that $q < 2 \cdot 10^{-9}$ for pointlike electric charges requires additional assumptions.

Another example follows. It is said that the spatial distribution of the electric charge inside a system can be extracted from measurements of form-factors in the electron scattering on this system. However, the information about the experiment is again given only in terms of momenta and conclusions about the spatial distribution can be drawn only if we assume additionally how the position operator is expressed in terms of momentum variables. On quantum level the physical meaning of such a spatial distribution is not fundamental.

In view of the above discussion, since the *results* of existing fundamental

quantum theories describing interactions on quantum level (QED, electroweak theory and QCD) are formulated exclusively in terms of the S-matrix in momentum space without any mentioning of space-time, *for investigating such stationary quantum problems as calculating energy levels, form-factors etc., the notion of the position operator is not needed.*

However, the choice of the position operator is important in nonstationary problems when evolution is described by the time dependent Schrödinger equation (with the nonrelativistic or relativistic Hamiltonian). As follows from the correspondence principle, quantum theory should reproduce the motion of a particle along the classical trajectory defined by classical equations of motion. Hence the position operator is needed only in semiclassical approximation and it should be *defined* from additional considerations.

In standard approaches to quantum theory the existence of space-time background is assumed from the beginning. Then the position operator for a particle in this background is the operator of multiplication by the particle radius-vector $\mathbf{r}$. As explained in textbooks on quantum mechanics (see e.g. Ref. [1]), the result $-i\hbar\partial/\partial\mathbf{r}$ for the momentum operator can be justified from the requirement that quantum theory should correctly reproduce classical results in semiclassical approximation. However, as noted above, this requirement does not define the operator unambiguously.

A standard approach to Poincare symmetry on quantum level follows. Since Poincare group is the group of motions of Minkowski space, quantum states should be described by representations of the Poincare group. In turn, this implies that the representation generators should commute according to the commutation relations of the Poincare group Lie algebra:

$$\begin{aligned}&[P^\mu, P^\nu] = 0 \quad [P^\mu, M^{\nu\rho}] = -i(\eta^{\mu\rho}P^\nu - \eta^{\mu\nu}P^\rho)\\ &[M^{\mu\nu}, M^{\rho\sigma}] = -i(\eta^{\mu\rho}M^{\nu\sigma} + \eta^{\nu\sigma}M^{\mu\rho} - \eta^{\mu\sigma}M^{\nu\rho} - \eta^{\nu\rho}M^{\mu\sigma})\end{aligned} \tag{2}$$

where P^μ are the operators of the four-momentum, $M^{\mu\nu}$ are the operators of Lorentz angular momenta, the diagonal metric tensor $\eta^{\mu\nu}$ has the nonzero components $\eta^{00} = -\eta^{11} = -\eta^{22} = -\eta^{33} = 1$ and $\mu, \nu = 0, 1, 2, 3$. It is usually said that the above relations are written in the system of units $c = \hbar = 1$. However, as we argue in Ref. [11], quantum theory should not contain c and $\hbar$ at all; those quantities arise only because we wish to measure velocities in m/s and angular momenta in $kg \times m^2/s$.

The above approach is in the spirit of Klein's Erlangen program in mathematics. However, as we argue in Refs. [11, 12], quantum theory should not be based on classical space-time background. The notion of space-time background contradicts the basic principle of physics that a definition of a physical quantity is a description of how this quantity should be measured. Indeed one cannot measure coordinates of a manifold which exists only in our imagination.

As we argue in Refs. [11, 12] and other publications, the approach should be the opposite. Each system is described by a set of independent operators. By

definition, the rules how these operators commute with each other define the symmetry algebra. In particular, *by definition*, Poincare symmetry on quantum level means that the operators commute according to Eq. (2). This definition does not involve Minkowski space at all. Such a definition of symmetry on quantum level is in the spirit of Dirac's paper [13].

The fact that an elementary particle in quantum theory is described by an IR of the symmetry algebra can be treated as a definition of the elementary particle. In Poincare invariant theory the IRs can be implemented in a space of functions $\chi(\mathbf{p})$ such that $\int |\chi(\mathbf{p})|^2 d^3\mathbf{p} < \infty$ (see Sec. 5). In this representation the momentum operator $\mathbf{P}$ is defined *unambiguously* and is simply the operator of multiplication by $\mathbf{p}$. A standard *assumption* is that the position operator in this representation is $i\hbar\partial/\partial\mathbf{p}$. However, as argued above, this assumption is not consistent.

In the present paper we propose a consistent definition of the position operator. As a consequence, in our approach WPS in directions perpendicular to the particle momentum is absent regardless of whether the particle is nonrelativistic or relativistic. Moreover, for an ultrarelativistic particle the effect of WPS is absent at all. In our approach different components of the position operator do not commute with each other and, as a consequence, there is no wave function in coordinate representation.

Our presentation is self-contained and for reproducing the results of the calculations no special knowledge is needed. The paper is organized as follows. In Secs. 2 and 5 we discuss the approach to the position operator in standard nonrelativistic and relativistic quantum theory, respectively. An inevitable consequence of this approach is the effect of WPS of the coordinate wave function which is discussed in Secs. 3 and 6 for the nonrelativistic and relativistic cases, respectively. In Sec. 8 we discuss a relation between the WPS effects for a classical wave packet and for photons comprising this packet. In Sec. 9 the problem of WPS in coherent states is discussed. In Sec. 10 we show that the WPS effect leads to several paradoxes and, as discussed in Sec. 11, in standard theory it is not possible to avoid those paradoxes. Our approach to a consistent definition of the position operator and its application to WPS are discussed in Secs. 12-14. Finally, in Sec. 15 we discuss implications of the results for entanglement, quantum locality and the problem of time in quantum theory.

2 Position operator in nonrelativistic quantum mechanics

In quantum theory, states of a system are represented by elements of a projective Hilbert space. The fact that a Hilbert space H is projective means that if $\psi \in H$ is a state then $const \cdot \psi$ is the same state. The matter is that not the probability itself but only relative probabilities of different measurement outcomes

have a physical meaning. In this paper we will work with states ψ normalized to one, i.e. such that $||\psi|| = 1$ where $||...||$ is a norm. It is defined such that if $(..., ...)$ is a scalar product in H then $||\psi|| = (\psi, \psi)^{1/2}$.

In quantum theory every physical quantity is described by a selfadjoint operator. Each selfadjoint operator is Hermitian i.e. satisfies the property $(\psi_2, A\psi_1) = (A\psi_2, \psi_1)$ for any states belonging to the domain of A. If A is an operator of some quantity then the mean value of the quantity and its uncertainty in state ψ are given by $\bar{A} = (\psi, A\psi)$ and $\Delta A = ||(A - \bar{A})\psi||$, respectively. The condition that a quantity corresponding to the operator A is semiclassical in state ψ can be defined such that $\Delta A \ll |\bar{A}|$. This implies that the quantity can be semiclassical only if $|\bar{A}|$ is rather large. In particular, if $\bar{A} = 0$ then the quantity cannot be semiclassical.

Let B be an operator corresponding to another physical quantity and $\bar{B}$ and ΔB be the mean value and the uncertainty of this quantity, respectively. We can write $AB = \{A, B\}/2 + [A, B]/2$ where the commutator $[A, B] = AB - BA$ is anti-Hermitian and the anticommutator $\{A, B\} = AB + BA$ is Hermitian. Let $[A, B] = -iC$ and $\bar{C}$ be the mean value of the operator C.

A question arises whether two physical quantities corresponding to the operators A and B can be simultaneously semiclassical in state ψ. Since $||\psi_1||||\psi_2|| \geq |(\psi_1, \psi_2)|$, we have that

$$\Delta A \Delta B \geq \frac{1}{2}|(\psi, (\{A - \bar{A}, B - \bar{B}\} + [A, B])\psi)| \tag{3}$$

Since $(\psi, \{A - \bar{A}, B - \bar{B}\}\psi)$ is real and $(\psi, [A, B]\psi)$ is imaginary, we get

$$\Delta A \Delta B \geq \frac{1}{2}|\bar{C}| \tag{4}$$

This condition is known as a general uncertainty relation between two quantities. A well-known special case is that if P is the x component of the momentum operator and X is the operator of multiplication by x then $[P, X] = -i\hbar$ and $\Delta p \Delta x \geq \hbar/2$. The states where $\Delta p \Delta x = \hbar/2$ are called coherent ones. They are treated such that the momentum and the coordinate are simultaneously semiclassical in a maximal possible extent. A well-known example is that if

$$\psi(x) = \frac{1}{a^{1/2}\pi^{1/4}} exp[\frac{i}{\hbar}p_0 x - \frac{1}{2a^2}(x - x_0)^2]$$

then $\bar{X} = x_0$, $\bar{P} = p_0$, $\Delta x = a/\sqrt{2}$ and $\Delta p = \hbar/(a\sqrt{2})$.

Consider first a one dimensional motion. In standard textbooks on quantum mechanics, the presentation starts with a wave function $\psi(x)$ in coordinate space since it is implicitly assumed that the meaning of space coordinates is known. Then a question arises why $P = -i\hbar d/dx$ should be treated as the momentum operator. The explanation follows.

Consider wave functions having the form $\psi(x) = exp(ip_0x/\hbar)a(x)$ where the amplitude $a(x)$ has a sharp maximum near $x = x_0 \in [x_1, x_2]$ such that $a(x)$ is not small only when $x \in [x_1, x_2]$. Then Δx is of the order $x_2 - x_1$ and the condition that the coordinate is semiclassical is $\Delta x \ll |x_0|$. Since $-i\hbar d\psi(x)/dx = p_0\psi(x) - i\hbar exp(ip_0x/\hbar)da(x)/dx$, we see that $\psi(x)$ will be approximately the eigenfunction of $-i\hbar d/dx$ with the eigenvalue p_0 if $|p_0a(x)| \gg \hbar|da(x)/dx|$. Since $|da(x)/dx|$ is of the order of $|a(x)/\Delta x|$, we have a condition $|p_0\Delta x| \gg \hbar$. Therefore if the momentum operator is $-i\hbar d/dx$, the uncertainty of momentum Δp is of the order of $\hbar/\Delta x$, $|p_0| \gg \Delta p$ and this implies that the momentum is also semiclassical. At the same time, $|p_0\Delta x|/2\pi\hbar$ is approximately the number of oscillations which the exponent makes on the segment $[x_1, x_2]$. Therefore the number of oscillations should be much greater than unity. In particular, semiclassical approximation cannot be valid if Δx is very small, but on the other hand, Δx cannot be very large since it should be much less than x_0. Another justification of the fact that $-i\hbar d/dx$ is the momentum operator is that in the formal limit $\hbar \to 0$ the Schrödinger equation becomes the Hamilton-Jacobi equation.

We conclude that the choice of $-i\hbar d/dx$ as the momentum operator is justified from the requirement that in semiclassical approximation this operator becomes the classical momentum. However, it is obvious that this requirement does not define the operator uniquely: any operator $\tilde{P}$ such that $\tilde{P} - P$ disappears in semiclassical limit, also can be called the momentum operator.

One might say that the choice $P = -i\hbar d/dx$ can also be justified from the following considerations. In nonrelativistic quantum mechanics we assume that the theory should be invariant under the action of the Galilei group, which is a group of transformations of Galilei space-time. The x component of the momentum operator should be the generator corresponding to spatial translations along the x axis and $-i\hbar d/dx$ is precisely the required operator. In this consideration one assumes that the space-time background has a physical meaning while, as discussed in Refs. [11, 12] and references therein, this is not the case.

As noted in Refs. [11, 12] and references therein, one should start not from space-time but from a symmetry algebra. Therefore in nonrelativistic quantum mechanics we should start from the Galilei algebra and consider its IRs. For simplicity we again consider a one dimensional case. Let $P_x = P$ be one of representation operators in an IR of the Galilei algebra. We can implement this IR in a Hilbert space of functions $\chi(p)$ such that $\int_{-\infty}^{\infty} |\chi(p)|^2 dp < \infty$ and P is the operator of multiplication by p, i.e. $P\chi(p) = p\chi(p)$. Then a question arises how the operator of the x coordinate should be defined. In contrast to the momentum operator, the coordinate one is not defined by the representation and so it should be defined from additional assumptions. Probably a future quantum theory of measurements will make it possible to construct operators of physical quantities from the rules how these quantities should be measured. However, at present we can construct necessary operators only from rather intuitive considerations.

By analogy with the above discussion, one can say that semiclassical wave functions should be of the form $\chi(p) = exp(-ix_0p/\hbar)a(p)$ where the amplitude $a(p)$ has a sharp maximum near $p = p_0 \in [p_1, p_2]$ such that $a(p)$ is not small only when $p \in [p_1, p_2]$. Then Δp is of the order of $p_2 - p_1$ and the condition that the momentum is semiclassical is $\Delta p \ll |p_0|$. Since $i\hbar d\chi(p)/dp = x_0\chi(p) + i\hbar exp(-ix_0p/\hbar)da(p)/dp$, we see that $\chi(p)$ will be approximately the eigenfunction of $i\hbar d/dp$ with the eigenvalue x_0 if $|x_0a(p)| \gg \hbar|da(p)/dp|$. Since $|da(p)/dp|$ is of the order of $|a(p)/\Delta p|$, we have a condition $|x_0\Delta p| \gg \hbar$. Therefore if the coordinate operator is $X = i\hbar d/dp$, the uncertainty of coordinate Δx is of the order of $\hbar/\Delta p$, $|x_0| \gg \Delta x$ and this implies that the coordinate defined in such a way is also semiclassical. We can also note that $|x_0\Delta p|/2\pi\hbar$ is approximately the number of oscillations which the exponent makes on the segment $[p_1, p_2]$ and therefore the number of oscillations should be much greater than unity. It is also clear that semiclassical approximation cannot be valid if Δp is very small, but on the other hand, Δp cannot be very large since it should be much less than p_0. By analogy with the above discussion, the requirement that the operator $i\hbar d/dp$ becomes the coordinate in classical limit does not define the operator uniquely. In nonrelativistic quantum mechanics it is assumed that the coordinate is a well defined physical quantity even on quantum level and that $i\hbar d/dp$ is the most pertinent choice.

The above results can be formally generalized to the three-dimensional case. For example, if the coordinate wave function is chosen in the form

$$\psi(\mathbf{r}) = \frac{1}{\pi^{3/4}a^{3/2}} exp[-\frac{(\mathbf{r}-\mathbf{r}_0)^2}{2a^2} + \frac{i}{\hbar}\mathbf{p}_0\mathbf{r}] \tag{5}$$

then the momentum wave function is

$$\chi(\mathbf{p}) = \int exp(-\frac{i}{\hbar}\mathbf{pr})\psi(\mathbf{r})\frac{d^3\mathbf{r}}{(2\pi\hbar)^{3/2}} = \frac{a^{3/2}}{\pi^{3/4}\hbar^{3/2}} exp[-\frac{(\mathbf{p}-\mathbf{p}_0)^2a^2}{2\hbar^2} - \frac{i}{\hbar}(\mathbf{p}-\mathbf{p}_0)\mathbf{r}_0] \tag{6}$$

It is easy to verify that

$$||\psi||^2 = \int |\psi(\mathbf{r})|^2 d^3\mathbf{r} = 1, \quad ||\chi||^2 = \int |\chi(\mathbf{p})|^2 d^3\mathbf{p} = 1, \tag{7}$$

the uncertainty of each component of the coordinate operator is $a/\sqrt{2}$ and the uncertainty of each component of the momentum operator is $\hbar/(a\sqrt{2})$. Hence one might think that Eqs. (5) and (6) describe a state which is semiclassical in a maximal possible extent.

Let us make the following remark about semiclassical vector quantities. We defined a quantity as semiclassical if its uncertainty is much less than its mean value. In particular, as noted above, a quantity cannot be semiclassical if its mean value is small. In the case of vector quantities we have sets of three physical quantities. Some of them can be small and for them it is meaningless to discuss whether they are semiclassical or not. We say that a vector quantity is semiclassical if all its

components which are not small are semiclassical and there should be at least one semiclassical component.

For example, if the mean value of the momentum $\mathbf{p}_0$ is directed along the z axes then the xy components of the momentum are not semiclassical but the three-dimensional vector quantity $\mathbf{p}$ can be semiclassical if $\mathbf{p}_0$ is rather large. However, in that case the definitions of the x and y components of the position operator as $x = i\hbar\partial/\partial p_x$ and $y = i\hbar\partial/\partial p_y$ become inconsistent. The situation when the validity of an operator depends on the choice of directions of the coordinate axes is not acceptable and hence the above definition of the position operator is at least problematic.

Let us note that semiclassical states can be constructed not only in momentum or coordinate representations. For example, instead of momentum wave functions $\chi(\mathbf{p})$ one can work in the representation where the quantum numbers (p, l, μ) in wave functions $\chi(p, l, \mu)$ mean the magnitude of the momentum p, the orbital quantum number l (such that a state is the eigenstate of the orbital momentum squared $\mathbf{L}^2$ with the eigenvalue $l(l+1)$) and the magnetic quantum number μ (such that a state is the eigenvector or L_z with the eigenvalue μ). A state described by a $\chi(p, l, \mu)$ will be semiclassical with respect to those quantum numbers if $\chi(p, l, \mu)$ has a sharp maximum at $p = p_0$, $l = l_0$, $\mu = \mu_0$ and the widths of the maxima in p, l and μ are much less than p_0, l_0 and μ_0, respectively. However, by analogy with the above discussion, those widths cannot be arbitrarily small if one wishes to have other semiclassical variables (e.g. the coordinates). Examples of such situations will be discussed in Sec. 13.

3 Wave packet spreading in nonrelativistic quantum mechanics

As noted by Pauli (see p. 63 of Ref. [14]), at early stages of quantum theory some authors treated time t as the operator commuting with the Hamiltonian as $[H, t] = i\hbar$ but such a treatment is not correct. For example, one cannot construct the eigenstate of the time operator with the eigenvalue 5000 BC or 3000 AD. Hence the quantity t can be only a classical parameter (see also Ref. [15]). We see that the principle of quantum theory that every physical quantity is defined by an operator does not apply to time. The problem of time in quantum theory is discussed in a wide literature and remarks on this problem are made in Sec. 15. However, for now we assume that standard treatment of time is valid, i.e. that time is a classical parameter such that the dependence of the wave function on time is defined by the Hamiltonian according to the Schrödinger equation.

In nonrelativistic quantum mechanics the Hamiltonian of a free particle with the mass m is $H = \mathbf{p}^2/2m$ and hence, as follows from Eq. (6), in the model discussed above the dependence of the momentum wave function on t is

$$\chi(\mathbf{p}, t) = \frac{a^{3/2}}{\pi^{3/4}\hbar^{3/2}} exp[-\frac{(\mathbf{p}-\mathbf{p}_0)^2 a^2}{2\hbar^2} - \frac{i}{\hbar}(\mathbf{p}-\mathbf{p}_0)\mathbf{r}_0 - \frac{i\mathbf{p}^2 t}{2m\hbar}] \tag{8}$$

It is easy to verify that for this state the mean value of the operator $\mathbf{p}$ and the uncertainty of each momentum component are the same as for the state $\chi(\mathbf{p})$, i.e. those quantities do not change with time.

Consider now the dependence of the coordinate wave function on t. This dependence can be calculated by using Eq. (8) and the fact that

$$\psi(\mathbf{r},t) = \int exp(\frac{i}{\hbar}\mathbf{pr})\chi(\mathbf{p},t)\frac{d^3\mathbf{p}}{(2\pi\hbar)^{3/2}} \tag{9}$$

The result of a direct calculation is

$$\psi(\mathbf{r},t) = \frac{1}{\pi^{3/4}a^{3/2}}(1+\frac{i\hbar t}{ma^2})^{-3/2}exp[-\frac{(\mathbf{r}-\mathbf{r}_0-\mathbf{v}_0t)^2}{2a^2(1+\frac{\hbar^2t^2}{m^2a^4})}(1-\frac{i\hbar t}{ma^2})+\frac{i}{\hbar}\mathbf{p}_0\mathbf{r}-\frac{i\mathbf{p}_0^2t}{2m\hbar}] \tag{10}$$

where $\mathbf{v}_0 = \mathbf{p}_0/m$ is the classical velocity. This result shows that the semiclassical wave packet is moving along the classical trajectory $\mathbf{r}(t) = \mathbf{r}_0 + \mathbf{v}_0t$. At the same time, it is now obvious that the uncertainty of each coordinate depends on time as

$$\Delta x_j(t) = \Delta x_j(0)(1+\hbar^2t^2/m^2a^4)^{1/2}, \quad (j=1,2,3) \tag{11}$$

where $\Delta x_j(0) = a/\sqrt{2}$, i.e. the width of the wave packet in coordinate representation is increasing. This fact, known as the wave-packet spreading (WPS), is described in many textbooks and papers (see e.g. the textbooks [4, 5] and references therein). It shows that if a state was semiclassical in the maximal extent at $t = 0$, it will not have this property at $t > 0$ and the accuracy of semiclassical approximation will decrease with the increase of t. The characteristic time of spreading can be defined as $t_* = ma^2/\hbar$. For macroscopic bodies this is an extremely large quantity and hence in macroscopic physics the WPS effect can be neglected. In the formal limit $\hbar \to 0$, t_* becomes infinite, i.e. spreading does not take place. This shows that WPS is a pure quantum phenomenon. For the first time the result (10) has been obtained by Darwin in Ref. [16].

One might pose a problem whether the WPS effect is specific only for Gaussian wave functions. One might expect that this effect will take place in general situations since each component of the standard position operator $i\hbar\partial/\partial\mathbf{p}$ does not commute with the Hamiltonian and so the distribution of the corresponding physical quantity will be time dependent. A good example showing inevitability of WPS follows. If at $t = 0$ the coordinate wave function is $\psi_0(\mathbf{r})$ then, as follows from Eqs. (6) and (9),

$$\psi(\mathbf{r},t) = \int exp\{\frac{i}{\hbar}[\mathbf{p}(\mathbf{r}-\mathbf{r}') - \frac{\mathbf{p}^2t}{2m}]\}\psi_0(\mathbf{r}')\frac{d^3\mathbf{r}'d^3\mathbf{p}}{(2\pi\hbar)^3} \tag{12}$$

As follows from this expression, if $\psi_0(\mathbf{r}) \neq 0$ only if $\mathbf{r}$ belongs to a finite vicinity of some vector $\mathbf{r}_0$ then at any $t > 0$ the support of $\psi(\mathbf{r},t)$ belongs to the whole three-dimensional space, i.e. the wave function spreads out with an infinite speed. One

might think that in nonrelativistic theory this is not unacceptable since this theory can be treated as a formal limit $c \to \infty$ of relativistic theory. In the next sections we will discuss an analogous situation in relativistic theory.

As shown in Ref. [17] titled "Nonspreading wave packets", for a one-dimensional wave function in the form of an Airy function, spreading does not take place and the maximum of the quantity $|\psi(x)|^2$ propagates with constant acceleration even in the absence of external forces. Those properties of Airy packets have been observed in optical experiments [18]. However, since such a wave function is not normalizable, the term "wave packet" in the given situation might be misleading since the mean values and uncertainties of the coordinate and momentum cannot be calculated in a standard way. Such a wave function can be constructed only in a limited region of space. As explained in Ref. [17], this wave function describes not a particle but rather families of particle orbits. As shown in Ref. [17], one can construct a normalized state which is a superposition of Airy functions with Gaussian coefficients and "eventually the spreading due to the Gaussian cutoff takes over". This is an additional argument that the effect of WPS is an inevitable consequence of standard quantum theory.

Since quantum theory is invariant under time reversal, one might ask the following question: is it possible that the width of the wave packet in coordinate representation is decreasing with time? From the formal point of view, the answer is "yes". Indeed, the solution given by Eq. (10) is valid not only when $t \geq 0$ but when $t < 0$ as well. Then, as follows from Eq. (11), the uncertainty of each coordinate is decreasing when t changes from some negative value to zero. However, eventually the value of t will become positive and the quantities $\Delta x_j(t)$ will grow to infinity. In the present paper we consider situations when a photon is created on atomic level and hence one might expect that its initial coordinate uncertainties are not large. However, when the photon travels a long distance to Earth, those uncertainties become much greater, i.e. the term WPS reflects the physics adequately.

4 Mott-Heisenberg problem and its generalization

In 1929 Mott and Heisenberg considered the following problem. Let an alpha-particle be emitted by a nucleus in a radioactive decay. Suppose, for simplicity, that the particle has been emitted in a state with zero angular momentum. Then the momentum wave function is spherically symmetric and all directions of the momentum have equal probabilities. However, when the particle is detected in Wilson's cloud chamber, the registered trajectory is always linear as if the particle moved along a classical trajectory. The explanation of the paradox has been given in Ref. [19]. In this section we consider a general case when it is not assumed that the partical wave function is spherically symmetric.

Consider the state (12) after a long period of time such that $D \gg a$ where $D = \hbar t/(ma)$. As follows from Eq. (12), at this condition the width of the coordinate

wave function is of the order D. Suppose that the particle is emitted at the origin such that $\mathbf{r}_0 = 0$. Suppose that a measuring device is at the point $\mathbf{r}_1$ and the size of the device is of the order of d. Although the device is macroscopic, we assume that D is already so large that $D \gg d$. A problem arises at which momentum range the particle will be detected.

For solving this problem we first project the coordinate wave function onto the region of space belonging to the device. Assume that the projected wave function is

$$\tilde{\psi}(\mathbf{r}, t) = exp[-\frac{(\mathbf{r} - \mathbf{r}_1)^2}{2d^2}]\psi(\mathbf{r}, t) \tag{13}$$

A direct calculation shows that the norm of this state is

$$||\tilde{\psi}||^2 = (\frac{d}{D})^3 exp[-\frac{(\mathbf{r}_1 - \mathbf{v}_0 t)^2}{D^2}] \tag{14}$$

This result is obvious because the wave function of the packet is not negligible only in the region having the volume of the order of D^3 and so if $\mathbf{r}_1$ is inside this region then the probability to detect the particle is of the order of $(d/D)^3$.

If the particle is detected by the device then the measured momentum range is defined by the Fourier transform of $\tilde{\psi}(\mathbf{r}, t)$. A direct calculation gives

$$\tilde{\psi}(\mathbf{p}, t) = \frac{1}{(2\pi\hbar)^{3/2}} \int exp(\frac{-i}{\hbar}\mathbf{pr})\tilde{\psi}(\mathbf{r}, t)d^3\mathbf{r} =$$
$$f(\mathbf{p}, t)exp[-\frac{d^2D^2a^2(\mathbf{p} - m\mathbf{r}_1/t)^2}{2\hbar^2(D^2a^2 + d^4)}] \tag{15}$$

where $f(\mathbf{p}, t)$ contains the dependence on $\mathbf{p}$ only in the exponent with the imaginary index. Therefore the probabilities of different momenta are defined by the last exponent which shows that the distribution of momenta has a sharp peak around the vector $m\mathbf{r}_1/t$ pointing to the device. While the width of the momentum distribution in the initial packet is of order of $\hbar/a$ (see Eq. (8)), the width given by Eq. (15) is much narrower. If for example $D^2a^2 \gg d^4$ then the width is of the of the order of $\hbar/d$ and in the opposite case the width is of the of the order of $\hbar d/(Da)$.

As discussed in Sec. 2, in semiclassical approximation the value of the momentum can be found by applying the operation $-i\hbar\partial/\partial\mathbf{r}$ to the rapidly oscillating exponent. In general the momentum distribution can be rather wide. However, if the particle is detected in a vicinity of the point $\mathbf{r}$ then, as follows from Eq. (15), it will be detected with the momentum close to $m\mathbf{r}/t$. This result has the following qualitative explanation. The operation $-i\hbar\partial/\partial\mathbf{r}$ applied to the imaginary index of the exponent in Eq. (10) gives exactly $m\mathbf{r}/t$.

The above results gives the solution of the Mott-Heisenberg problem when the particle is in the state (10). However, in this case the wave function can be spherically symmetric only if $\mathbf{p}_0 = 0$. This case is of no interest because typically a particle created in the spherically symmetric state has a nonzero kinetic energy. We

now consider a model where, instead of Eq. (6), the initial particle momentum wave function is

$$\chi(\mathbf{p}) = \frac{f(\mathbf{p}/p)}{p} exp[-\frac{1}{2\hbar^2}a^2(p-p_0)^2] \tag{16}$$

where $p = |\mathbf{p}|$ and the quantaties p_0 and a are positive. We assume that $p_0 a \gg \hbar$. Then, with a good accuracy, integrals over p from 0 to ∞ containing the exponent can be replaced by integrals from $-\infty$ to ∞. By analogy with the calculation in Sec. 2, one can easily show that $\bar{p} \approx p_0$ and $\Delta p \approx \hbar/(a\sqrt{2})$ and therefore the p-distribution is semiclassical. The dependence of the momentum wave function on t is the same as in Eq. (8).

The coordinate wave function is again given by Eq. (9). For calculating this function in the case when the initial momentum wave function is given by Eq. (16) we need the following auxiliary results:

$$\int_0^\infty exp[-\frac{1}{2\hbar^2}a^2(p-p_0)^2 + \frac{i}{\hbar}pr\xi]dp \approx \frac{\hbar}{a}(\frac{2\pi}{1+iD/a})^{1/2} exp[-\frac{(r\xi - p_0 t/m)^2}{2a^2(1+iD/a)}] \tag{17}$$

where $r = |\mathbf{r}|$ and

$$exp(\frac{i}{\hbar}\mathbf{pr}) = 4\pi \sum_{l\mu} i^l j_l(pr/\hbar) Y^*_{l\mu}(\mathbf{p}/p) Y_{l\mu}(\mathbf{r}/r) \tag{18}$$

The last expression is the well-known decomposition of the flat wave. Here $Y_{l\mu}$ is the spherical function corresponding to the orbital angular momentum l and its z-projection μ and j_l is the spherical Bessel function. Its asymptotic expression when the argument is large is $j_l(x) \approx sin(x - \pi l/2)/x$.

Let $f(\mathbf{p}/p) = \sum_{l\mu} c_{lm} Y_{l\mu}(\mathbf{p}/p)$ be the decomposition of the function f in Eq. (16) over spherical functions. Then it follows from the orthogonality of spherical functions, Eqs. (16-18) and the above remarks that if $(pr/\hbar) \gg 1$ then

$$\psi(\mathbf{r},t) = -\frac{i}{ar}(\frac{\hbar}{1+iD/a})^{1/2} exp(-\frac{p_0^2 t}{2m\hbar}) \sum_{l\mu} c_{l\mu} Y_{l\mu}(\mathbf{r}/r)$$
$$\{exp[-\frac{(r-p_0 t/m)^2}{2a^2(1+iD/a)}] - (-1)^l exp[-\frac{(r+p_0 t/m)^2}{2a^2(1+iD/a)}]\} \tag{19}$$

At large distances and times the second term in the figure brackets is negligible and the final result is

$$\psi(\mathbf{r},t) = -\frac{i}{ar}(\frac{\hbar}{1+iD/a})^{1/2} exp(-\frac{p_0^2 t}{2m\hbar}) f(\mathbf{r}/r) exp[-\frac{(r-p_0 t/m)^2(1-iD/a)}{2(a^2+D^2)}] \tag{20}$$

Therefore for the initial momentum wave function (16) the coordinate wave function at large distances and times has the same angular dependence as the momentum wave function and the radial wave functions spreads out by analogy with Eq. (10).

The result (20) gives an obvious solution of the Mott-Heisenberg problem in the case when the angular dependence of the wave function is arbitrary. Indeed, suppose that a particle is created at the origin and a measuring device is seen from the origin in the narrow angular range defined by the function $\tilde{f}(\mathbf{r}/r)$. Suppose that the support of $\tilde{f}(\mathbf{r}/r)$ is within the range defined by $f(\mathbf{r}/r)$. Then the projection of the wave function (20) onto the device is given by the same expression where $f(\mathbf{r}/r)$ is replaced by $\tilde{f}(\mathbf{r}/r)$. Since the angular wave functions in coordinate and momentum representations are the same, the momenta measured by the device will be in the angular range defined by the function $\tilde{f}(\mathbf{p}/p)$.

5 Position operator in relativistic quantum mechanics

The problem of the position operator in relativistic quantum theory has been discussed in a wide literature and different authors have different opinions on this problem. In particular, some authors state that in relativistic quantum theory no position operator exists. As already noted, the results of fundamental quantum theories are formulated only in terms of the S-matrix in momentum space without any mentioning of space-time. This is in the spirit of the Heisenberg S-matrix program that in relativistic quantum theory it is possible to describe only transitions of states from the infinite past when $t \to -\infty$ to the distant future when $t \to +\infty$. On the other hand, since quantum theory is treated as a theory more general than classical one, it is not possible to fully avoid space and time in quantum theory. For example, quantum theory should explain how photons from distant objects travel to Earth and even how macroscopic bodies are moving along classical trajectories. Hence we can conclude that: a) in quantum theory (nonrelativistic and relativistic) we must have a position operator and b) this operator has a physical meaning only in semiclassical approximation.

Let us first consider the definition of elementary particle. Although theory of elementary particles exists for a rather long period of time, there is no commonly accepted definition of elementary particle in this theory. In Refs. [11, 12] and references therein we argue that, in the spirit of Wigner's approach to Poincare symmetry [20], a general definition, not depending on the choice of the classical background and on whether we consider a local or nonlocal theory, is that a particle is elementary if the set of its wave functions is the space of an IR of the symmetry algebra in the given theory.

There exists a wide literature describing how IRs of the Poincare algebra can be constructed. In particular, an IR for a spinless particle can be implemented in a space of functions $\xi(\mathbf{p})$ satisfying the condition

$$\int |\xi(\mathbf{p})|^2 d\rho(\mathbf{p}) < \infty, \quad d\rho(\mathbf{p}) = \frac{d^3\mathbf{p}}{\epsilon(\mathbf{p})} \tag{21}$$

where $\epsilon(\mathbf{p}) = (m^2 + \mathbf{p}^2)^{1/2}$ is the energy of the particle with the mass m. The convenience of the above requirement is that the volume element $d\rho(\mathbf{p})$ is Lorentz invariant. In that case it can be easily shown by direct calculations (see e.g. Ref. [21]) that the representation operators have the form

$$\mathbf{L} = -i\mathbf{p} \times \frac{\partial}{\partial \mathbf{p}}, \quad \mathbf{N} = -i\epsilon(\mathbf{p})\frac{\partial}{\partial \mathbf{p}}, \quad \mathbf{P} = \mathbf{p}, \quad E = \epsilon(\mathbf{p}) \tag{22}$$

where $\mathbf{L}$ is the orbital angular momentum operator, $\mathbf{N}$ is the Lorentz boost operator, $\mathbf{P}$ is the momentum operator, E is the energy operator and these operators are expressed in terms of the operators in Eq. (2) as

$$\mathbf{L} = (M^{23}, M^{31}, M^{12}),\ \mathbf{N} = (M^{10}, M^{20}, M^{30}),\ \mathbf{P} = (P^1, P^2, P^3),\ E = P^0$$

For particles with spin these results are modified as follows. For a massive particle with spin s the functions $\xi(\mathbf{p})$ also depend on spin projections which can take $2s + 1$ values $-s, -s + 1, ...s$. If $\mathbf{s}$ is the spin operator then the total angular momentum has an additional term $\mathbf{s}$ and the Lorentz boost operator has an additional term $(\mathbf{s} \times \mathbf{p})/(\epsilon(\mathbf{p}) + m)$ (see e.g. Eq. (2.5) in Ref. [21]). Hence corrections of the spin terms to the quantum numbers describing the angular momentum and the Lorentz boost do not exceed s. We assume as usual that in semiclassical approximation the quantum numbers characterizing the angular momentum and the Lorentz boost are much greater than unity and hence in this approximation spin effects can be neglected. For a massless particle with the spin s the spin projections can take only values $-s$ and s and those quantum numbers have the meaning of helicity. In this case the results for the representation operators can be obtained by taking the limit $m \to 0$ if the operators are written in the light front variables (see e.g. Eq. (25) in Ref. [11]). As a consequence, in semiclassical approximation the spin corrections in the massless case can be neglected as well. Hence for investigating the position operator we will neglect spin effects and will not explicitly write the dependence of wave functions on spin projections.

In the above IRs the representation operators are Hermitian as it should be for operators corresponding to physical quantities. In standard theory (over complex numbers) such IRs of the Lie algebra can be extended to unitary IRs of the Poincare group. In particular, in the spinless case the unitary operator $U(\Lambda)$ corresponding to the Lorentz transformation Λ acts in H as (see e.g. Ref. [21])

$$U(\Lambda)\xi(p) = \xi(\Lambda^{-1}p) \tag{23}$$

In the literature elementary particles are described not only by such IRs but also by nonunitary representations induced from the Lorentz group [22]. Since the factor space of the Poincare group over the Lorentz group is Minkowski space, the elements of such representations are fields $\Psi(x)$ depending on four-vectors x in Minkowski space and possibly on spin variables. Since those functions describe

nonunitary representations, their probabilistic interpretation is problematic. The Pauli theorem [23] states that for fields with an integer spin it is impossible to define a positive definite charge density and for fields with a half-integer spin it is impossible to define a positive definite energy density.

Hence a problem arises whether such fields have a physical meaning. The answer is that in QFT after quantizing they become quantum fields defining the stress-energy and angular momentum tensors. Then the Hermitian operators P^μ and $M^{\mu\nu}$ are defined by integrals of those tensors over a space-like hyperplane. So the quantity x in local fields is only an integration parameter and a problem of whether there are quantum operators corresponding to x does not arise. This is clear also from the fact that quantized fields are operators in Fock spaces describing systems with an infinite number of particles and hence x does not refer to any specific particle. Therefore local quantum fields (in this situation even the term "local" is not clear) are only auxiliary tools for constructing physical operators in QFT.

Let us note that although QFT has achieved very impressive successes in explaining many experimental data, a problem of its mathematical substantiation has not been solved yet. The main mathematical inconsistency of QFT is that since interacting local quantum fields can be treated only as operatorial distributions, their products at the same space-time points are not well defined (see e.g. Ref. [24]). One of ideas of the string theory is that if products of fields at the same points (zero-dimensional objects) are replaced by products where arguments belong to strings (one-dimensional objects) then there is hope that infinities will be less singular. In view of such controversial properties of local quantum fields, many authors posed a question whether local fields will survive in the future quantum theory. Nevertheless, in the literature the problem of position operator is mainly discussed in the approach when elementary particles are described by local fields rather than unitary IRs. Below we discuss the both approaches but first we consider the case of unitary IRs.

As follows from Eq. (2), the operator $I_2 = E^2 - \mathbf{P}^2$ is the Casimir operator of the second order, i.e. it is a bilinear combination of representation operators commuting with all the operators of the algebra. As follows from the well-known Schur lemma, all states belonging to an IR are the eigenvectors of I_2 with the same eigenvalue m^2. Note that Eq. (22) contains only m^2 but not m. The choice of the energy sign is only a matter of convention but not a matter of principle. Indeed, the energy can be measured only if the momentum $\mathbf{p}$ is measured and then it is only a matter of convention what sign of the square root should be chosen. However, it is important that the sign should be the same for all particles. For example, if we consider a system of two particles with the same values of m^2 and the opposite momenta $\mathbf{p}_1$ and $\mathbf{p}_2$ such that $\mathbf{p}_1 + \mathbf{p}_2 = 0$, we cannot define the energies of the particles as $\epsilon(\mathbf{p}_1)$ and $-\epsilon(\mathbf{p}_2)$, respectively, since in that case the total four-momentum of the two-particle system will be zero what contradicts experiment.

The notation $I_2 = m^2$ is justified by the fact that for all known particles $I_2 \geq 0$. Then the mass m is *defined* as the square root of m^2 and the sign of m is

only a matter of convention. The usual convention is that $m \geq 0$. However, from mathematical point of view, IRs with $I_2 < 0$ are not prohibited. If the velocity operator $\mathbf{v}$ is *defined* as $\mathbf{v} = \mathbf{P}/E$ then for known particles $|\mathbf{v}| \leq 1$, i.e. $|\mathbf{v}| \leq c$ in standard units. However, for IRs with $I_2 < 0$, $|\mathbf{v}| > c$ and, at least from the point of view of mathematical construction of IRs, this case is not prohibited. The hypothetical particles with such properties are called tachyons and their possible existence is widely discussed in the literature. If the tachyon mass m is also defined as the square root of m^2 then this quantity will be imaginary. However, this does not mean than the corresponding IRs are unphysical since all the operators of the Poincare group Lie algebra depend only on m^2.

As follows from Eqs. (21) and (22), in the nonrelativistic approximation $d\rho(\mathbf{p}) = d^3\mathbf{p}/m$ and $\mathbf{N} = -im\partial/\partial\mathbf{p}$. Therefore in this approximation $\mathbf{N}$ is proportional to *standard* position operator and one can say that the position operator is in fact present in the description of the IR.

The following remarks are in order. The choice of the volume element in the Lorentz invariant form $d\rho(\mathbf{p})$ (see Eq. (21)) might be convenient from the point of view that then the Hilbert space can be treated as a space of functions $\xi(p)$ depending on four-vectors p such that $p^0 = \epsilon(\mathbf{p})$ and the norm can be written in the covariant form (i.e. in the form depending only on Lorentz invariant quantities): $||\xi||^2 = \int |\xi(p)|^2 \delta(p^2 - m^2)\theta(p^0)d^4p$. However, the requirement of covariance does not have a fundamental physical meaning. In relativistic theory a necessary requirement is that symmetry is defined by operators satisfying the commutation relations (2) and this requirement can be implemented in different forms, not necessarily in covariant ones.

As an illustration, consider the following problem. Suppose that we wish to construct a single-particle coordinate wave function. Such a wave function cannot be defined on the whole Minkowski space. This is clear even from the fact that there is no time operator. The wave function can be defined only on a space-like hyperplane of the Minkowski space. For example, on the hyperplane $t = const$ the wave function depends only on $\mathbf{x}$. Hence for defining the wave function one has to choose the form of the position operator. By analogy with the nonrelativistic case, one might try to define the position operator as $i\partial/\partial\mathbf{p}$. However, if the Hilbert space is implemented as in Eq. (21) then this operator is not selfadjoint since $d\rho(\mathbf{p})$ is not proportional to $d^3\mathbf{p}$. One can perform a unitary transformation $\xi(\mathbf{p}) \to \chi(\mathbf{p}) = \xi(\mathbf{p})/\epsilon(\mathbf{p})^{1/2}$ such that the Hilbert space becomes the space of functions $\chi(\mathbf{p})$ satisfying the condition $\int |\chi(\mathbf{p})|^2 d^3\mathbf{p} < \infty$. It is easy to verify that in this implementation of the IR the operators $(\mathbf{L}, \mathbf{P}, E)$ will have the same form as in Eq. (22) but the expression for $\mathbf{N}$ will be

$$\mathbf{N} = -i\epsilon(\mathbf{p})^{1/2} \frac{\partial}{\partial \mathbf{p}} \epsilon(\mathbf{p})^{1/2} \tag{24}$$

In this case one can *define* $i\hbar\partial/\partial\mathbf{p}$ as a position operator but now we do not have a situation when the position operator is present among the other representation

operators.

A problem of the definition of the position operator in relativistic quantum theory has been discussed since the beginning of the 1930s and it has been noted that when quantum theory is combined with relativity the existence of the position operator with correct physical properties becomes a problem. The above definition has been proposed by Newton and Wigner in Ref. [25]. They worked in the approach when elementary particles are described by local fields $\Psi(x)$ defined on the whole Minkowski space rather than unitary IRs. As noted above, such fields cannot be treated as single-particle wave functions. The spacial Fourier transform of such fields at $t = const$ describes states where the energy can be positive and negative and this is interpreted such that local quantum fields describe a particle and its antiparticle simultaneously. Newton and Wigner first discuss the spinless case and consider only states on the upper Lorentz hyperboloid where the energy is positive. For such states the representation operators act in the same way as in the case of spinless unitary IRs. With this definition the coordinate wave function $\psi(\mathbf{r})$ can be again defined by Eq. (5) and a question arises whether such a position operator has all the required properties.

For example, in the introductory section of the textbook [26] the following arguments are given in favor of the statement that in relativistic quantum theory it is not possible to define a physical position operator. Suppose that we measure coordinates of an electron with the mass m. When the uncertainty of coordinates is of the order of $\hbar/mc$, the uncertainty of momenta is of the order of mc, the uncertainty of energy is of the order of mc^2 and hence creation of electron-positron pairs is allowed. As a consequence, it is not possible to localize the electron with the accuracy better than its Compton wave length $\hbar/mc$. Hence, for a particle with a nonzero mass exact measurement is possible only either in the nonrelativistic limit (when $c \to \infty$) or classical limit (when $\hbar \to 0$). In the case of the photon, as noted by Pauli (see p. 191 of Ref. [14]), the coordinate cannot be measured with the accuracy better than $\hbar/p$ where p is the magnitude of the photon momentum. The quantity $\lambda = 2\pi\hbar/p$ is called the photon wave length although, as noted in Sec. 1, the meaning of this quantity in quantum case might be fully different than in classical one. Since $\lambda \to 0$ in the formal limit $\hbar \to 0$, Pauli concludes that "Only within the confines of the classical ray concept does the position of the photon have a physical significance".

Another argument that the Newton-Wigner position operator does not have all the required properties follows. Since the energy operator acts on the function $\chi(\mathbf{p})$ as $E\chi(\mathbf{p}) = \epsilon(\mathbf{p})\chi(\mathbf{p})$ (see Eq. (22)) and the energy is an operator corresponding to infinitesimal time translations, the dependence of the wave function $\chi(\mathbf{p})$ on t is given by

$$\chi(\mathbf{p}, t) = exp(-\frac{i}{\hbar}Et)\chi(\mathbf{p}) = exp(-\frac{i}{\hbar}\epsilon(\mathbf{p})t)\chi(\mathbf{p}) \tag{25}$$

Then a relativistic analog of Eq. (12) is

$$\psi(\mathbf{r},t)=\int exp\{\frac{i}{\hbar}[\mathbf{p}(\mathbf{r}-\mathbf{r}')-\epsilon(\mathbf{p})t]\}\psi_0(\mathbf{r}')\frac{d^3\mathbf{r}'d^3\mathbf{p}}{(2\pi\hbar)^3} \tag{26}$$

As a consequence, the Newton-Wigner position operator has the "tail property": if $\psi_0(\mathbf{r})\neq 0$ only if $\mathbf{r}$ belongs to a finite vicinity of some vector $\mathbf{r}_0$ then at any $t>0$ the function $\psi(\mathbf{r},t)$ has a tail belonging to the whole three-dimensional space, i.e. the wave function spreads out with an infinite speed. Hence at any $t>0$ the particle can be detected at any point of the space and this contradicts the requirement that no information should be transmitted with the speed greater than c.

The tail property of the Newton-Wigner position operator has been known for a long time (see e.g. Ref. [27] and references therein). It is characterized as nonlocality leading to the action at a distance. Hegerfeldt argues [27] that this property is rather general because it can be proved assuming that energy is positive and without assuming a specific choice of the position operator. The Hegerfeldt theorem [27] is based on the assumption that there exists an operator $N(V)$ whose expectation defines the probability to find a particle inside the volume V. However, the meaning of time on quantum level is not clear and for the position operator proposed in the present paper such a probability does not exist because there is no wave function in coordinate representation (see Sec. 12 and the discussion in Sec. 15).

One might say that the requirement that no signal can be transmitted with the speed greater than c has been obtained in Special Relativity which is a classical (i.e. nonquantum) theory operating only with classical space-time coordinates. For example, in classical theory the velocity of a particle is defined as $\mathbf{v}=d\mathbf{r}/dt$ but, as noted above, the velocity *should be defined* as $\mathbf{v}=\mathbf{p}/E$ (i.e. without mentioning space-time) and then on classical level it can be shown that $\mathbf{v}=d\mathbf{r}/dt$. In QFT local quantum fields separated by space-like intervals commute or anticommute (depending on whether the spin is integer or half-integer) and this is treated as a requirement of causality and that no signal can be transmitted with the speed greater than c. However, as noted above, the physical meaning of space-time coordinates on quantum level is not clear. Hence from the point of view of quantum theory the existence of tachyons is not prohibited. Note also that when two electrically charged particles exchange by a virtual photon, a typical situation is that the four-momentum of the photon is space-like, i.e. the photon is the tachyon. We conclude that although in relativistic theory such a behavior might seem undesirable, there is no proof that it must be excluded. Also, as argued by Griffiths (see Ref. [28] and references therein), with a consistent interpretation of quantum theory there are no nonlocality and superluminal interactions. In Sec. 15 we argue that the position operator proposed in the present paper sheds a new light on this problem.

Another striking example is a photon emitted in the famous 21cm transition line between the hyperfine energy levels of the hydrogen atom. The phrase that the lifetime of this transition is of the order of $\tau=10^7$ years implies that the width

of the level is of the order of $\hbar/\tau$, i.e. experimentally the uncertainty of the photon energy is $\hbar/\tau$. Hence the uncertainty of the photon momentum is $\hbar/(c\tau)$ and with the above definition of the coordinate operators the uncertainty of the longitudinal coordinate is $c\tau$, i.e. of the order of 10^7 light years. Then there is a nonzero probability that immediately after its creation at point A the photon can be detected at point B such that the distance between A and B is 10^7 light years.

A problem arises how this phenomenon should be interpreted. On one hand, one might say that in view of the above discussion it is not clear whether or not the requirement that no information should be transmitted with the speed greater than c should be a must in relativistic quantum theory. On the other hand (as pointed out to me by Alik Makarov), we can know about the photon creation only if the photon is detected and when it was detected at point B at the moment of time $t = t_0$, this does not mean that the photon travelled from A to B with the speed greater than c since the time of creation has an uncertainty of the order of 10^7 years. Note also that in this situation a description of the system (atom + electric field) by the wave function (e.g. in the Fock space) depending on a continuous parameter t has no physical meaning (since roughly speaking the quantum of time in this process is of the order of 10^7 years). If we accept this explanation then we should acknowledge that in some situations a description of evolution by a continuous classical parameter t is not physical and this is in the spirit of the Heisenberg S-matrix program. However, this example describes a pure quantum phenomenon while, as noted above, a position operator is needed only in semiclassical approximation.

For particles with nonzero spin, the number of states in local fields is typically by a factor of two greater than in the case of unitary IRs (since local fields describe a particle and its antiparticle simultaneously) but those components are not independent since local fields satisfy a covariant equation (Klein-Gordon, Dirac etc.). In Ref. [25] Newton and Wigner construct a position operator in the massive case but say that in the massless one they have succeeded in constructing such an operator only for Klein-Gordon and Dirac particles while in the case of the photon the position operator does not exist. On the other hand, as noted above, in the case of unitary IRs different spin components are independent and in semiclassical approximation spin effects are not important. So in this approach one might adopt the Newton-Wigner position operator for particles with any spin and any mass.

We now consider the following problem. Since the Newton-Wigner position operator formally has the same form as in nonrelativistic quantum mechanics, the coordinate and momentum wave functions also are related to each other by the same Fourier transform as in nonrelativistic quantum mechanics (see Eq. (9)). One might think that this relation is not Lorentz covariant and pose a question whether in relativistic theory this is acceptable. As noted above, for constructing the momentum wave function covariance does not have a fundamental physical meaning and is not necessary. A question arises whether the same is true for constructing the coordinate wave function.

Let us note first that if the four-vector x is such that $x = (t, \mathbf{x})$ then the wave function $\psi(x) = \psi(\mathbf{x}, t)$ can have a physical meaning only if we accept that (at least in some approximations) a position operator is well defined. Then the function $\psi(\mathbf{x}, t)$ describes amplitudes of probabilities for different values of $\mathbf{x}$ at a fixed value of t. This function cannot describe amplitudes of probabilities for different values of t because there is no time operator.

For discussing Lorentz covariance of the coordinate wave function it is important to note that, in view of the above remarks, this function can be defined not in the whole Minkowski space but only on space-like hyperplanes of that space (by analogy with the fact that in QFT the operators $(P^\mu, M^{\mu\nu})$ are defined by integrals over such hyperplanes). They are defined by a time-like unit vector n and the evolution parameter τ such that the corresponding hyperplane is a set of points with the coordinates x satisfying the condition $nx = \tau$. Wave functions $\psi(x)$ on this hyperplane satisfy the requirement $\int |\psi(x)|^2 \delta(nx - \tau) d^4x < \infty$. In a special case when $n^0 = 1$, $\mathbf{n} = 0$ the hyperplane is a set of points $(t = \tau, \mathbf{x})$ and the wave functions satisfy the usual requirement $\int |\psi(\mathbf{x}, t)|^2 d^3\mathbf{x} < \infty$. In the literature coordinate wave functions are usually considered without discussions of the position operator and without mentioning the fact that those functions are defined on space-like hyperplanes (see e.g. Refs. [29, 30]).

By analogy with the construction of the coordinate wave function in Refs. [29, 31], it can be defined as follows. Let $\tilde{x}_0$ be a four-vector and p and p_0 be four-vectors $(\epsilon(\mathbf{p}), \mathbf{p})$ and $(\epsilon(\mathbf{p}_0), \mathbf{p}_0)$, respectively. We will see below that momentum wave functions describing wave packets can be chosen in the form

$$\xi(p, p_0, \tilde{x}_0) = f(p, p_0) exp(\frac{i}{\hbar} p\tilde{x}_0) \tag{27}$$

where $f(p, p_0)$ as a function of p has a sharp maximum in the vicinity of $p = p_0$, $\tilde{x}_0 = x_0 - (nx_0)n$ and the four-vector x_0 has the coordinates $(t, \mathbf{r}_0)$. Then the coordinate wave function can be defined as

$$\psi(x, p_0, \tilde{x}_0) = \frac{1}{(2\pi\hbar)^{3/2}} \int \xi(p, p_0, \tilde{x}_0) exp(-\frac{i}{\hbar} px) d\rho(\mathbf{p}) \tag{28}$$

Suppose that $f(p, p_0)$ is a covariant function of its arguments, i.e. it can depend only on p^2, p_0^2 and pp_0. Then, as follows from Eq. (23), the function $\psi(x, p_0, \tilde{x}_0)$ is covariant because its Lorentz transformation is $\psi(x, p_0, \tilde{x}_0) \to \psi(\Lambda^{-1}x, p_0, \tilde{x}_0)$.

The choice of $f(p, p_0)$ in the covariant form might encounter the following problem. For example, the authors of Ref. [31] propose to consider $f(p, p_0)$ in the form

$$f(p, p_0) = const\, exp[\frac{(p - p_0)^2}{4\sigma^2}] \tag{29}$$

The exponent in this expression has the maximum at $\mathbf{p} = \mathbf{p}_0$ and in the vicinity of the maximum

$$(p - p_0)^2 = -(\mathbf{p} - \mathbf{p}_0)^2 + [\frac{(\mathbf{p}_0, \mathbf{p} - \mathbf{p}_0)}{\epsilon(\mathbf{p}_0)}]^2 + o(|\mathbf{p} - \mathbf{p}_0|^2) \tag{30}$$

If $\mathbf{p}_0$ is directed along the z axis and the subscript $\perp$ is used to denote the projection of the vector onto the xy plane then

$$(p-p_0)^2 = -(\mathbf{p}_\perp - \mathbf{p}_{0\perp})^2 - [\frac{m}{\epsilon(\mathbf{p}_0)}]^2(p_z - p_{0z})^2 + o(|\mathbf{p}-\mathbf{p}_0|^2) \tag{31}$$

It follows from this expression that if the particle is ultrarelativistic then the width of the momentum distribution in the longitudinal direction is much greater that in transverse ones and for massless particles the former becomes infinite. We conclude that for massless particles the covariant parametrization of $f(p,p_0)$ is problematic.

As noted above, the only fundamental requirement on quantum level is that the representation operators should satisfy the commutation relations (2) while covariance is not fundamental. Nevertheless, the above discussion shows that covariance of coordinate wave functions can be preserved if one takes into account the fact that they are defined on space-like hyperplanes. In particular, covariance of functions f can be preserved if one assumes that they depend not only on p and p_0 but also on n. In what follows we consider only the case when the vector n is such that $n^0 = 1$ and $\mathbf{n} = 0$. Let us replace $f(p,p_0)$ by $f(\tilde{p},\tilde{p}_0)$ where $\tilde{p} = p - (pn)n$ and $\tilde{p}_0 = p_0 - (p_0 n)n$. Then the four-vectors $\tilde{p}$ and $\tilde{p}_0$ have only nonzero spatial components equal $\mathbf{p}$ and $\mathbf{p}_0$, respectively. As a consequence, any rotationally invariant combination of $\mathbf{p}$ and $\mathbf{p}_0$ can be treated as a Lorentz covariant combination of $\tilde{p}$ and $\tilde{p}_0$.

We conclude that with the above choice of the vector n one can work with momentum and coordinate wave functions in full analogy with nonrelativistic quantum mechanics and in that case Lorentz covariance is satisfied. In particular in that case Eq. (28) can be written in the form of Eq. (9).

We now consider the photon case in greater details. The coordinate photon wave function has been discussed by many authors. A question arises in what situations this function is needed. As already noted, since the fundamental theory of electromagnetic interactions is QED, and this theory does not contain space-time at all, for solving quantum problems in the framework of QED the coordinate photon wave function is not needed. However, this function is used in some special problems, for example for describing single-photon interference and diffraction by analogy with classical theory.

In the present paper we consider only the case of free photons. If we consider a motion of a free particle, it is not important in what interactions this particle participates and, as explained above, if the particle is described by its IR in semiclassical approximation then the particle spin is not important. Hence the effect of WPS for an ultrarelativistic particle does not depend on the nature of the particle, i.e. on whether the particle is the photon, the proton, the electron etc. For this reason we are interested in papers on the photon coordinate wave function mainly from the point of view how the position operator for the free ultrarelativistic particle is defined.

Note that in classical theory the notion of field, as well as that of wave, is used for describing systems of many particles by their mean characteristics. For

example, the electromagnetic field consists of many photons. In classical theory each photon is not described individually but the field as a whole is described by the field strengths $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$ which can be measured (in principle) by using macroscopic test bodies such that the quantities $\mathbf{r}$ and t refer to positions of such bodies at time t. In quantum theory one can formally define corresponding quantized field operators but the meaning of $(\mathbf{r},t)$ for elementary particles is not clear. In addition, in view of the above remarks, the physical meaning of electric and magnetic fields of a free photon is problematic.

For the first time the coordinate photon wave function has been discussed by Landau and Peierls in Ref. [32]. However, in the literature it has been stated (see e.g. Refs. [33] and [29]) that in QED there is no way to define a coordinate photon wave function. A section in the textbook [33] is titled "Impossibility of introducing the photon wave function in coordinate representation". The arguments follow. The electric and magnetic fields of the photon in coordinate representation are proportional to the Fourier transforms of $|\mathbf{p}|^{1/2}\chi(\mathbf{p})$, rather than $\chi(\mathbf{p})$. As a consequence, the quantities $\mathbf{E}(\mathbf{r})$ and $\mathbf{B}(\mathbf{r})$ are defined not by $\psi(\mathbf{r})$ but by integrals of $\psi(\mathbf{r})$ over a region of the order of the wave length. However, this argument also does not exclude the possibility that $\psi(\mathbf{r})$ can have a physical meaning in semiclassical approximation since, as noted above, the notions of the electric and magnetic fields of a single photon are problematic. In addition, since $\lambda \to 0$ in the formal limit $\hbar \to 0$, one should not expect that any position operator in semiclassical approximation can describe coordinates with the accuracy better than the wave length. Another arguments in favor of the existence of the coordinate photon wave function have been given by Bialynicki-Birula [34].

A detailed discussion of the photon position operator can be found in papers by Margaret Hawton and references therein (see e.g. Ref. [35]). In this approach the photon is described by a local field and the momentum and coordinate representations are related to each other by standard Fourier transform. The author of Ref. [35] discusses generalizations of the photon position operator proposed by Pryce [36]. However, the Pryce operator and its generalizations discussed in Refs. [34, 35] differ from the Newton-Wigner operator only by terms of the order of the wave length. Hence in semiclassical approximation all those operators are equivalent.

The above discussion shows that on quantum level the physical meaning of the coordinate is a difficult problem but in view of a) and b) (see the beginning of this section) one can conclude that in semiclassical approximation all the existing proposals for the position operator are equivalent to the Newton-Wigner operator $i\hbar\partial/\partial\mathbf{p}$. An additional argument in favor of this operator is that the relativistic nature of the photon might be somehow manifested in the longitudinal direction while in transverse directions the behavior of the wave function should be similar to that in standard nonrelativistic quantum mechanics. Another argument is that the photon wave function in coordinate representation constructed by using this operator satisfies the wave equation in agreement with classical electrodynamics (see Sec. 7).

For all the reasons described above, in the next section we consider what happens if the space-time evolution of relativistic wave packets is described by using the Newton-Wigner position operator.

6 Wave packet spreading in relativistic quantum mechanics

Consider first a construction of the wave packet for a particle with nonzero mass. A possible way of the construction follows. We first consider the particle in its rest system, i.e. in the reference frame where the mean value of the particle momentum is zero. The wave function $\chi_0(\mathbf{p})$ in this case can be taken as in Eq. (6) with $\mathbf{p}_0 = 0$. As noted in Sec. 2, such a state cannot be semiclassical. However, it is possible to obtain a semiclassical state by applying a Lorentz transformation to $\chi_0(\mathbf{p})$. As a consequence of Eq. (23) and the relation between the functions ξ and χ

$$U(\Lambda)\chi_0(\mathbf{p}) = [\frac{\epsilon(\mathbf{p}')}{\epsilon(\mathbf{p})}]^{1/2}\chi_0(\mathbf{p}') \tag{32}$$

where $\mathbf{p}'$ is the momentum obtained from $\mathbf{p}$ by the Lorentz transformation Λ^{-1}. If Λ is the Lorentz boost along the z axis with the velocity v then

$$\mathbf{p}'_\perp = \mathbf{p}_\perp, \quad p'_z = \frac{p_z - v\epsilon(\mathbf{p})}{(1-v^2)^{1/2}} \tag{33}$$

As follows from this expression, $exp(-\mathbf{p}'^2a^2/2\hbar^2)$ as a function of $\mathbf{p}$ has the maximum at $\mathbf{p}_\perp = 0$, $p_z = p_{z0} = v[(m^2+\mathbf{p}_\perp^2)/(1-v^2)]^{1/2}$ and near the maximum

$$exp(-\frac{a^2\mathbf{p}'^2}{2\hbar^2}) \approx exp\{-\frac{1}{2\hbar^2}[a^2\mathbf{p}_\perp^2 + b^2(p_z - p_{z0})^2]\}$$

where $b = a(1-v^2)^{1/2}$ what represents the effect of the Lorentz contraction. If $mv \gg \hbar/a$ (in units where $c = 1$) then $m \gg |\mathbf{p}_\perp|$ and $p_{z0} \approx mv/(1-v^2)^{1/2}$. In this case the transformed state is semiclassical and the mean value of the momentum is exactly the classical (i.e. nonquantum) value of the momentum of a particle with mass m moving along the z axis with the velocity v. However, in the opposite case when $m \ll \hbar/a$ the transformed state is not semiclassical since the uncertainty of p_z is of the same order as the mean value of p_z.

If the photon mass is exactly zero then the photon cannot have the rest state. However, even if the photon mass is not exactly zero, it is so small that the condition $m \ll \hbar/a$ is certainly satisfied for any realistic value of a. Hence a semiclassical state for the photon or a particle with a very small mass cannot be obtained by applying the Lorentz transformation to $\chi_0(\mathbf{p})$ and considering the case when v is very close to unity. An analogous problem with the covariant description

of the massless wave function has been discussed in the preceding section (see Eq. (31)).

The above discussion shows that in the relativistic case the momentum distribution in transverse directions is the same as in the nonrelativistic case (see also Eq. (31)) and the difference arises only for the momentum distribution in the longitudinal direction. Let us consider the ultrarelativistic case when $|\mathbf{p}_0| = p_0 \gg m$ and suppose that $\mathbf{p}_0$ is directed along the z axis. As noted in the preceding section, the formal requirement of Lorentz covariance will be satisfied if one works with rotationally invariant combinations of $\mathbf{p}$ and $\mathbf{p}_0$. The quantities $\mathbf{p}_\perp^2$ and $(p_z - p_0)^2$ satisfy this condition because

$$\mathbf{p}_\perp^2 = [\mathbf{p} - \mathbf{p}_0 \frac{(\mathbf{p}\mathbf{p}_0)}{p_0^2}]^2, \quad (p_z - p_0)^2 = \frac{1}{p_0^2}[(\mathbf{p}\mathbf{p}_0) - p_0^2]^2$$

We will describe an ultrarelativistic semiclassical state by a wave function which is a generalization of the function (6) (see also Eq. (27)):

$$\chi(\mathbf{p}, 0) = \frac{ab^{1/2}}{\pi^{3/4}\hbar^{3/2}} exp[-\frac{\mathbf{p}_\perp^2 a^2}{2\hbar^2} - \frac{(p_z - p_0)^2 b^2}{2\hbar^2} - \frac{i}{\hbar}\mathbf{p}_\perp \mathbf{r}_{0\perp} - \frac{i}{\hbar}(p_z - p_0) z_0] \qquad (34)$$

In the general case the parameters a and b defining the momentum distributions in the transverse and longitudinal directions, respectively, can be different. In that case the uncertainty of each transverse component of momentum is $\hbar/(a\sqrt{2})$ while the uncertainty of the z component of momentum is $\hbar/(b\sqrt{2})$. In view of the above discussion one might think that, as a consequence of the Lorentz contraction, the parameter b should be very small. However, the notion of the Lorentz contraction has a physical meaning only if $m \gg \hbar/a$ while for the photon the opposite relation takes place. We will see below that in typical situations the quantity b is large and much greater than a.

In relativistic quantum theory the situation with time is analogous to that in the nonrelativistic case (see Sec. 3) and time can be treated only as a good approximate parameter describing the evolution according to the Schrödinger equation with the relativistic Hamiltonian. Then, as a consequence of Eq. (25), we have that in the ultrarelativistic case (i.e. when $p = |\mathbf{p}| \gg m$)

$$\chi(\mathbf{p}, t) = exp(-\frac{i}{\hbar} pct)\chi(\mathbf{p}, 0) \qquad (35)$$

Since at different moments of time the wave functions in momentum space differ each other only by a phase factor, the mean value and uncertainty of each momentum component do not depend on time. In other words, there is no WPS for the wave function in momentum space. As noted in Sec. 3, the same is true in the nonrelativistic case.

As noted in the preceding section, in the relativistic case the function $\psi(\mathbf{r}, t)$ can be again defined by Eq. (9) where now $\chi(\mathbf{p}, t)$ is defined by Eq. (35).

If the variable p_z in the integrand is replaced by $p_0 + p_z$ then as follows from Eqs. (9,34,35)

$$\psi(\mathbf{r},t) = \frac{ab^{1/2}exp(i\mathbf{p}_0\mathbf{r}/\hbar)}{\pi^{3/4}\hbar^{3/2}(2\pi\hbar)^{3/2}} \int exp\{-\frac{\mathbf{p}_\perp^2 a^2}{2\hbar^2} - \frac{p_z^2 b^2}{2\hbar^2} + \frac{i}{\hbar}\mathbf{p}(\mathbf{r}-\mathbf{r}_0)$$
$$-\frac{ict}{\hbar}[(p_z+p_0)^2 + \mathbf{p}_\perp^2]^{1/2}\}d^3\mathbf{p} \tag{36}$$

In contrast to the nonrelativistic case where the energy is the quadratic function of momenta and the integration in Eq. (10) can be performed analytically, here the analytical integration is a problem in view of the presence of square root in Eq. (36). We will perform the integration by analogy with the Fresnel approximation in optics and with Ref. [37] where a similar approximation has been used for discussing the WPS effect in classical electrodynamics. The Fresnel approximation describes some important features of the relativistic WPS effect but, as will be noted below, in this approximation some important features of this effect are lost.

The approximation is based on the fact that in semiclassical approximation the quantity p_0 should be much greater than uncertainties of the momentum in the longitudinal and transversal directions, i.e. $p_0 \gg p_z$ and $p_0 \gg |\mathbf{p}_\perp|$. Hence with a good accuracy one can expand the square root in the integrand in powers of $|\mathbf{p}|/p_0$. Taking into account the linear and quadratic terms in the square root we get

$$[(p_z+p_0)^2 + \mathbf{p}_\perp^2]^{1/2} \approx p_0 + p_z + \mathbf{p}_\perp^2/2p_0 \tag{37}$$

This is analogous to the approximation $(m^2+\mathbf{p}^2)^{1/2} \approx m + \mathbf{p}^2/2m$ in nonrelativistic case. Then the integral over $d^3\mathbf{p}$ can be calculated as a product of integrals over $d^2\mathbf{p}_\perp$ and dp_z and the calculation is analogous to that in Eq. (10). The result of the calculation is

$$\psi(\mathbf{r},t) = [\pi^{3/4}ab^{1/2}(1+\frac{i\hbar ct}{p_0a^2})]^{-1}exp[\frac{i}{\hbar}(\mathbf{p}_0\mathbf{r} - p_0ct)]$$
$$exp[-\frac{(\mathbf{r}_\perp - \mathbf{r}_{0\perp})^2(1-\frac{i\hbar ct}{p_0a^2})}{2a^2(1+\frac{\hbar^2c^2t^2}{p_0^2a^4})} - \frac{(z-z_0-ct)^2}{2b^2}] \tag{38}$$

This result shows that the wave packet describing an ultrarelativistic particle (including a photon) is moving along the classical trajectory $z(t) = z_0 + ct$, in the longitudinal direction there is no spreading while in transverse directions spreading is characterized by the function

$$a(t) = a(1+\frac{\hbar^2c^2t^2}{p_0^2a^4})^{1/2} \tag{39}$$

The characteristic time of spreading can be defined as $t_* = p_0a^2/\hbar c$. The fact that $t_* \to \infty$ in the formal limit $\hbar \to 0$ shows that in relativistic case WPS also is a

pure quantum phenomenon (see the end of Sec. 3). From the formal point of view the result for t_* is the same as in nonrelativistic theory but m should be replaced by E/c^2 where E is the energy of the ultrarelativistic particle. This fact could be expected since, as noted above, it is reasonable to think that spreading in directions perpendicular to the particle momentum is similar to that in standard nonrelativistic quantum mechanics. However, in the ultrarelativistic case spreading takes place only in these directions. If $t \gg t_*$ the transverse width of the packet is $a(t) = \hbar ct/(p_0 a)$.

Hence the speed of spreading in perpendicular directions is $v_* = \hbar c/p_0 a$. In the nonrelativistic case different points of the packet are moving with different velocities and this is not a problem but in the case of the photon one expects that each point is moving with the speed c. However, the Fresnel approximation creates a problem because different points are moving with different velocities such that their magnitudes are in the range $[c, (c^2 + v_*^2)^{1/2}]$.

We now consider a model where

$$\chi(\mathbf{p}) = f(\mathbf{p}/p)F(p)/p \tag{40}$$

and assume that $f(\mathbf{p}/p) = \sum_{l\mu} c_{l\mu} Y_{l\mu}(\mathbf{p}/p)$ is the decomposition of the function f over spherical functions. The dependence of the momentum wave function on t is now defined by Eq. (35). In full analogy with the derivation of Eq. (19) we now get that

$$\psi(\mathbf{r}, t) = \frac{-i}{(2\pi\hbar)^{1/2} r} \sum_{l\mu} c_{l\mu} Y_{l\mu}(\mathbf{r}/r)[G(ct - r) - (-1)^l G(ct + r)] \tag{41}$$

where

$$G(\xi) = \int_0^\infty F(p) exp(\frac{-i}{\hbar}\xi p) dp \tag{42}$$

For reasonable choices of $F(p)$ we will have that at large distances and times $G(ct - r) \gg G(ct + r)$. Indeed if, for example, the quantities p_0 and b are such that $p_0 b \gg \hbar$ then possible (F, G) choices are:

$$F(p) = exp(-\frac{|p - p_0|b}{\hbar}), \quad G(\xi) = \frac{exp(-ip_0\xi/\hbar)}{b^2 + \xi^2};$$
$$F(p) = exp(-\frac{(|p - p_0|b)^2}{2\hbar^2}), \quad G(\xi) = (2\pi)^{1/2}\frac{\hbar}{b} exp(-\frac{ip_0\xi}{\hbar} - \frac{\xi^2}{2b^2}) \tag{43}$$

As follows from Eq. (41), in those cases

$$\psi(\mathbf{r}, t) = \frac{-i}{(2\pi\hbar)^{1/2} r} f(\mathbf{r}/r) G(ct - r) \tag{44}$$

Therefore at each moment of time t the coordinate wave function is not negligible only inside a narrow sphere with the radius ct and the width of the order of b.

The conclusion is that, in contrast to the nonrelativistic case, in the ultrarelativistic one there is no WPS in the radial direction (by analogy with the Fresnel approximation) and, by analogy with the result (20), at large distances and times the angular distributions in momentum and coordinate wave functions are the same. Therefore, in full analogy with the Mott-Heisenberg problem (see Sec. 4), the momenta of particles detected by a measuring device will be in the angular range defined not by the function $f(\mathbf{r}/r)$ but by the function $\tilde{f}(\mathbf{r}/r)$ characterizing the angles at which the device is seen from the origin. In addition, the angular distribution of momenta characterized by the function f does not depend on time, as well as in the nonrelativistic case.

If the function f is essentially different from zero only in the range where angles between momenta and the z-axis are small then the model (40) gives the same qualitative predictions as the Fresnel approximation. Indeed, suppose that this function is essentially different from zero for angles which are of the order of α or less, and $\alpha \ll 1$. Then the parameter b in Eq. (43) is similar to the parameter b in Eq. (34). The characteristic magnitude of the transverse momentum is of the order of $p_\perp \approx \alpha p_0$. Let a be defined such that $p_\perp = \hbar/a$. When the time is greater than a characteristic time for which the transition from Eq. (41) to Eq. (44) is legitimate (this time can differ from t_* for the Fresnel model) then, since the angular distributions in the momentum and coordinate wave functions are the same, the transversal width of the packet is of the order of $\alpha ct \approx ct\hbar/(p_0 a)$ in agreement with the Fresnel approximation. Therefore *if t is greater than some characteristic time then the width $a(t)$ of the packet is inversely proportional to the initial width $a(0) = a$.* It is also possible to define v_* by the same expression as in the Fresnel approximation. If $v_* \ll c$ the only difference between the two models is that in the Fresnel approximation different points of the packet are moving with different speeds while in the model (40) they are moving with the same speed c. In fact the Fresnel approximation is such that a small arc representing the front of the wave function in the model (40) is replaced by a segment.

7 Geometrical optics

The relation between quantum and classical electrodynamics is well-known and is described in textbooks (see e.g. Ref. [4, 33]). As already noted, classical electromagnetic field consists of many photons and in classical electrodynamics the photons are not described individually. Instead, classical electromagnetic field is described by field strengths which represent mean characteristics of a large set of photons. For constructing the field strengths one can use the photon wave functions $\chi(\mathbf{p}, t)$ or $\psi(\mathbf{r}, t)$ where E is replaced by $\hbar\omega$ and $\mathbf{p}$ is replaced by $\hbar\mathbf{k}$. In this connection it is interesting to note that since ω is a classical quantity used for describing a classical electromagnetic field, the photon is a pure quantum particle since its energy disappears in the formal limit $\hbar \to 0$. Even this fact shows that the photon cannot be treated as a classical particle and the effect of WPS for the photon cannot be neglected.

With the above replacements the functions χ and ψ do not contain any dependence on $\hbar$ (note that the normalization factor $\hbar^{-3/2}$ in $\chi(\mathbf{k},t)$ will disappear since the normalization integral for $\chi(\mathbf{k},t)$ is now over $d^3\mathbf{k}$, not $d^3\mathbf{p}$). The quantities ω and $\mathbf{k}$ are now treated, respectively, as the frequency and the wave vector of the classical electromagnetic field, and the functions $\chi(\mathbf{k},t)$ and $\psi(\mathbf{r},t)$ are interpreted not such that they describe probabilities for a single photon but such that they describe classical electromagnetic field $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$ which be constructed from these functions as described in textbooks on QED (see e.g. Ref. [33]).

An additional argument in favor of the choice of $\psi(\mathbf{r},t)$ as the coordinate photon wave function is that in classical electrodynamics the quantities $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$ for the free field should satisfy the wave equation $\partial^2\mathbf{E}/c^2\partial t^2 = \Delta\mathbf{E}$ and analogously for $\mathbf{B}(\mathbf{r},t)$. Hence if $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$ are constructed from $\psi(\mathbf{r},t)$ as described in textbooks (see e.g. Ref. [33]), they will satisfy the wave equation since, as follows from Eqs. (9,34,35), $\psi(\mathbf{r},t)$ also satisfies this equation.

The geometrical optics approximation implies that if $\mathbf{k}_0$ and $\mathbf{r}_0$ are the mean values of the wave vector and the spatial radius vector for a wave packet describing the electromagnetic wave then the uncertainties Δk and Δr, which are the mean values of $|\mathbf{k}-\mathbf{k}_0|$ and $|\mathbf{r}-\mathbf{r}_0|$, respectively, should satisfy the requirements $\Delta k \ll |\mathbf{k}_0|$ and $\Delta r \ll |\mathbf{r}_0|$. In full analogy with the derivation of Eq. (4), one can show that for each $j = 1, 2, 3$ the uncertainties of the corresponding projections of the vectors $\mathbf{k}$ and $\mathbf{r}$ satisfy the requirement $\Delta k_j \Delta r_j \geq 1/2$ (see e.g. Ref. [2]). In particular, an electromagnetic wave satisfies the approximation of geometrical optics in the greatest possible extent if $\Delta k \Delta r$ is of the order of unity.

The above discussion confirms what has been mentioned in Sec. 1 that *the effect of WPS in transverse directions takes place not only in quantum theory but even in classical electrodynamics*. Indeed, since the function $\psi(\mathbf{r},t)$ satisfies the classical wave equation, the above consideration can be also treated as an example showing that *even for a free wave packet in classical electrodynamics the WPS effect is inevitable*. In the language of classical waves the parameters of spreading can be characterized by the function $a(t)$ (see Eq. (39)) and the quantities t_* and v_* such that in terms of the wave length $\lambda = 2\pi c/\omega_0$

$$a(t) = a(1 + \frac{\lambda^2 c^2 t^2}{4\pi^2 a^4})^{1/2}, \quad t_* = \frac{2\pi a^2}{\lambda c}, \quad v_* = \frac{\lambda c}{2\pi a} \tag{45}$$

The last expression can be treated such that if $\lambda \ll a$ then the momentum has the angular uncertainty of the order of $\alpha = \lambda/(2\pi a)$. This result is natural from the following consideration. Let the mean value of the momentum be directed along the z-axis and the uncertainty of the transverse component of the momentum be $\Delta p_\perp$. Then $\Delta p_\perp$ is of the order of $\hbar/a$, $\lambda = 2\pi\hbar/p_0$ and hence α is of the order of $\Delta p_\perp/p_0 \approx \lambda/(2\pi a)$. This is analogous to the well-known result in classical optics that the best angular resolution of a telescope with the dimension d is of the order of λ/d. Another well-known result of classical optics is that if a wave encounters an obstacle

having the dimension d then the direction of the wave diverges by the angle of the order of λ/d.

The inevitability of WPS for a free wave packet in classical electrodynamics is obvious from the following consideration. Suppose that a classical wave packet does not have a definite value of the momentum. Then if a is the initial width of the packet in directions perpendicular to the mean momentum, one might expect that the width will grow as $a(t) = a + \alpha ct$ and for large values of t, $a(t) \approx \alpha ct$. As follows from Eq. (45), if $t \gg t_*$ then indeed $a(t) \approx \alpha ct$. In standard quantum theory we have the same result because the coordinate and momentum wave functions are related to each other by the same Fourier transform as the coordinate and **k** distributions in classical electrodynamics.

The quantity $N_{||} = b/\lambda$ shows how many oscillations the oscillating exponent in Eq. (38) makes in the region where the wave function or the amplitude of the classical wave is significantly different from zero. As noted in Sec. 2, for the validity of semiclassical approximation this quantity should be very large. In nonrelativistic quantum mechanics a and b are of the same order and hence the same can be said about the quantity $N_\perp = a/\lambda$. As noted above, in the case of the photon we do not know the relation between a and b. In terms of the quantity $N_\perp$ we can rewrite the expressions for t_* and v_* in Eq. (45) as

$$t_* = 2\pi N_\perp^2 T, \quad v_* = \frac{c}{2\pi N_\perp} \tag{46}$$

where T is the period of the classical wave. Hence the accuracy of semiclassical approximation (or the geometrical optics approximation in classical electrodynamics) increases with the increase of $N_\perp$.

In Ref. [37] the problem of WPS for classical electromagnetic waves has been discussed in the Fresnel approximation for a two-dimensional wave packet. Equation (25) of Ref. [37] is a special case of Eq. (37) and the author of Ref. [37] shows that, in his model the wave packet spreads out in the direction perpendicular to the group velocity of the packet. As noted in the preceding section, in the ultrarelativistic case the function $a(t)$ is given by the same expression as in the nonrelativistic case but m is replaced by E/c^2. Hence if the results of the preceding section are reformulated in terms of classical waves then m should be replaced by $\hbar\omega_0/c^2$ and this fact has been pointed out in Ref. [37].

8 Wave packet width paradox

We now consider the following important question. We assume that a classical wave packet is a collection of photons. Let a_{cl} be the quantity a for the classical packet and a_{ph} be a typical value of a for the photons. What is the relation between a_{cl} and a_{ph}?

My observation is that physicists answer this question in different ways. Quantum physicists usually say that in typical situations $a_{ph} \ll a_{cl}$ because a_{cl} is of macroscopic size while in semiclassical approximation the quantity a_{ph} for each photon can be treated as the size of the region where the photon has been created. On the other hand, classical physicists usually say that $a_{ph} \gg a_{cl}$ and the motivation follows.

Consider a decomposition of some component of classical electromagnetic field into the Fourier series:

$$A(x) = \sum_{\sigma} \int [a(\mathbf{p},\sigma)u(\mathbf{p},\sigma)exp(-ipx) + a(\mathbf{p},\sigma)^* u(\mathbf{p},\sigma)^* exp(ipx)]d^3\mathbf{p} \qquad (47)$$

where σ is the polarization, x and p are the four-vectors such that $x = (ct, \mathbf{x})$ and $p = (|\mathbf{p}|c, \mathbf{p})$, the functions $a(\mathbf{p},\sigma)$ are the same for all the components, the functions $u(\mathbf{p},\sigma)$ depend on the component and * is used to denote the complex conjugation. Then photons arise as a result of quantization when $a(\mathbf{p},\sigma)$ and $a(\mathbf{p},\sigma)^*$ are understood not as usual function but as operators of annihilation and creation of the photon with the quantum numbers $(\mathbf{p},\sigma)$ and * is now understood as Hermitian conjugation. Hence the photon is described by a plane wave which has the same magnitude in all points of the space. In other words, a_{ph} is infinitely large and a finite width of the classical wave packet arises as a result of interference of different plane waves.

The above definition of the photon has at least the following inconsistency. If the photon is treated as a particle then its wave function should be normalizable while the plane wave is not normalizable. In textbooks this problem is often circumvented by saying that we consider our system in a finite box. Then the spectrum of momenta becomes finite and instead of Eq. (47) one can write

$$A(x) = \sum_{\sigma} \sum_{j} [a(\mathbf{p}_j,\sigma)u(\mathbf{p}_j,\sigma)exp(-ip_j x) + a(\mathbf{p}_j,\sigma)^* u(\mathbf{p}_j,\sigma)^* exp(ip_j x)] \qquad (48)$$

where j enumerates the points of the momentum spectrum.

One can now describe quantum electromagnetic field by states in the Fock space where the vacuum vector Φ_0 satisfies the condition $a(\mathbf{p}_j,\sigma)\Phi_0 = 0$, $||\Phi_0|| = 1$ and the operators commute as

$$[a(\mathbf{p}_i,\sigma_k), a(\mathbf{p}_j,\sigma_l)] = [a(\mathbf{p}_i,\sigma_k)^*, a(\mathbf{p}_j,\sigma_l)^*] = 0, \quad [a(\mathbf{p}_i,\sigma_k), a(\mathbf{p}_j,\sigma_l)^*] = \delta_{ij}\delta_{kl} \qquad (49)$$

Then any state can be written as

$$\Psi = \sum_{n=0}^{\infty} \sum_{\sigma_1 \ldots \sigma_n} \sum_{\mathbf{p}_1,\ldots\mathbf{p}_n} \chi(\mathbf{p}_1,\sigma_1,\ldots\mathbf{p}_n,\sigma_n)a(\mathbf{p}_1,\sigma_1)^* \cdots a(\mathbf{p}_n,\sigma_n)^*\Phi_0 \qquad (50)$$

Classical states are understood such that although the number of photons is large, it is much less than the number of possible momenta and in Eq. (50) all the photons have different momenta (this is analogous to the situation in classical

statistics where mean occupation numbers are much less than unity). Then it is not important whether the operators (a, a^*) commute or anticommute. However, according to the Pauli theorem on spin-statistics connection [23], they should commute and this allows the existence of coherent states where many photons have the same quantum numbers. Such states can be created in lasers and they are not described by classical electrodynamics. In the next section we consider position operator for coherent states while in this section we consider only quantum description of states close to classical.

Note that even in some textbooks on quantum optics (see e.g. Ref. [38]) classical and quantum states are characterized in the opposite way: it is stated that classical states are characterized by large occupation numbers while quantum states - by small ones. The question what states should be called classical or quantum is not a matter of convention since in quantum theory there are rigorous criteria for that purpose. In particular, as explained in textbooks on quantum theory, the exchange interaction is a pure quantum phenomenon which does not have classical analogs. That's why the Boltzmann statistics (which works when mean occupation numbers are much less than unity and the exchange interaction is negligible) is classical while the Fermi-Dirac and Bose-Einstein statistics (which work when mean occupation numbers are of the order of unity or greater and the exchange interaction is important) are quantum.

The next problem is that one should take into account that in standard theory the photon momentum spectrum is continuous. Then the above construction can be generalized as follows. The vacuum state Φ_0 satisfies the same conditions $||\Phi_0|| = 1$ and $a(\mathbf{p}, \sigma)\Phi_0 = 0$ while the operators (a, a^*) satisfy the following commutation relations

$$[a(\mathbf{p}, \sigma), a(\mathbf{p}', \sigma')] = [a(\mathbf{p}, \sigma)^*, a(\mathbf{p}', \sigma')^*] = 0, \quad [a(\mathbf{p}, \sigma), a(\mathbf{p}', \sigma')^*] = \delta^{(3)}(\mathbf{p} - \mathbf{p}')\delta_{\sigma\sigma'} \tag{51}$$

Then a general quantum state can be written as

$$\Psi = \sum_{n=0}^{\infty} \sum_{\sigma_1 ... \sigma_n} \int ... \int \chi(\mathbf{p}_1, \sigma_1, ...\mathbf{p}_n, \sigma_n) a(\mathbf{p}_1, \sigma_1)^* \cdots a(\mathbf{p}_n, \sigma_n)^* d^3\mathbf{p}_1 \cdots d^3\mathbf{p}_n \Phi_0 \tag{52}$$

In the approximation when a classical wave packet is understood as a collection of independent photons (see the discussion in Sec. 11), the state of this packet has the form

$$\Psi = \sum_{n=0}^{\infty} c_n \prod_{j=1}^{n} \{\sum_{\sigma_j} \int \chi_j(\mathbf{p}_j, \sigma_j) a(\mathbf{p}_j, \sigma_j)^* d^3\mathbf{p}_j\} \Phi_0 \tag{53}$$

where χ_j is the wave function of the jth photon and intersections of supports of wave functions of different photons can be neglected. This is an analog of the above situation with the discrete case where it is assumed that different photons in a classical wave packet have different momenta. In other words, while the wave function of each

photon can be treated as an interference of plane waves, different photons can interfere only in coherent states but not in classical wave packets.

We now describe a well-known generalization of the results on IRs of the Poincare algebra to the description in the Fock space (see e.g. Ref. [39] for details). If A is an operator in the space of the photon IR then a generalization of this operator to the case of the Fock space can be constructed as follows. Any operator in the space of IR can be represented as an integral operator acting on the wave function as

$$A\chi(\mathbf{p},\sigma)=\sum_{\sigma'}\int A(\mathbf{p},\sigma,\mathbf{p}',\sigma')\chi(\mathbf{p}',\sigma')d^3\mathbf{p}' \tag{54}$$

For example, if $\mathbf{A}\chi(\mathbf{p},\sigma)=\partial\chi(\mathbf{p},\sigma)/\partial\mathbf{p}$ then $\mathbf{A}$ is the integral operator with the kernel

$$\mathbf{A}(\mathbf{p},\sigma,\mathbf{p}',\sigma')=\frac{\partial\delta^{(3)}(\mathbf{p}-\mathbf{p}')}{\partial\mathbf{p}}\delta_{\sigma\sigma'}$$

We now require that if the action of the operator A in the space of IR is defined by Eq. (54) then in the case of the Fock space this action is defined as

$$A=\sum_{\sigma\sigma'}\int A(\mathbf{p},\sigma,\mathbf{p}',\sigma')a(\mathbf{p},\sigma)^*a(\mathbf{p}',\sigma')d^3\mathbf{p}d^3\mathbf{p}' \tag{55}$$

Then it is easy to verify that if A, B and C are operators in the space of IR satisfying the commutation relation $[A,B]=C$ then the generalizations of these operators in the Fock space satisfy the same commutation relation. It is also easy to verify that the operators generalized to the action in the Fock space in such a way are additive, i.e. for a system of n photons they are sums of the corresponding single-particle operators. In particular, the energy of the n-photon system is a sum of the energies of the photons in the system and analogously for the other representation operators of the Poincare algebra - momenta, angular momenta and Lorentz boosts.

We are interested in calculating mean values of different combinations of the momentum operator. Since this operator does not act over spin variables, we will drop such variables in the (a,a^*) operators and in the functions χ_j. Then the explicit form of the momentum operator is $\mathbf{P}=\int\mathbf{p}a(\mathbf{p})^*a(\mathbf{p})d^3\mathbf{p}$. Since this operator does not change the number of photons, the mean values can be independently calculated in each subspace where the number of photons is N.

Suppose that the momentum of each photon is approximately directed along the z-axis and the quantity p_0 for each photon approximately equals $2\pi\hbar/\lambda$. If $\Delta p_\perp$ is a typical uncertainty of the transversal component of the momentum for the photons then a typical value of the angular uncertainty for the photons is $\alpha_{ph}=\Delta p_\perp/p_0\approx\lambda/(2\pi a_{ph})$. The total momentum of the classical wave packet consisting of N photons is a sum of the photon momenta: $\mathbf{P}=\sum_{i=1}^N\mathbf{p}^{(i)}$. Suppose that the mean value of $\mathbf{P}$ is directed along the z-axis and its magnitude P_0 is such that $P_0\approx Np_0$.

The uncertainty of the x component of $\mathbf{P}$ is $\Delta P_x = \overline{P_x^2}^{1/2}$ where

$$\overline{P_x^2} = \sum_{i=1}^{N} \overline{(p_x^{(i)})^2} + \sum_{i\neq j;i,j=1}^{N} \overline{p_x^{(i)} p_x^{(j)}}$$

Then in the approximation of independent photons (see the remarks after Eq. (53))

$$\overline{P_x^2} = \sum_{i=1}^{N} \overline{(p_x^{(i)})^2} + \sum_{i\neq j;i,j=1}^{N} \overline{p_x^{(i)}} \cdot \overline{p_x^{(j)}} = \sum_{i=1}^{N} [\overline{(p_x^{(i)})^2} - \overline{p_x^{(i)}}^2] = \sum_{i=1}^{N} (\Delta p_x^{(i)})^2$$

where we have taken into account that $\overline{P_x} = \sum_{i=1}^{N} \overline{p_x^{(i)}} = 0$.

As a consequence, if typical values of $\Delta p_\perp^{(i)}$ have the the same order of magnitude equal to $\Delta p_\perp$ then $\Delta P_\perp \approx N^{1/2} \Delta p_\perp$ and the angular divergence of the classical vave packet is

$$\alpha_{cl} = \Delta P_\perp / P_0 \approx \Delta p_\perp / (p_0 N^{1/2}) = \alpha_{ph} / N^{1/2} \tag{56}$$

Since the classical wave packet is described by the same wave equation as the photon wave function, its angular divergence can be expressed in terms of the parameters λ and a_{cl} such that $\alpha_{cl} = \lambda/(2\pi a_{cl})$. Hence $a_{cl} \approx N^{1/2} a_{ph}$ and we conclude that $a_{ph} \ll a_{cl}$.

Note that in this derivation no position operator has been used. Although the quantities λ and a_{ph} have the dimension of length, they are defined only from considering the photon in momentum space because, as noted in Sec. 5, for individual photons λ is understood only as $2\pi\hbar/p_0$, a_{ph} defines the width of the photon momentum wave function (see Eq. (34)) and is of the order of $\hbar/\Delta p_\perp$. As noted in Secs. 3 and 6, the momentum distribution does not depend on time and hence the result $a_{ph} \ll a_{cl}$ does not depend on time too. If photons in a classical wave packet could be treated as (almost) pointlike particles then photons do not experience WPS while the WPS effect for a classical wave packet is a consequence of the fact that different photons in the packet have different momenta.

However, in standard quantum theory this scenario does not take place for the following reason. Let $a_{cl}(t)$ be the quantity $a(t)$ for the classical wave packet and $a_{ph}(t)$ be a typical value of the quantity $a(t)$ for individual photons. With standard position operator the quantity $a_{ph}(t)$ is interpreted as the spatial width of the photon coordinate wave function in directions perpendicular to the photon momentum and this quantity is time dependent. As shown in Secs. 6 and 7, $a(0) = a$ but if $t \gg t_*$ then $a(t)$ is *inversely proportional* to a and the coefficient of proportionality is the same for the classical wave packet and individual photons (see Eq. (45)). Hence *in standard quantum theory we have a paradox that after some period of time* $a_{ph}(t) \gg a_{cl}(t)$ i.e. individual photons in a classical wave packet spread out in a much greater extent than the wave packet as a whole. We call this situation the wave packet width (WPW) paradox (as noted above, different photons in a classical wave packet do not interfere

with each other). The reason of the paradox is obvious: if the law that the angular divergence of a wave packet is of the order of λ/a is applied to both, a classical wave packet and photons comprising it then the paradox follows from the fact that the quantities a for the photons are much less than the quantity a for the classical wave packet. Note that in classical case the quantity a_{cl} does not have the meaning of $\hbar/\Delta P_\perp$ and λ is not equal to $2\pi\hbar/P_0$.

9 Wave packet spreading in coherent states

In textbooks on quantum optics the laser emission is described by the following model (see e.g. Refs. [38, 40]). Consider a set of photons having the same momentum $\mathbf{p}$ and polarization σ and, by analogy with the discussion in the preceding section, suppose that the momentum spectrum is discrete. Consider a quantum superposition $\Psi = \sum_{n=0}^{\infty} c_n[a(\mathbf{p},\sigma)^*]^n\Phi_0$ where the coefficients c_n satisfy the condition that Ψ is an eigenstate of the annihilation operator $a(\mathbf{p},\sigma)$. Then the product of the coordinate and momentum uncertainties has the minimum possible value $\hbar/2$ and, as noted in Sec. 2, such a state is called coherent. However, the term coherent is sometimes used meaning that the state is a quantum superposition of many-photon states $[a(\mathbf{p},\sigma)^*]^n\Phi_0$.

In the above model it is not taken into account that (in standard theory) photons emitted by a laser can have only a continuous spectrum of momenta. Meanwhile for the WPS effect the width of the momentum distribution is important. In this section we consider a generalization of the above model where the fact that photons have a continuous spectrum of momenta is taken into account. This will make it possible to consider the WPS effect in coherent states.

In the above formalism coherent states can be defined as follows. We assume that all the photons in the state Eq. (52) have the same polarization. Hence for describing such states we can drop the quantum number σ in wave functions and a-operators. We also assume that all photons in coherent states have the same momentum distribution. These conditions can be satisfied by requiring that coherent states have the form

$$\Psi = \sum_{n=0}^{\infty} c_n[\int \chi(\mathbf{p})a(\mathbf{p})^* d^3\mathbf{p}]^n\Phi_0 \tag{57}$$

where c_n are some coefficients. Finally, by analogy with the description of coherent states in standard textbooks on quantum optics one can require that they are eigenstates of the operator $\int a(\mathbf{p})d^3\mathbf{p}$.

The dependence of the state Ψ in Eq. (57) on t is $\Psi(t) = exp(-iEt/\hbar)\Psi$ where, as follows from Eqs. (22) and (55), the action of the energy operator in the Fock space is $E = \int pca(\mathbf{p})^* a(\mathbf{p})d^3\mathbf{p}$. Since $exp(iEt/\hbar)\Phi_0 = \Phi_0$, it readily follows from Eq. (51) that

$$\Psi(t) = \sum_{n=0}^{\infty} c_n[\int \chi(\mathbf{p},t)a(\mathbf{p})^* d^3\mathbf{p}]^n\Phi_0 \tag{58}$$

where the relation between $\chi(\mathbf{p},t)$ and $\chi(\mathbf{p})=\chi(\mathbf{p},0)$ is given by Eq. (35).

A problem arises how to define the position operator in the Fock space. If this operator is defined by analogy with the above construction then we get an unphysical result that each coordinate of the n-photon system as a whole is a sum of the corresponding coordinates of the photons in the system. This is an additional argument that the position operator is less fundamental than the representation operators of the Poincare algebra and its action should be defined from additional considerations. In textbooks on quantum optics the position operator for coherent states is usually defined by analogy with the position operator in nonrelativistic quantum mechanics for the harmonic oscillator problem. The motivation follows. If the energy levels $\hbar\omega(n+1/2)$ of the harmonic oscillator are treated as states of n quanta with the energies $\hbar\omega$ then the harmonic oscillator problem can be described by the operators a and a^* which are expressed in terms of the one-dimensional position and momentum operators q and p as $a=(\omega q+ip)/(2\hbar\omega)^{1/2}$ and $a^*=(\omega q-ip)/(2\hbar\omega)^{1/2}$, respectively. However, as noted above, the model description of coherent states in those textbooks is one-dimensional because the continuous nature of the momentum spectrum is not taken into account. In addition, the above results on WPS give indications that the position operator in standard theory is not consistently defined. For all these reasons a problem arises whether the requirement that the state Ψ in Eq. (57) is an eigenvector of the operator $\int a(\mathbf{p})d^3\mathbf{p}$ has a physical meaning. In what follows this requirement is not used.

In nonrelativistic classical mechanics the radius vector of a system of n particles as a whole (the radius vector of the center of mass) is defined as $\mathbf{R}=(m_1\mathbf{r}_1+...+m_n\mathbf{r}_n)/(m_1+...+m_n)$ and in works on relativistic classical mechanics it is usually defined as $\mathbf{R}=(\epsilon_1(\mathbf{p}_1)\mathbf{r}_1+...+\epsilon_n(\mathbf{p}_n)\mathbf{r}_n)/(\epsilon_1(\mathbf{p}_1)+...+\epsilon_n(\mathbf{p}_n))$ where $\epsilon_i(\mathbf{p}_i)=(m_i^2+\mathbf{p}_i^2)^{1/2}$. Hence if all the particles have the same masses and momenta, $\mathbf{R}=(\mathbf{r}_1+...+\mathbf{r}_n)/n$.

These remarks make it reasonable to define the position operator for coherent states as follows. Let x_j be the jth component of the position operator in the space of IR and $A_j(\mathbf{p},\mathbf{p}')$ be the kernel of this operator. Then in view of Eq. (55) the action of the operator X_j on the state $\Psi(t)$ in Eq. (57) can be defined as

$$X_j\Psi(t)=\sum_{n=1}^{\infty}\frac{c_n}{n}\int\int A_j(\mathbf{p}",\mathbf{p}')a(\mathbf{p}")^*a(\mathbf{p}')d^3\mathbf{p}"d^3\mathbf{p}'[\int\chi(\mathbf{p},t)a(\mathbf{p})^*d^3\mathbf{p}]^n\Phi_0 \qquad (59)$$

If $\overline{x_j}(t)$ and $\overline{x_j^2}(t)$ are the mean values of the operators x_j and x_j^2, respectively then as follows from the definition of the kernel of the operator x_j

$$\overline{x_j}(t)=\int\int\chi(\mathbf{p},t)^*A_j(\mathbf{p},\mathbf{p}')\chi(\mathbf{p}',t)d^3\mathbf{p}d^3\mathbf{p}'$$
$$\overline{x_j^2}(t)=\int\int\int\chi(\mathbf{p}",t)^*A_j(\mathbf{p},\mathbf{p}")^*A_j(\mathbf{p},\mathbf{p}')\chi(\mathbf{p}',t)d^3\mathbf{p}d^3\mathbf{p}"d^3\mathbf{p}' \qquad (60)$$

and in the case of IR the uncertainty of the quantity x_j is $\Delta x_j(t)=[\overline{x_j^2}(t)-\overline{x_j}(t)^2]^{1/2}$. At the same time, if $\overline{X_j}(t)$ and $\overline{X_j^2}(t)$ are the mean values of the operators X_j and

X_j^2, respectively then

$$\overline{X_j}(t) = (\Psi(t), X_j\Psi(t)), \quad \overline{X_j^2}(t) = (\Psi(t), X_j^2\Psi(t)) \tag{61}$$

and the uncertainty of the quantity X_j is $\Delta X_j(t) = [\overline{X_j^2}(t) - \overline{X_j}(t)^2]^{1/2}$. Our goal is to express $\Delta X_j(t)$ in terms of $\overline{x_j}(t)$, $\overline{x_j^2}(t)$ and $\Delta x_j(t)$.

If the function $\chi(\mathbf{p}, t)$ is normalized to one (see Eq. (7)) then, as follows from Eq. (51), $||\Psi(t)|| = 1$ if

$$\sum_{n=0}^{\infty} n!|c_n|^2 = 1 \tag{62}$$

A direct calculation using Eqs. (51), (59), (60) and (61) gives

$$\overline{X_j}(t) = \overline{x_j}(t) \sum_{n=1}^{\infty} n!|c_n|^2$$

$$\overline{X_j^2}(t) = \sum_{n=1}^{\infty}(n-1)!|c_n|^2[\overline{x_j^2}(t) + (n-1)\overline{x_j}(t)^2] \tag{63}$$

It now follows from Eq. (62) and the definitions of the quantities $\Delta x_j(t)$ and $\Delta X_j(t)$ that

$$\Delta X_j(t)^2 = (1 - |c_0|^2)|c_0|^2\overline{x_j}(t)^2 + \sum_{n=1}^{\infty}(n-1)!|c_n|^2\Delta x_j(t)^2 \tag{64}$$

Equation (64) is the key result of this section. It has been derived without using a specific choice of the single photon position operator. The consequence of this result follows. If the main contribution to the state $\Psi(t)$ in Eq. (58) is given by very large values of n then $|c_0|$ is very small and the first term in this expression can be neglected. Suppose that the main contribution is given by terms where n is of the order of $\bar{n}$. Then, as follows from Eqs. (62) and (64), $\Delta X_j(t)$ is of the order of $\Delta x_j(t)/\bar{n}^{1/2}$. This means that for coherent states where the main contribution is given by very large numbers of photons the effect of WPS is pronounced in a much less extent than for single photons.

It is interesting to note that the relation between $\Delta X_j(t)$ and $\Delta x_j(t)$ is analogous to (56) although those relations describe fully difference situations. In both of them relative uncertainties for a system of many particles are much less than for a single particle. Since the WPS effect for photons in laser beams is very small, divergence of the laser beam is only a consequence of the fact that different photons have different momenta.

10 Experimental consequences of WPS in standard theory

The problem of explaining the redshift phenomenon has a long history. Different competing approaches can be divided into two big sets which we call Theory A and

Theory B. In Theory A the redshift has been originally explained as a manifestation of the Doppler effect but in recent years the cosmological and gravitational redshifts have been added to the consideration. In this theory the interaction of photons with the interstellar medium is treated as practically not important. On the contrary, in Theory B, which is often called the tired-light theory, the interaction of photons with the interstellar medium is treated as the main reason for the redshift. At present the majority of physicists believe that Theory A explains the astronomical data better than Theory B. Even some physicists working on Theory B acknowledged that any sort of scattering of light would predict more blurring than is seen (see e.g. the article "Tired Light" in Wikipedia).

As follows from these remarks, in Theory A it is assumed that with a good accuracy we can treat photons as propagating in empty space. It is also reasonable to expect (see the discussion in the next section) that photons from distant stars practically do not interact with each other. Hence the effect of WPS can be considered for each photon independently and the results of the preceding sections make it possible to understand what experimental consequences of WPS are.

A question arises what can be said about characteristics of photons coming to Earth from distance objects. Since wave lengths of such photons are typically much less than all characteristic dimensions in question one might think that the radiation of stars can be described in the geometrical optics approximation. As discussed in Sec. 7, this approximation is similar to semiclassical approximation in quantum theory. This poses a question whether this radiation can be approximately treated as a collection of photons moving along classical trajectories. However, as noted below, not all photons in the radiation can be treated in such a way.

Consider, for example, the Lyman transition $2P \to 1S$ in the hydrogen atom, which plays an important role in the star radiation. We first consider the case when the atom is at rest. Then the mean energy of the photon is $E_0 = 10.2eV$, its wave length is $\lambda = 121.6nm$ and the lifetime is $\tau = 1.6 \cdot 10^{-9}s$. The phrase that the lifetime is τ is interpreted such that the uncertainty of the energy is $\hbar/\tau$. This implies that the uncertainty of the momentum magnitude is $\hbar/c\tau$ and b is of the order of $c\tau \approx 0.48m$. In this case the photon has a very narrow energy distribution since the mean value of the momentum $p_0 = E_0/c$ satisfies the condition $p_0 b \gg \hbar$. At the same time, since the orbital angular momentum of the photon is a small quantity, the function $f(\theta) = f(\mathbf{p}/p)$ in Eq. (43) has the same order of magnitude at all angles and the direction of the photon momentum cannot be semiclassical. If the atom is not at rest those conclusions remain valid because typically the speed of the atom is much less than c.

As pointed out in Sec. 6, it follows from Eq. (44) that even if the function $f(\theta)$ describes a broad angular distribution, the star will be visible only in the angular range of the order of R/L where R is the radius of the star and L is the distance to the star. The experimental verification of this prediction is problematic since the quantities R/L are very small and at present star radii cannot be measured directly.

Conclusions about them are made from the data on luminosity and temperature assuming that the major part of the radiation from stars comes not from transitions between atomic levels but from processes which can be approximately described as a blackbody radiation.

A theoretical model describing blackbody radiation (see e.g. Ref. [41]) is such that photons are treated as an ideal Bose gas weakly interacting with matter and such that typical photon energies are not close to energies of absorption lines for that matter (hence the energy spectrum of photons is almost continuous). It is also assumed that the photons are distributed over states with definite values of momenta. With these assumptions one can derive the famous Planck formula for the spectral distribution of the blackbody radiation (this formula is treated as marking the beginning of quantum theory). As explained in Ref. [41], when the photons leave the black body, their distribution in the phase space can be described by the Liouville theorem; in particular it implies that the photons leaving stars are moving along classical trajectories.

If we accept those arguments then the main part of photons emitted by stars can be described in the formalism considered in Sec. 6. In that case we cannot estimate the quantity b as above and it is not clear what criteria can be used for estimating the quantity a. The estimation $a \approx b \approx 0.48m$ seems to be extremely favorable since one might expect that the value of a is of atomic size, i.e. much less than $0.48m$. With this estimation for yellow light (with $\lambda = 580nm$) $N_\perp = a/\lambda \approx 8 \cdot 10^5$. So the value of $N_\perp$ is rather large and in view of Eq. (46) one might think that the effect of spreading is not important.

However, this is not the case because, as follows from Eq. (46), $t_* \approx 0.008s$. Even in the case of the Sun the distance to the Earth is approximately $t = 8$ light minutes, and this time is much greater than t_*. Then the value of $a(t)$ (which can be called the half-width of the wave packet) when the packet arrives to the Earth is $v_* t \approx 28km$. In this case standard geometrical interpretation does not apply. In addition, if we assume that the initial value of a is of the order of several wave lengths then the value of $N_\perp$ is much less and the width of the wave packet coming to the Earth even from the Sun is much greater. An analogous estimation shows that even in the favorable scenario the half-width of the wave packet coming to the Earth from Sirius will be approximately equal to $15 \cdot 10^6 km$ but in less favorable situations the half-width will be much greater. Hence we come to the conclusion that even in favorable scenarios the assumption that photons are moving along classical trajectories does not apply and a problem arises whether or not this situation is in agreement with experiment.

As already noted, even if the function $f(\theta)$ describes a broad angular distribution, a star will be visible only in the angular range of the order of R/L. Hence one might think that the absence of classical trajectories does not contradict observations. We now consider this problem in greater details. For simplicity we first assume that the photon wave function is spherically symmetric, i.e. $f(\mathbf{r}/r) = const.$

As follows from Eqs. (43) and (44), the wave function of the photon coming to Earth from a distant star is not negligible only within a narrow sphere with the radius ct and the width of the order of b. On its way to Earth the sphere passes *all* stars, planets and other objects the distance from which to the star is less than L (in particular, even those objects which are from the star in directions opposite to the direction to Earth). A problem arises how to explain the fact that the photon was detected on Earth and escaped detection by those stars, planets etc.

One might think that the event when the photon was detected on Earth is purely probabilistic. The fact that the photon was not detected by the objects on its way to Earth can be explained such that since the photon wave function has a huge size (of the order of light years or more) the probability of detection even by stars is extremely small and so it was only a favorable accident that the photon was detected on Earth.

If we accept this explanation then a new problem arises. If the photon passed stars, planets and other objects on its way to Earth then with approximately the same probability it can pass Earth and can be detected on the opposite side of the Earth. In that case we could see stars even through the Earth.

Moreover, consider the following experiment. Suppose that we first look at a star and then place a small screen between the eye and the star. Then the experiment shows that the star will not be visible. However, since the photon wave function passed many big objects without interacting with them then with approximately the same probability it can pass the screen. In that case we could see the star through the screen.

Another possibility is to try to avoid the above paradoxes by using an analogy with classical diffraction theory. Here the general problem statement requires solutions of Maxwell's equations with boundary conditions depending on the shape of the body and its material. In practice this problem is tackled assuming that deviation from geometrical optics is small (see e.g. Ref. [2]). When a classical wave encounters a macroscopic object it is also assumed that in optical phenomena the wave cannot penetrate inside the object. Then we get a picture that the wave far from the object does not change, right after the object the wave has a hole but when the length is much greater than the Rayleigh one the hole disappears and the wave function is practically the same as without diffration. Those results are natural from the point of view that classical waves consist of many almost pointlike photons.

Let us now consider a single-photon experiment on the Earth such the photon encounters a classical object and the transversal width of the photon coordinate wave function is much greater than the size of the object. One might think that the classical diffraction theory can be used even in this case. The justification involves arguments similar to those in Dirac's textbook [4] and in Sec. 7 that in some cases the classical and quantum theories involve the same formulas but they have different interpretations. Then the behavior of the photon wave function after passing the object will be similar to the behavior of the wave in classical diffraction

theory.

However, any change of the photon transverse wave function implies that the photon somehow interacted with the object. For example, when the photon is absorbed by an atom and then reemitted, the size of its wave function is defined by the atom; so the photon will not have a broad wave function anymore or in other words the photon wave function will collapse. Another example is that in the Compton scattering the photon is first absorbed by a charge particle, in the virtual intermediate state there is no photon and it is reemitted. So again the wave function will not have a large transverse size and the collapse will occur. In general, any Feynman diagram containing photons consists only of vertices with one photon. So in any interaction the photon will be first absorbed and hence the reemitted photon will not have the wave function with a large transverse size. In summary, since the phenomenon of wave function collapse exists only in quantum theory, in the single-photon experiment discussed above the behavior of the photon wave after passing the object cannot be similar to the behavior of the wave in classical diffraction theory.

Let us now return to the case when a photon with a wave function having a cosmic size encounters an object. In addition to the above arguments one can notice the following. The assumption that the photon wave function cannot penetrate inside the macroscopic object is reasonable in experiments on the Earth but in the given case it is highly problematic.

For example, our understanding of neutrino physics implies that neutrinos not only can pass the Earth practically without problems but even neutrinos created in the center of the Sun can easily reach the Earth. The major neutrino detectors are under the Earth surface and, for example, in the OPERA and ICARUS experiments neutrinos created at CERN reached Gran Sasso (Italy) after traveling 730km under the Earth surface. The explanation is that the probability of the neutrino interaction with the particle comprising the Sun and the Earth is very small.

At small energies the electromagnetic interaction is much stronger than the weak one but, as follows from the discussion in Secs. 4 and 6, the probability of interaction for photons having cosmic sizes contains the factor $|\tilde{f}/f|^2 = (d/D)^2$. Therefore it is reasonable to expect that for such photons the probability of interaction with particles comprising an object is even much less than in the above experiments with neutrinos. Hence the requirement that the photon wave function cannot penetrate a classical object is not justified. In addition, by analogy with the above consideration, after every interaction of the photon with particles comprising the object the photon wave function will collapse and will not have a cosmic size anymore. Such a photon can reach Earth only if its momentum considerably differs from the original one but this contradicts Theory A. So the assumption that the above paradoxes can be explained by analogy with classical diffraction theory is not justified.

If $f(\mathbf{r}/r) \neq const$ then, as follows from Eqs. (43) and (44), the radial part of the wave function is the same as in the spherically symmetric case and, as follows from the above discussion, the photon coordinate wave function still has a cosmic

size. Therefore on its way to Earth the photon wave function will also pass stars, planets and other objects (even if they are far from the line connecting the star and Earth) and the same inconsistencies arise.

In summary, since according to standard theory photons emitted by stars have coordinate wave functions with cosmic sizes, we treat the above arguments as a strong indication that the theory contradicts observational data.

In the infrared and radio astronomy wave lengths are much greater than in the optical region but typical values of a_{ph} are expected to be much greater. As a consequence, here standard quantum theory encounters the same problems that in the optical region.

In the case of gamma-ray bursts (GRBs) wave lengths are much less than in the optical region but this is outweighed by the facts that, according to the present understanding of the GRB phenomenon (see e.g. Ref. [42]), gamma quanta created in GRBs typically travel to Earth for billions of years and typical values of a_{ph} are expected to be much less than in the optical region. The location of sources of GBRs are determined with a good accuracy and the data can be explained only assuming that the gamma quanta are focused into narrow jets which are observable when Earth lies along the path of those jets. However, in view of the above discussion, the results on WPS predicted by standard quantum theory are incompatible with the data on GRBs because, as a consequence of WPS, the probability to detect photons from GRBs would be negligible.

Consider now WPS effects for radio wave photons. In radiolocation it is important that a beam from a directional antenna has a narrow angular distribution and a narrow distribution of wave lengths. This makes it possible to communicate even with very distant space probes. For this purpose a set of radio telescopes can be used but for simplicity we consider a model where signals from a space probe are received by one radio telescope having the diameter D of the dish.

The Cassini spacecraft can transmit to Earth at three radio wavelengths: 14cm, 4cm and 1cm [43]. A radio telescope on Earth can determine the position of Cassini with a good accuracy if it detects photons having momenta in the angular range of the order of D/L where L is the distance to Cassini. The main idea of using a system of radio telescopes is to increase the effective value of D. As a consequence of the fact that the radio signal sent from Cassini has an angular divergence which is much greater than D/L, only a small part of photons in the signal can be detected. We consider a case when Cassini was 7AU away from the Earth.

Consider first the problem on classical level. For the quantity $a = a_{cl}$ we take the value of $1m$ which is of the order of the radius of the Cassini antenna. If $\alpha = \lambda/(2\pi a)$ and $L(t)$ is the length of the classical path then, as follows from Eq. (45), $a_{cl}(t) \approx L(t)\alpha$. As a result, even for $\lambda = 1cm$ we have $a_{cl}(t) \approx 1.6 \cdot 10^6 km$. Hence one might expect that only a $[D/a_{cl}(t)]^2$ part of the photons can be detected.

Consider now the problem on quantum level. The condition $t \gg t_*$ is satisfied for both, the classical and quantum problems. Then, as follows from Eq.

(45), $a_{ph}(t) = a_{cl}(t)a_{cl}/a_{ph}$, i.e. the quantity $a_{ph}(t)$ is typically greater than $a_{cl}(t)$ and in Sec. 8 this effect is called the WPW paradox. The fact that only photons in the angular range D/L can be detected can be described by projecting the states $\chi = \chi(\mathbf{p}, t)$ (see Eqs. (34), and (35)) onto the states $\chi_1 = \mathcal{P}\chi$ where $\chi_1(\mathbf{p}, t) = \rho(\mathbf{p})\chi(\mathbf{p}, t)$ and the form factor $\rho(\mathbf{p})$ is significant only if $\mathbf{p}$ is in the needed angular range. We choose $\rho(\mathbf{p}) = exp(-\mathbf{p}_\perp^2 a_1^2/2\hbar^2)$ where a_1 is of the order of $\hbar L/(p_0 D)$. Since $a_1 \gg a_{ph}$, it follows from Eqs. (34), and (35) that $||\mathcal{P}\chi||^2 = (a_{ph}/a_1)^2$. Then, as follows from Eq. (45), $(a_{ph}/a_1)^2$ is of the order of $[D/a_{ph}(t)]^2$ as expected and this quantity is typically much less than $[D/a_{cl}(t)]^2$. Hence the WPW paradox would make communications with space probes much more difficult.

We now consider the following problem. The parameter γ in General Relativity (GR) is extracted from experiments on deflection of light from distant stars by the Sun and from the effect called Shapiro time delay. The meaning of the effect follows. An antenna on Earth sends a signal to Mercury, Venus or an interplanetary space probe and receives the reflected signal. If the path of the signal nearly grazes the Sun then the gravitational influence of the Sun deflects the path from a straight line. As a result, the path becomes longer by $S \approx 75km$ and the signals arrive with a delay $S/c \approx 250\mu s$. This effect is treated as the fourth test of GR.

The consideration of the both effects in GR is based on the assumption that the photon is a pointlike classical particle moving along classical trajectory. In the first case the photon wave function has a cosmic size. In the second case the available experimental data are treated such that the best test of γ has been performed in measuring the Shapiro delay when signals from the DSS-25 antenna [44] were sent to the Cassini spacecraft when it was 7AU away from the Earth. As noted above, in that case case, even in the most favorable scenario $a_{cl}(t) \approx 1.6 \cdot 10^6 km$ and the quantity $a_{ph}(t)$ is expected to be much greater. Therefore a problem arises whether the classical consideration in GR is compatible with the fact that the photon coordinate wave functions have very large sizes.

One might think that the compatibility is not a problem because when we detect a photon with the momentum pointing to the area near the Sun we know that this photon moved to us on the trajectory bending near the Sun. The results of Sec. 6 indeed show that even if the photon momentum wave function has a broad distribution, the photon detected by a measuring device can be detected only at the moment of time close to L/c and momentum of the detected photon will point to the star which emitted this photon. However, quantum formalism does not contain any information about the photon trajectory from the moment of emission to the moment of detection. One might guess that the required trajectory will give the main contribution in the Feynman path integral formulation but the proof of this guess is rather complicated.

In summary, by analogy with the consideration in Subsec. 1.3, one can conclude that quantum theory does not contain any information about trajectories. The notion of trajectories in quantum theory is a reasonable approximation only in

semiclassical approximation when a choice of the position operator has been made. However, in the case of packets with broad coordinate distributions the notion of trajectories does not have a physical meaning and one cannot avoid quantum consideration of the problem. In particular, the results of GR on the deflection of light and on the Shapiro delay are meaningful only if there is no considerable WPS in quantum theory. In addition, in view of the WPW paradox, the probability to detect reflected photons in the Shapiro delay experiments can be very small.

One might think that the WPS effect is important only if a particle travels a rather long distance. Hence one might expect that in experiments on the Earth this effect is negligible. Indeed, one might expect that in typical experiments on the Earth the time t is so small that $a(t)$ is much less than the size of any macroscopic source of light. However, a conclusion that the effect of WPS is negligible for any experiment on the Earth might be premature.

As an example, consider the case of protons in the LHC accelerator. According to Ref. [45], protons in the LHC ring injected at the energy $E = 450GeV$ should be accelerated to the energy $E = 7TeV$ within one minute during which the protons will turn around the $27km$ ring approximately 674729 times. Hence the length of the proton path is of the order of $18 \cdot 10^6 km$. The protons cannot be treated as free particles since they are accelerated by strong magnets. A problem of how the width of the proton wave function behaves in the presence of strong electromagnetic field is very complicated and the solution of the problem is not known yet. It is always assumed that the WPS effect for the protons can be neglected.

We first consider a model problem of the WPS for a free proton which moves for $t_1 = 1min$ with the energy in the range $[0.45, 7]\, TeV$. In nuclear physics the size of the proton is usually assumed to be a quantity of the order of $10^{-13}cm$. Therefore for estimations we take $a = 10^{-13}cm$. Then the quantity t_* defined after Eq. (39) is not greater than $10^{-19}s$, i.e. $t_* \ll t_1$. Hence, as follows from Eq. (39), the quantity $a(t_1)$ is of the order of $500km$ if $E = 7\, TeV$ and by a factor of $7/0.45 \approx 15.6$ greater if $E = 450\, GeV$.

This fully unrealistic result cannot be treated as a paradox since, as noted above, the protons in the LHC ring are not free. In the real situation the protons interact with many real and virtual photons emitted by magnets. For example, this might lead to the collaps of the proton wave function each time when the proton interacts with the real or virtual photon. This phenomenon is not well studied yet and so a problem of what standard theory predicts on the width of proton wave functions in the LHC ring is far from being obvious.

The last example follows. The astronomical objects called pulsars are treated such that they are neutron stars with radii much less than radii of ordinary stars. Therefore if mechanisms of pulsar electromagnetic radiation were the same as for ordinary stars then the pulsars would not be visible. The fact that pulsars are visible is explained as a consequence of the fact that they emit beams of light which can only be seen when the light is pointed in the direction of the observer with some

periods which are treated as periods of rotation of the neutron stars. In popular literature this is compared with the light of a lighthouse. However, by analogy with the case of a signal sent from Cassini, only a small part of photons in the beam can reach the Earth. At present the pulsars have been observed in different regions of the electromagnetic spectrum but the first pulsar called PSR B1919+21 was discovered in 1967 as a radio wave radiation with $\lambda \approx 3.7m$ [46]. This pulsar is treated as the neutron star with the radius $R = 0.97km$ and the distance from the pulsar to the Earth is 2283 light years. If for estimating $a_{cl}(t)$ we assume that $a_{cl} = R$ then we get $\alpha \approx 6 \cdot 10^{-4}$ and $a_{cl}(t) \approx 1.3ly \approx 12 \cdot 10^{12}km$. Such an extremely large value of spreading poses a problem whether even predictions of classical electrodynamics are compatible with the fact that pulsars are observable. However, in view of the WPW paradox, the value of $a_{ph}(t)$ will be even much greater and no observation of pulsars would be possible.

Our conclusion is that we have several fundamental paradoxes indicating that predictions of standard quantum theory for the WPS effect contradict experimental data.

11 Discussion: is it possible to avoid the WPS paradoxes in standard theory?

As shown in the preceding section, if one assumes that photons coming to Earth do not interact with the interstellar or interplanetary medium and with each other then a standard treatment of the WPS effect leads to several paradoxes. Hence a question arises whether this assumption is legitimate.

As shown in textbooks on quantum optics (see e.g. Refs. [38, 40]), quantum states describing the laser emission are strongly coherent and the approximation of independent photons is not legitimate. As shown in Sec. 9, the WPS effect in coherent states is pronounced in a much less extent than for individual photons. However, laser emission can be created only at very special conditions when energy levels are inverted, the emission is amplified in the laser cavity etc. At the same time, the main part of the radiation emitted by stars is understood such that it can be approximately described as the blackbody radiation and in addition a part of the radiation consists of photons emitted from different atomic energy levels. In that case the emission of photons is spontaneous rather than induced and one might think that the photons can be treated independently. Several authors (see e.g. Ref. [47] and references therein) discussed a possibility that at some conditions the inverted population and amplification of radiation in stellar atmospheres might occur and so a part of the radiation can be induced. This problem is now under investigation. Hence we adopt a standard assumption that a main part of the radiation from stars is spontaneous. In addition, there is no reason to think that radiation of GRBs, radio antennas, space probes or pulsars is laser like.

The next question is whether the interaction of photons in the above phenomena is important or not. As explained in standard textbooks on QED (see e.g. Ref. [33]), the photon-photon interaction can go only via intermediate creation of virtual electron-positron or quark-antiquark pairs. If ω is the photon frequency, m is the mass of the charged particle in the intermediate state and e is the electric charge of this particle then in the case when $\hbar\omega \ll mc^2$ the total cross section of the photon-photon interaction is [33]

$$\sigma = \frac{56}{5\pi m^2}\frac{139}{90^2}(\frac{e^2}{\hbar c})^4(\frac{\hbar\omega}{mc^2})^6 \tag{65}$$

For photons of visible light the quantities $\hbar\omega/(mc^2)$ and σ are very small and for radio waves they are even smaller by several orders of magnitude. At present the effect of the direct photon-photon interaction has not been detected, and experiments with strong laser fields were only able to determine the upper limit of the cross section [48].

The problem of WPS in the ultrarelativistic case has been discussed in a wide literature. As already noted, in Ref. [37] the effect of WPS has been discussed in the Fresnel approximation for a two-dimensional model and the author shows that in the direction perpendicular to the group velocity of the wave spreading is important. He considers WPS in the framework of classical electrodynamics. We believe that considering this effect from quantum point of view is even simpler since the photon wave function satisfies the relativistic Schrödinger equation which is linear in $\partial/\partial t$. As noted in Sec. 7, this function also satisfies the wave equation but it is simpler to consider an equation linear in $\partial/\partial t$ than that quadratic in $\partial/\partial t$. However, in classical theory there is no such an object as the photon wave function and hence one has to solve either a system of Maxwell equations or the wave equation. There is also a number of works where the authors consider WPS in view of propagation of classical waves in a medium such that dissipation is important (see e.g. Ref. [49]). In Ref. [50] the effect of WPS has been discussed in view of a possible existence of superluminal neutrinos. The authors consider only the dynamics of the wave packet in the longitudinal direction in the framework of the Dirac equation. They conclude that wave packets describing ultrarelativistic fermions do not experience WPS in this direction. However, the authors do not consider WPS in perpendicular directions.

In view of the above discussion, standard treatment of WPS leads to several fundamental paradoxes. To the best of our knowledge, those paradoxes have never been discussed in the literature. For resolving the paradoxes one could discuss several possibilities. One of them might be such that the interaction of light with the interstellar or interplanetary medium cannot be neglected. On quantum level a process of propagation of photons in the medium is rather complicated because several mechanisms of propagation should be taken into account. For example, a possible process is such that a photon can be absorbed by an atom and reemitted. This process makes it clear why the speed of light in the medium is less than c: because the atom

which absorbed the photon is in an excited state for some time before reemitting the photon. However, this process is also important from the following point of view: even if the coordinate photon wave function had a large width before absorption, as a consequence of the collapse of the wave function, the wave function of the emitted photon will have in general much smaller dimensions since after detection the width is defined only by parameters of the corresponding detector. If the photon encounters many atoms on its way, this process does not allow the photon wave function to spread out significantly. Analogous remarks can be made about other processes, for example about rescattering of photons on large groups of atoms, rescattering on elementary particles if they are present in the medium etc. However, such processes have been discussed in Theory B and, as noted in Sec. 10, they probably result in more blurring than is seen.

The interaction of photons with the interstellar or interplanetary medium might also be important in view of hypotheses that the density of the medium is much greater than usually believed. Among the most popular scenarios are dark energy, dark matter etc. As shown in our papers (see e.g. Refs. [11, 12, 51] and references therein), the phenomenon of the cosmological acceleration can be easily and naturally explained from first principles of quantum theory without involving dark energy, empty space-background and other artificial notions. However, the other scenarios seem to be more realistic and one might expect that they will be intensively investigated. A rather hypothetical possibility is that the propagation of photons in the medium has something in common with the induced emission when a photon induces emission of other photons in practically the same direction. In other words, the interstellar medium amplifies the emission as a laser. This possibility seems to be not realistic since it is not clear why the energy levels in the medium might be inverted.

We conclude that at present in standard theory there are no realistic scenarios which can explain the WPS paradoxes. In the remaining part of the paper we propose a solution of the problem proceeding from a consistent definition of the position operator.

12 Consistent construction of position operator

The above results give grounds to think that the reason of the paradoxes which follow from the behavior of the coordinate photon wave function in perpendicular directions is that standard definition of the position operator in those directions does not correspond to realistic measurements of coordinates. Before discussing a consistent construction, let us make the following remark. On elementary level students treat the mass m and the velocity $\mathbf{v}$ as primary quantities such that the momentum is $m\mathbf{v}$ and the kinetic energy is $m\mathbf{v}^2/2$. However, from the point of view of Special Relativity, the primary quantities are the momentum $\mathbf{p}$ and the total energy E and then the mass and velocity are defined as $m^2c^4 = E^2 - \mathbf{p}^2c^2$ and $\mathbf{v} = \mathbf{p}c^2/E$, re-

spectively. This example has the following analogy. In standard quantum theory the primary operators are the position and momentum operators and the orbital angular momentum operator is defined as their cross product. However, the operators $\mathbf{P}$ and $\mathbf{L}$ are consistently defined as representation operators of the Poincare algebra while the definition of the position operator is a problem. Hence a question arises whether the position operator can be defined in terms of $\mathbf{P}$ and $\mathbf{L}$.

One might seek the position operator such that on classical level the relation $\mathbf{r}\times\mathbf{p}=\mathbf{L}$ will take place. Note that on quantum level this relation is not necessary. Indeed, the very fact that some elementary particles have a half-integer spin shows that the total angular momentum for those particles does not have the orbital nature but on classical level the angular momentum can be always represented as a cross product of the radius-vector and standard momentum. However, if the values of $\mathbf{p}$ and $\mathbf{L}$ are known and $\mathbf{p}\neq 0$ then the requirement that $\mathbf{r}\times\mathbf{p}=\mathbf{L}$ does not define $\mathbf{r}$ uniquely. One can define parallel and perpendicular components of $\mathbf{r}$ as $\mathbf{r}=r_{||}\mathbf{p}/p+\mathbf{r}_\perp$ where $p=|\mathbf{p}|$. Then the relation $\mathbf{r}\times\mathbf{p}=\mathbf{L}$ defines uniquely only $\mathbf{r}_\perp$. Namely, as follows from this relation, $\mathbf{r}_\perp=(\mathbf{p}\times\mathbf{L})/p^2$. In view of the fact that on quantum level the operators $\mathbf{p}$ and $\mathbf{L}$ do not commute, on this level $\mathbf{r}_\perp$ should be replaced by a selfadjoint operator $\mathcal{R}_\perp=(\mathbf{p}\times\mathbf{L}-\mathbf{L}\times\mathbf{p})/(2p^2)$. Therefore

$$\mathcal{R}_{\perp j}=\frac{\hbar}{2p^2}e_{jkl}(p_kL_l+L_lp_k)=\frac{\hbar}{p^2}e_{jkl}p_kL_l-\frac{i\hbar}{p^2}p_j$$
$$=i\hbar\frac{\partial}{\partial p_j}-i\frac{\hbar}{p^2}p_jp_k\frac{\partial}{\partial p_k}-\frac{i\hbar}{p^2}p_j \tag{66}$$

where e_{jkl} is the absolutely antisymmetric tensor, $e_{123}=1$, a sum over repeated indices is assumed and we assume that if $\mathbf{L}$ is given by Eq. (22) then the orbital momentum is $\hbar\mathbf{L}$.

We define the operators $\mathbf{F}$ and $\mathbf{G}$ such that $\mathcal{R}_\perp=\hbar\mathbf{F}/p$ and $\mathbf{G}$ is the operator of multiplication by the unit vector $\mathbf{n}=\mathbf{p}/p$. A direct calculation shows that these operators satisfy the following relations:

$$[L_j,F_k]=ie_{jkl}F_l,\quad [L_j,G_k]=ie_{jkl}F_l,\quad \mathbf{G}^2=1,\quad \mathbf{F}^2=\mathbf{L}^2+1$$
$$[G_j,G_k]=0,\quad [F_j,F_k]=-ie_{jkl}L_l\quad e_{jkl}\{F_k,G_l\}=2L_j$$
$$\mathbf{LG}=\mathbf{GL}=\mathbf{LF}=\mathbf{FL}=0,\quad \mathbf{FG}=-\mathbf{GF}=i \tag{67}$$

The first two relations show that $\mathbf{F}$ and $\mathbf{G}$ are the vector operators as expected. The result for the anticommutator shows that on classical level $\mathbf{F}\times\mathbf{G}=\mathbf{L}$ and the last two relations show that on classical level the operators in the triplet $(\mathbf{F},\mathbf{G},\mathbf{L})$ are mutually orthogonal.

Note that if the momentum distribution is narrow and such that the mean value of the momentum is directed along the z axis then it does not mean that on the operator level the z component of the operator $\mathcal{R}_\perp$ should be zero. The matter is that the direction of the momentum does not have a definite value. One might expect that only the mean value of the operator $\mathcal{R}_\perp$ will be zero or very small.

In addition, an immediate consequence of the definition (66) follows: *Since the momentum and angular momentum operators commute with the Hamiltonian, the distribution of all the components of* $\mathbf{r}_\perp$ *does not depend on time. In particular, there is no WPS in directions defined by* $\mathcal{R}_\perp$. This is also clear from the fact that $\mathcal{R}_\perp = \hbar\mathbf{F}/p$ where the operator $\mathbf{F}$ acts only over angular variables and the Hamiltonian depends only on p. On classical level the conservation of $\mathcal{R}_\perp$ is obvious since it is defined by the conserving quantities $\mathbf{p}$ and $\mathbf{L}$. It is also obvious that since a free particle is moving along a straight line, a vector from the origin perpendicular to this line does not change with time.

The above definition of the perpendicular component of the position operator is well substantiated since on classical level the relation $\mathbf{r} \times \mathbf{p} = \mathbf{L}$ has been verified in numerous experiments. However, this relation does not make it possible to define the parallel component of the position operator and a problem arises what physical arguments should be used for that purpose.

A direct calculation shows that if $\partial/\partial\mathbf{p}$ is written in terms of p and angular variables then

$$i\hbar\frac{\partial}{\partial\mathbf{p}} = \mathbf{G}\mathcal{R}_{||} + \mathcal{R}_\perp \tag{68}$$

where the operator $\mathcal{R}_{||}$ acts only over the variable p:

$$\mathcal{R}_{||} = i\hbar(\frac{\partial}{\partial p} + \frac{1}{p}) \tag{69}$$

The correction $1/p$ is related to the fact that the operator $\mathcal{R}_{||}$ is Hermitian since in variables $(p, \mathbf{n})$ the scalar product is given by

$$(\chi_2, \chi_1) = \int \chi_2(p, \mathbf{n})^* \chi_1(p, \mathbf{n}) p^2 dp do \tag{70}$$

where do is the element of the solid angle.

While the components of standard position operator commute with each other, the operators $\mathcal{R}_{||}$ and $\mathcal{R}_\perp$ satisfy the following commutation relations:

$$[\mathcal{R}_{||}, \mathcal{R}_\perp] = -\frac{i\hbar}{p}\mathcal{R}_\perp, \quad [\mathcal{R}_{\perp j}, \mathcal{R}_{\perp k}] = -\frac{i\hbar^2}{p^2}e_{jkl}L_l \tag{71}$$

An immediate consequence of these relations follows: *Since the operator* $\mathcal{R}_{||}$ *and different components of* $\mathcal{R}_\perp$ *do not commute with each other, the corresponding quantities cannot be simultaneously measured and hence there is no wave function* $\psi(r_{||}, \mathbf{r}_\perp)$ *in coordinate representation.*

In standard theory $-\hbar^2(\partial/\partial\mathbf{p})^2$ is the operator of the quantity $\mathbf{r}^2$. As follows from Eq. (67), the two terms in Eq. (68) are not strictly orthogonal and on the operator level $-\hbar^2(\partial/\partial\mathbf{p})^2 \neq \mathcal{R}_{||}^2 + \mathcal{R}_\perp^2$. A direct calculation using Eqs. (67) and (68) gives

$$\frac{\partial^2}{\partial\mathbf{p}^2} = \frac{\partial^2}{\partial p^2} + \frac{2}{p}\frac{\partial}{\partial p} - \frac{\mathbf{L}^2}{p^2}, \quad -\hbar^2\frac{\partial^2}{\partial\mathbf{p}^2} = \mathcal{R}_{||}^2 + \mathcal{R}_\perp^2 - \frac{\hbar^2}{p^2} \tag{72}$$

in agreement with the expression for the Laplacian in spherical coordinates. In semiclassical approximation, $(\hbar^2/p^2) \ll \mathcal{R}_\perp^2$ since the eigenvalues of $\mathbf{L}^2$ are $l(l+1)$, in semiclassical states $l \gg 1$ and, as follows from Eq. (67), $\mathcal{R}_\perp^2 = [\hbar^2(l^2+l+1)/p^2]$.

As follows from Eq. (71), $[\mathcal{R}_{||}, p] = -i\hbar$, i.e. in the longitudinal direction the commutation relation between the coordinate and momentum is the same as in standard theory. One can also calculate the commutators between the different components of $\mathcal{R}_\perp$ and $\mathbf{p}$. Those commutators are not given by such simple expressions as in standard theory but it is easy to see that all of them are of the order of $\hbar$ as it should be.

Equation (68) can be treated as an implementation of the relation $\mathbf{r} = r_{||}\mathbf{p}/|\mathbf{p}| + \mathbf{r}_\perp$ on quantum level. As argued in Secs. 1 and 2, standard position operator $i\hbar\partial/\partial p_j$ in the direction j is not consistently defined if p_j is not sufficiently large. One might think however that since the operator $\mathcal{R}_{||}$ contains $i\hbar\partial/\partial p$, it is defined consistently if the magnitude of the momentum is sufficiently large.

In summary, we propose to define the position operator not by the set $(i\hbar\partial/\partial p_x, i\hbar\partial/\partial p_y, i\hbar\partial/\partial p_z)$ but by the operators $\mathcal{R}_{||}$ and $\mathcal{R}_\perp$. Those operators are defined from different considerations. As noted above, the definition of $\mathcal{R}_\perp$ is based on solid physical facts while the definition of $\mathcal{R}_{||}$ is expected to be more consistent than the definition of standard position operator. However, this does not guarantee that the operator $\mathcal{R}_{||}$ is consistently defined in all situations. As argued in Ref. [52], in a quantum theory over a Galois field an analogous definition is not consistent *for macroscopic bodies* (even if p is large) since in that case semiclassical approximation is not valid. In the remaining part of the paper we assume that for elementary particles the above definition of $\mathcal{R}_{||}$ is consistent in situations when semiclassical approximation applies.

One might pose the following question. What is the reason to work with the parallel and perpendicular components of the position operator separately if, according to Eq. (68), their sum is the standard position operator? The explanation follows.

In quantum theory every physical quantity corresponds to a selfadjoint operator but the theory does not define explicitly how a quantity corresponding to a specific operator should be measured. There is no guaranty that for each selfadjoint operator there exists a physical quantity which can be measured in real experiments.

Suppose that there are three physical quantities corresponding to the selfadjoint operators A, B and C such that $A + B = C$. Then in each state the mean values of the operators are related as $\bar{A} + \bar{B} = \bar{C}$ but in situations when the operators A and B do not commute with each other there is no direct relation between the distributions of the physical quantities corresponding to the operators A, B and C. For example, in situations when the physical quantities corresponding to the operators A and B are semiclassical and can be measured with a good accuracy, there is no guaranty that the physical quantity corresponding to the operator C can be measured in real measurements. As an example, the physical meaning of the quantity corre-

sponding to the operator $L_x + L_y$ is problematic. Another example is the situation with WPS in directions perpendicular to the particle momentum. Indeed, as noted above, the physical quantity corresponding to the operator $\mathcal{R}_\perp$ does not experience WPS and, as shown in Sec. 14, in the case of ultrarelativistic particles there is no WPS in the parallel direction as well. However, standard position operator is a sum of noncommuting operators corresponding to well defined physical quantities and, as a consequence, there are situations when standard position operator defines a quantity which cannot be measured in real experiments.

13 New position operator and semiclassical states

As noted in Sec. 2, in standard theory states are treated as semiclassical in greatest possible extent if $\Delta r_j \Delta p_j = \hbar/2$ for each j and such states are called coherent. The existence of coherent states in standard theory is a consequence of commutation relations $[p_j, r_k] = -i\hbar\delta_{jk}$. Since in our approach there are no such relations, a problem arises how to construct states in which all physical quantities p, $r_{||}$, $\mathbf{n}$ and $\mathbf{r}_\perp$ are semiclassical.

One can calculate the mean values and uncertainties of the operator $\mathcal{R}_{||}$ and all the components of the operator $\mathcal{R}_\perp$ in the state defined by Eq. (34). The calculation is not simple since it involves three-dimensional integrals with Gaussian functions divided by p^2. The result is that these operators are semiclassical in the state (34) if $p_0 \gg \hbar/b$, $p_0 \gg \hbar/a$ and r_{0z} has the same order of magnitude as r_{0x} and r_{0y}.

However, a more natural approach follows. Since $\mathcal{R}_\perp = \hbar\mathbf{F}/p$, the operator $\mathbf{F}$ acts only over the angular variable $\mathbf{n}$ and $\mathcal{R}_{||}$ acts only over the variable p, it is convenient to work in the representation where the Hilbert space is the space of functions $\chi(p, l, \mu)$ such that the scalar product is

$$(\chi_2, \chi_1) = \sum_{l\mu} \int_0^\infty \chi_2(p, l, \mu)^* \chi_1(p, l, \mu) dp \tag{73}$$

and l and μ are the orbital and magnetic quantum numbers, respectively, i.e.

$$\mathbf{L}^2 \chi(p, l, \mu) = l(l+1)\chi(p, l, \mu), \quad L_z \chi(p, l, \mu) = \mu\chi(p, l, \mu) \tag{74}$$

The operator $\mathbf{L}$ in this space does not act over the variable p and the action of the remaining components is given by

$$L_+\chi(l, \mu) = [(l+\mu)(l+1-\mu)]^{1/2}\chi(l, \mu-1), \quad L_-\chi(l, \mu) = [(l-\mu)(l+1+\mu)]^{1/2}\chi(l, \mu+1) \tag{75}$$

where the $\pm$ components of vectors are defined such that $L_x = L_+ + L_-$, $L_y = -i(L_+ - L_-)$.

A direct calculation shows that, as a consequence of Eq. (66)

$$F_+\chi(l,\mu) = -\frac{i}{2}[\frac{(l+\mu)(l+\mu-1)}{(2l-1)(2l+1)}]^{1/2} l\chi(l-1,\mu-1)$$
$$-\frac{i}{2}[\frac{(l+2-\mu)(l+1-\mu)}{(2l+1)(2l+3)}]^{1/2}(l+1)\chi(l+1,\mu-1)$$
$$F_-\chi(l,\mu) = \frac{i}{2}[\frac{(l-\mu)(l-\mu-1)}{(2l-1)(2l+1)}]^{1/2} l\chi(l-1,\mu+1)$$
$$+\frac{i}{2}[\frac{(l+2+\mu)(l+1+\mu)}{(2l+1)(2l+3)}]^{1/2}(l+1)\chi(l+1,\mu+1)$$
$$F_z\chi(l,\mu) = i[\frac{(l-\mu)(l+\mu)}{(2l-1)(2l+1)}]^{1/2} l\chi(l-1,\mu)$$
$$-i[\frac{(l+1-\mu)(l+1+\mu)}{(2l+1)(2l+3)}]^{1/2}(l+1)\chi(l+1,\mu) \qquad (76)$$

The operator $\mathbf{G}$ acts on such states as follows

$$G_+\chi(l,\mu) = \frac{1}{2}[\frac{(l+\mu)(l+\mu-1)}{(2l-1)(2l+1)}]^{1/2}\chi(l-1,\mu-1)$$
$$-\frac{1}{2}[\frac{(l+2-\mu)(l+1-\mu)}{(2l+1)(2l+3)}]^{1/2}\chi(l+1,\mu-1)$$
$$G_-\chi(l,\mu) = -\frac{1}{2}[\frac{(l-\mu)(l-\mu-1)}{(2l-1)(2l+1)}]^{1/2}\chi(l-1,\mu+1)$$
$$+\frac{1}{2}[\frac{(l+2+\mu)(l+1+\mu)}{(2l+1)(2l+3)}]^{1/2}\chi(l+1,\mu+1)$$
$$G_z\chi(l,\mu) = -[\frac{(l-\mu)(l+\mu)}{(2l-1)(2l+1)}]^{1/2}\chi(l-1,\mu)$$
$$-[\frac{(l+1-\mu)(l+1+\mu)}{(2l+1)(2l+3)}]^{1/2}\chi(l+1,\mu) \qquad (77)$$

and now the operator $\mathcal{R}_{||}$ has a familiar form $\mathcal{R}_{||} = i\hbar\partial/\partial p$.

Therefore by analogy with Secs. 2 and 3 one can construct states which are coherent with respect to $(r_{||}, p)$, i.e. such that $\Delta r_{||}\Delta p = \hbar/2$. Indeed (see Eq. (6)), the wave function

$$\chi(p) = \frac{b^{1/2}}{\pi^{1/4}\hbar^{1/2}} exp[-\frac{(p-p_0)^2 b^2}{2\hbar^2} - \frac{i}{\hbar}(p-p_0)r_0] \qquad (78)$$

describes a state where the mean values of p and $r_{||}$ are p_0 and r_0, respectively and their uncertainties are $\hbar/(b\sqrt{2})$ and $b/\sqrt{2}$, respectively. Strictly speaking, the analogy between the given case and that discussed in Secs. 2 and 3 is not full since in the given case the quantity p can be in the range $[0,\infty)$, not in $(-\infty,\infty)$ as momentum

variables used in those sections. However, if $p_0 b/\hbar \gg 1$ then the formal expression for $\chi(p)$ at $p < 0$ is extremely small and so the normalization integral for $\chi(p)$ can be formally taken from $-\infty$ to ∞.

In such an approximation one can define wave functions $\psi(r)$ in the $r_{||}$ representation. By analogy with the consideration in Secs. 2 and 3 we define

$$\psi(r) = \int exp(\frac{i}{\hbar} pr)\chi(p)\frac{dp}{(2\pi\hbar)^{1/2}} \tag{79}$$

where the integral is formally taken from $-\infty$ to ∞. Then

$$\psi(r) = \frac{1}{\pi^{1/4} b^{1/2}} exp[-\frac{(r-r_0)^2}{2b^2} + \frac{i}{\hbar} p_0 r] \tag{80}$$

Note that here the quantities r and r_0 have the meaning of coordinates in the direction parallel to the particle momentum, i.e. they can be positive or negative.

Consider now states where the quantities $\mathbf{F}$ and $\mathbf{G}$ are semiclassical. One might expect that in semiclassical states the quantities l and μ are very large. In this approximation, as follows from Eqs. (76) and (77), the action of the operators $\mathbf{F}$ and $\mathbf{G}$ can be written as

$$\begin{aligned}
F_+\chi(l,\mu) &= -\frac{i}{4}(l+\mu)\chi(l-1,\mu-1) - \frac{i}{4}(l-\mu)\chi(l+1,\mu-1) \\
F_-\chi(l,\mu) &= \frac{i}{4}(l-\mu)\chi(l-1,\mu+1) + \frac{i}{4}(l+\mu)\chi(l+1,\mu+1) \\
F_z\chi(l,\mu) &= -\frac{i}{2l}(l^2-\mu^2)^{1/2}[\chi(l+1,\mu) + \chi(l-1,\mu)] \\
G_+\chi(l,\mu) &= \frac{l+\mu}{4l}\chi(l-1,\mu-1) - \frac{l-\mu}{4l}\chi(l+1,\mu-1) \\
G_-\chi(l,\mu) &= -\frac{l-\mu}{4l}\chi(l-1,\mu+1) + \frac{l+\mu}{4l}\chi(l+1,\mu+1) \\
G_z\chi(l,\mu) &= -\frac{1}{2l}(l^2-\mu^2)^{1/2}[\chi(l+1,\mu) + \chi(l-1,\mu)]
\end{aligned} \tag{81}$$

In view of the remark in Sec. 2 about semiclassical vector quantities, consider a state $\chi(l,\mu)$ such that $\chi(l,\mu) \neq 0$ only if $l \in [l_1, l_2]$, $\mu \in [\mu_1, \mu_2]$ where $l_1, \mu_1 > 0$, $\delta_1 = l_2 + 1 - l_1$, $\delta_2 = \mu_2 + 1 - \mu_1$, $\delta_1 \ll l_1$, $\delta_2 \ll \mu_1$ $\mu_2 < l_1$ and $\mu_1 \gg (l_1 - \mu_1)$. This is the state where the quantity μ is close to its maximum value l. As follows from Eqs. (74) and (75), in this state the quantity L_z is much greater than L_x and L_y and, as follows from Eq. (81), the quantities F_z and G_z are small. So on classical level this state describes a motion of the particle in the xy plane. The quantity L_z in this state is obviously semiclassical since $\chi(l,\mu)$ is the eigenvector of the operator L_z with the eigenvalue μ. As follows from Eq. (81), the action of the operators (F_+, F_-, G_+, G_-) on this state can be described by the following approximate formulas:

$$\begin{aligned}
F_+\chi(l,\mu) &= -\frac{il_0}{2}\chi(l-1,\mu-1), \quad F_-\chi(l,\mu) = \frac{il_0}{2}\chi(l+1,\mu+1) \\
G_+\chi(l,\mu) &= \frac{1}{2}\chi(l-1,\mu-1), \quad G_-\chi(l,\mu) = \frac{1}{2}\chi(l+1,\mu+1)
\end{aligned} \tag{82}$$

where l_0 is a value from the interval $[l_1, l_2]$.

Consider a simple model when $\chi(l,\mu) = exp[i(l\alpha - \mu\beta)]/(\delta_1\delta_2)^{1/2}$, $l \in [l_1, l_2]$ and $\mu \in [\mu_1, \mu_2]$. Then a simple direct calculation using Eq. (82) gives

$$\bar{G}_x = cos\gamma, \quad \bar{G}_y = -sin\gamma \quad \bar{F}_x = -l_0 sin\gamma \quad \bar{F}_y = -l_0 cos\gamma$$
$$\Delta G_x = \Delta G_y = (\frac{1}{\delta_1} + \frac{1}{\delta_2})^{1/2}, \quad \Delta F_x = \Delta F_y = l_0(\frac{1}{\delta_1} + \frac{1}{\delta_2})^{1/2} \tag{83}$$

where $\gamma = \alpha - \beta$. Hence the vector quantities $\mathbf{F}$ and $\mathbf{G}$ are semiclassical since either $|cos\gamma|$ or $|sin\gamma|$ or both are much greater than $(\delta_1 + \delta_2)/(\delta_1\delta_2)$.

14 New position operator and wave packet spreading

If the space of states is implemented according to the scalar product (73) then the dependence of the wave function on t is

$$\chi(p,k,\mu,t) = exp[-\frac{i}{\hbar}(m^2c^2 + p^2)^{1/2}ct]\chi(p,k,\mu,t=0) \tag{84}$$

As noted in Secs. 3 and 6, there is no WPS in momentum space and this is natural in view of momentum conservation. Then, as already noted, the distribution of the quantity $\mathbf{r}_\perp$ does not depend on time and this is natural from the considerations described in Sec. 12.

At the same time, the dependence of the $r_{||}$ distribution on time can be calculated in full analogy with Sec. 3. Indeed, consider, for example a function $\chi(p,l,\mu,t=0)$ having the form

$$\chi(p,l,\mu,t=0) = \chi(p,t=0)\chi(l,\mu) \tag{85}$$

Then, as follows from Eqs. (79) and (84),

$$\psi(r,t) = \int exp[-\frac{i}{\hbar}(m^2c^2 + p^2)^{1/2}ct + \frac{i}{\hbar}pr]\chi(p,t=0)\frac{dp}{(2\pi\hbar)^{1/2}} \tag{86}$$

Suppose that the function $\chi(p,t=0)$ is given by Eq. (78). Then in full analogy with the calculations in Sec. 3 we get that in the nonrelativistic case the $r_{||}$ distribution is defined by the wave function

$$\psi(r,t) = \frac{1}{\pi^{1/4}b^{1/2}}(1 + \frac{i\hbar t}{mb^2})^{-1/2}exp[-\frac{(r - r_0 - v_0t)^2}{2b^2(1 + \frac{\hbar^2t^2}{m^2b^4})}(1 - \frac{i\hbar t}{mb^2}) + \frac{i}{\hbar}p_0r - \frac{ip_0^2t}{2m\hbar}] \tag{87}$$

where $v_0 = p_0/m$ is the classical speed of the particle in the direction of the particle momentum. Hence the WPS effect in this direction is similar to that given by Eq. (10) in standard theory.

In the opposite case when the particle is ultrarelativistic, Eq. (86) can be written as

$$\psi(r,t)=\int exp[\frac{i}{\hbar}p(r-ct)]\chi(p,t=0)\frac{dp}{(2\pi\hbar)^{1/2}} \tag{88}$$

Hence, as follows from Eq. (80):

$$\psi(r,t)=\frac{1}{\pi^{1/4}b^{1/2}}exp[-\frac{(r-r_0-ct)^2}{2b^2}+\frac{i}{\hbar}p_0(r-ct)] \tag{89}$$

In particular, for an ultrarelativistic particle there is no WPS in the direction of particle momentum and this is in agreement with the results of Sec. 6.

We conclude that in our approach an ultrarelativistic particle (e.g. the photon) experiences WPS neither in the direction of its momentum nor in perpendicular directions, i.e. the WPS effect for an ultrarelativistic particle is absent at all.

Let us note that the absence of WPS in perpendicular directions is simply a consequence of the fact that a consistently defined operator $\mathcal{R}_\perp$ commutes with the Hamiltonian. In quantum theory a physical quantity is called conserved if its operator commutes with the Hamiltonian. Therefore $\mathbf{r}_\perp$ is a conserved physical quantity. In contrast to classical theory, this does not mean that $\mathbf{r}_\perp$ should necessarily have only one value but means that the $\mathbf{r}_\perp$ distribution does not depend on time. On the other hand, the longitudinal coordinate is not a conserved physical quantity since a particle is moving along the direction of its momentum. However, in a special case of ultrarelativistic particle the absence of WPS is simply a consequence of the fact that the wave function given by Eq. (88) depends on r and t only via a combination of $r-ct$.

15 Discussion and conclusion

In the present paper we consider a problem of constructing position operator in quantum theory. As noted in Sec. 1, this operator is needed in situations where semiclassical approximation works with a high accuracy.

A standard choice of the position operator in momentum space is $i\hbar\partial/\partial\mathbf{p}$. A motivation for this choice is discussed in Sec. 2. We note that this choice is not consistent since $i\hbar\partial/\partial p_j$ cannot be a physical position operator in directions where the momentum is small. Physicists did not pay attention to the inconsistency probably for the following reason: as explained in textbooks, transition from quantum to classical theory can be performed such that if the coordinate wave function contains a rapidly oscillating exponent $exp(iS/\hbar)$ where S is the classical action then in the formal limit $\hbar\to 0$ the Schrödinger equation becomes the Hamilton-Jacobi equation.

However, an inevitable consequence of standard quantum theory is the effect of wave packet spreading (WPS). This fact has not been considered as a drawback of the theory. Probably the reasons are that for macroscopic bodies this effect

is extremely small while in experiments on the Earth with atoms and elementary particles spreading probably does not have enough time to manifest itself. However, for photons traveling to the Earth from distant objects this effect is considerable, and it seems that this fact has been overlooked by physicists.

As shown in Sec. 10, if the WPS effect for photons traveling to Earth from distant objects is as given by standard theory then we have several fundamental paradoxes. The most striking of them is that standard theory contradicts our experience on observations of stars.

We propose a new definition of the position operator which we treat as consistent for the following reasons. Our position operator is defined by two components - in the direction along the momentum and in perpendicular directions. The first part has a familiar form $i\hbar\partial/\partial p$ and is treated as the operator of the longitudinal coordinate if the magnitude of p is rather large. At the same condition the position operator in the perpendicular directions is defined as a quantum generalization of the relation $\mathbf{r}_\perp \times \mathbf{p} = \mathbf{L}$. So in contrast to the standard definition of the position operator, the new operator is expected to be physical only if the *magnitude* of the momentum is rather large.

As a consequence of our construction, WPS in directions perpendicular to the particle momentum is absent regardless of whether the particle is nonrelativistic or relativistic. Moreover, for an ultrarelativistic particle the effect of WPS is absent at all.

Different components of the new position operator commute with each other only in the formal limit $\hbar \to 0$. As a consequence, there is no wave function in coordinate representation. In particular, there is no quantum analog of the coordinate Coulomb potential (see the discussion in Sec. 1). A possibility that coordinates can be noncommutative has been first discussed by Snyder [53] and it is implemented in several modern theories. In those theories the measure of noncommutativity is defined by a parameter l called the fundamental length (the role of which can be played e.g. by the Planck length or the Schwarzschild radius). In the formal limit $l \to 0$ the coordinates become standard ones related to momenta by a Fourier transform. As shown in the present paper, this is unacceptable in view of the WPS paradoxes. One of ideas of those theories is that with a nonzero l it might be possible to resolve difficulties of standard theory where $l = 0$ (see e.g. Ref. [54] and references therein). At the same time, in our approach there can be no notion of fundamental length since commutativity of coordinates takes place only in the formal limit $\hbar \to 0$.

The absence of the coordinate wave function is not unusual. For example, there is no wave function in the angular momentum representation because different components of the angular momentum operator commute only in the formal limit $\hbar \to 0$. However, on classical level all the commutators can be neglected and different components of the position vector and angular momentum can be treated independently.

In our approach the uncertainties of each component of the photon mo-

mentum and each component of the photon coordinate do not change with time. If in some problem those quantities can be treated as small then the photon can be treated as a pointlike particle moving along classical trajectory. So in our approach the coordinate photon wave function never has a cosmic size and there can be no paradoxes discussed in Sec. 10.

In view of the absence of the coordinate wave function, such quantum problems as diffraction and interference of single photon should be considered only in momentum representation. In particular, if boundary conditions are needed they should be formulated in that representation. When a problem is solved and characteristic spatial dimensions in the problem are greater than uncertainties of all the coordinates one can discuss spatial features of the process.

As noted in Sec. 8, in standard quantum theory photons comprising a classical electromagnetic wave packet cannot be (approximately) treated as pointlike particles in view of the WPW paradox. However, in our approach, in view of the absence of WPS for massless particles, the usual intuition is restored and photons comprising a divergent classical wave packet can be (approximately) treated as pointlike particles. Moreover, the phenomenon of divergence of a classical wave packet can now be naturally explained simply as a consequence of the fact that different photons in the packet have different momenta.

Our consideration also poses a problem whether the results of classical electrodynamics can be applied for wave packets moving for a long period of time. For example, as noted in Sec. 10, even classical theory predicts that when a wave packet emitted in a gamma-ray burst or by a pulsar reaches the Earth, the width of the packet is extremely large (while the value predicted by standard quantum theory is even much greater) and this poses a problem whether such a packet can be detected. A natural explanation of why classical theory does not apply in this case follows. As noted in Sec. 5, classical electromagnetic fields should be understood as a result of taking mean characteristics for many photons. Then the fields will be (approximately) continuous if the density of the photons is high. However, for a divergent beam of photons their density decreases with time. Hence after a long period of time the mean characteristics of the photons in the beam cannot represent continuous fields. In other words, in this situation the set of photons cannot be effectively described by classical electromagnetic fields.

The new position operator might also have applications in the problem of neutrino oscillations. As pointed out by several authors (see e.g. Refs. [55, 56, 31]) this problem should be considered from the point of view that for describing observable neutrinos one should treat them as quantum superpositions of wave packets with different neutrino flavors. Then the choice of the position operator might play an important role.

The position operator proposed in the present paper is also important in view of the following. There exists a wide literature discussing the Einstein-Podolsky-Rosen paradox, locality in quantum theory, quantum entanglement, Bell's theorem

and similar problems (see e.g. Ref. [28] and references therein). Consider, for example, the following problem in standard theory. Let at $t = 0$ particles 1 and 2 be localized inside finite volumes V_1 and V_2, respectively, such that the volumes are very far from each other. Hence the particles don't interact with each other. However, as follows from Eq. (26), their wave functions will overlap at any $t > 0$ and hence the interaction can be transmitted even with an infinite speed. This is often characterized as quantum nonlocality, entanglement and/or action at a distance.

Consider now this problem in the framework of our approach. Since in this approach there is no wave function in coordinate representation, there is no notion of a particle localized inside a finite volume. Hence a problem arises whether on quantum level the notions of locality or nonlocality have a physical meaning. In addition, spreading does not take place in directions perpendicular to the particle momenta and for ultrarelativistic particles spreading does not occur at all. Hence, at least in the case of ultrarelativistic particles, this kind of interaction does not occur in agreement with classical intuition that no interaction can be transmitted with the speed greater than c. This example poses a problem whether the position operator should be modified not only in directions perpendicular to particle momenta but also in longitudinal directions such that the effect of WPS should be excluded at all.

A problem discussed in a wide literature is whether evolution of a quantum system can be always described by the time dependent Schrödinger equation. We will discuss this problem in view of the statements (see e.g. Refs. [57, 58]) that t cannot be treated as a fundamental physical quantity. The reason is that all fundamental physical laws do not require time and the quantity t is obsolete on fundamental level. A hypothesis that time is an independently flowing fundamental continuous quantity has been first proposed by Newton. However, a problem arises whether this hypothesis is compatible with the principle that the definition of a physical quantity is a description of how this quantity can be measured.

Consider first the problem of time in classical mechanics. A standard treatment of this theory is that its goal is to solve equations of motion and get classical trajectories where coordinates and momenta are functions of t. In Hamiltonian mechanics the action can be written as $S = S_0 - \int H dt$ where S_0 does not depend on t and is called the abbreviated action. Then, as explained in textbooks, the dependence of the coordinates and momenta on t can be obtained from a variational principle with the action S. Suppose now that one wishes to consider a problem which is usually treated as less general: to find not the dependence of the coordinates and momenta on t but only possible forms of trajectories in the phase space without mentioning time at all. If the energy is a conserved physical quantity then, as described in textbooks, this problem can be solved by using the Maupertuis principle involving only S_0.

However, the latter problem *is not* less general than the former one. For illustration we first consider the one-body case. Here the phase space can be described by the quantities $(r_{||}, \mathbf{r}_\perp, \mathbf{G}, p)$ discussed in Sec. 12. Suppose that by using the Maupertuis principle one has solved the problem with some initial values

of coordinates and momenta. One can choose $r_{||}$ such that it is zero at the initial point and increases along the trajectory. Then $r_{||} = s$ where s is the length along the spacial trajectory and a natural parametrization for the trajectory in the phase space is such that $(\mathbf{r}_\perp, \mathbf{G}, p)$ are functions of $r_{||} = s$. This is an additional indication that our choice of the position operator is more natural than standard one. At this stage the problem does not contain t yet. We can note that in standard case $ds/dt = |\mathbf{v}(s)| = |\mathbf{p}(s)|/E(s)$. Hence in the problem under consideration one can *define* t such that $dt = E(s)ds/|\mathbf{p}(s)|$ and hence the value of t at any point of the trajectory can be obtained by integration. In the case of many bodies one can define t by using the spatial trajectory of any body and the result does not depend on the choice of the body. Hence the general problem of classical mechanics can be formulated without mentioning t while if one wishes to work with t then, by definition, this value can flow only in positive direction.

Consider now the problem of time in quantum theory. In the case of one strongly quantum system (i.e. the system which cannot be described in classical theory) a problem arises whether there exists a quantum analog of the Maupertuis principle and whether time can be defined by using this analog. This is a difficult unsolved problem. A possible approach for solving this problem has been proposed in Ref. [57]. However, one can consider a situation when a quantum system under consideration is a small subsystem of a big system where the other subsystem - the environment, is strongly classical. Then one can define t for the environment as described above. The author of Ref. [58] considers a scenario when the system as a whole is described by the stationary Schrödinger equation $H\Psi = E\Psi$ but the small quantum subsystem is described by the time dependent Schrödinger equation where t is defined for the environment as $t = \partial S_0/\partial E$.

One might think that this scenario gives a natural solution of the problem of time in quantum theory. Indeed, in this scenario it is clear why a quantum system is described by the Schrödinger equation depending on the classical parameter t which is not an operator: because t is the physical quantity characterizing not the quantum system but the environment. This scenario seems also natural because it is in the spirit of the Copenhagen interpretation of quantum theory: the evolution of a quantum system can be characterized only in terms of measurements which in the Copenhagen interpretation are treated as interactions with classical objects. However, this scenario encounters the following problems. As noted in Ref. [58], it does not solve the problem of quantum jumps. For example, as noted in Sec. 5, the 21cm transition in the hydrogen atom cannot be described by the evolution operator depending on the continuous parameter t. Another problem is that the environment can be a classical object only in some approximation and hence t can be only an approximately continuous parameter. Finally, the Copenhagen interpretation cannot be universal in all situations. For example, if the Big Bang hypothesis is correct then at the early stage of the Universe there were no classical objects but nevertheless physics should somehow describe evolution even in this situation.

Our result for ultrarelativistic particles can be treated as ideal: quantum theory reproduces the motion along a classical trajectory without any spreading. However, this is only a special case of one free elementary particle. If quantum theory is treated as more general than the classical one then it should describe not only elementary particles and atoms but even the motion of macroscopic bodies in the Solar System and in the Universe. We believe that the assumption that the evolution of macroscopic bodies can be described by the Schrödinger equation is unphysical. For example, if the motion of the Earth is described by the evolution operator $exp[-iH(t_2 - t_1)/\hbar]$ where H is the Hamiltonian of the Earth then the quantity $H(t_2 - t_1)/\hbar$ becomes of the order of unity when $t_2 - t_1$ is a quantity of the order of $10^{-68}s$ if the Hamiltonian is written in nonrelativistic form and $10^{-76}s$ if it is written in relativistic form. Such time intervals seem to be unphysical and so in the given case the approximation when t is a continuous parameter seems to be unphysical too. In modern theories (e.g. in the Big Bang hypothesis) it is often stated that the Planck time $t_P \approx 10^{-43}s$ is a physical minimum time interval. However, at present there are no experiments confirming that time intervals of the order of $10^{-43}s$ can be measured.

The time dependent Schrödinger equation has not been experimentally verified and the major theoretical arguments in favor of this equation are as follows: a) the Hamiltonian is the generator of the time translation in the Minkowski space; b) this equation becomes the Hamilton-Jacobi one in the formal limit $\hbar \to 0$. However, as noted in Sec. 1, quantum theory should not be based on the space-time background and the conclusion b) is made without taking into account the WPS effect. Hence the problem of describing evolution in quantum theory remains open.

Let us now return to the problem of the position operator. As noted above, in directions perpendicular to the particle momentum the choice of the position operator is based only on the requirement that semiclassical approximation should reproduce the standard relation $\mathbf{r}_\perp \times \mathbf{p} = \mathbf{L}$. This requirement seems to be beyond any doubts since *on classical level* this relation is confirmed in numerous experiments. At the same time, the choice $i\hbar\partial/\partial p$ of the coordinate operator in the longitudinal direction is analogous to that in standard theory and hence one might expect that this operator is physical if the magnitude of p is rather large (see, however, the above remark about the entanglement caused by WPS).

It will be shown in a separate publication that the construction of the position operator described in this paper for the case of Poincare invariant theory can be generalized to the case of de Sitter (dS) invariant theory. In this case the interpretation of the position operator is even more important than in Poincare invariant theory. The reason is that even the free two-body mass operator in the dS theory depends not only on the relative two-body momentum but also on the distance between the particles.

As argued in Ref. [52], in dS theory over a Galois field the assumption that the dS analog of the operator $i\hbar\partial/\partial p$ is the operator of the longitudinal coordinate

is not valid *for macroscopic bodies* (even if p is large) since in that case semiclassical approximation is not valid. We have proposed a modification of the position operator such that quantum theory reproduces for the two-body mass operator the mean value compatible with the Newton law of gravity. Then a problem arises how quantum theory can reproduce classical evolution for macroscopic bodies.

The above examples show that at macroscopic level a consistent definition of the transition from quantum to classical theory is the fundamental open problem.

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