Waves in a Dispersive Exponential Half-Space[a)]

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(Dated: 21 January 2011)

Maxwell equations for electromagnetic waves propagating in dispersive media are studied as they are, without commonplace substituting a scalar function for electromagnetic field. A method of variables separation for the original system of equation is proposed. It is shown that in case of planar symmetry variables separate in systems of Cartesian and cylindric coordinates and Maxwell equations reduce to one-dimensional Schrödinger equation. Complete solutions are obtained for waves in medium with electric permittivity and magnetic permeability given as $\epsilon = e^{-\kappa z}$, $\mu = c^{-2}e^{-\lambda z}$.
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PACS numbers: 41.20.Jb, 42.25.Bs
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I. INTRODUCTION

The method of variables separation plays important role in physics providing the main opportunity to obtain complete solutions of equations of mathematical physics[1]. Separation of Maxwell equations is much more complicated procedure than that of scalar equations found in standard texts on mathematical physics. So far this procedure was used only in vacuum[2–5]. In this work we apply the method of separation of Maxwell equations in a dispersive medium, proposed in our work[6] and obtain complete set of pure states of electromagnetic waves in a matter possessing planar symmetry.

Maxwell equations in a uniform medium reduce to scalar d'Alembert equation in Cartesian coordinates and can be solved by the same methods as any scalar equation[7–9]. In other coordinate systems and even in Cartesian coordinates, but in non-uniform media with electric permittivity ϵ and magnetic permeability μ being arbitrary functions on the space, it is not so, because these functions enter the equations like a kind of potential on which the wave refracts an reflects as in analogous phenomena studied in quantum mechanics[10]. Since these functions play more complicated role than potential in quantum mechanics, details of these phenomena cannot be studied properly in the same approach. Therefore, description of waves propagation in non-uniform media, requires solutions of Maxwell equations obtained by solving the entire system as it stands, without replacing vector electromagnetic field by a scalar one. Such a substitution does not help even in the simplest case when the medium possesses planar symmetry so that ϵ and μ are specified as single-variable functions in Cartesian coordinates.

At the same time, methods for solving Maxwell equations as they stand, have been elaborated in general relativity[3] where electromagnetic waves propagate under similar conditions. These methods may well be applied to the problem of electromagnetic waves propagation in media specified by their dielectric factors ϵ and μ as functions on the space. Obtaining complete solutions would change contents of classical electrodynamics and make it similar to all the rest linear theories, for example, quantum mechanics, in which solution of any problem has the form of orthogonal expansion over functional space endowed with basis of particular solutions called "pure states" or "orthogonal modes". This can be done, at least, in the simplest case of medium possessing planar symmetry specified by certain functions $\epsilon(z)$ and $\mu(z)$.

II. COMPLEX POTENTIAL

In this work we consider Maxwell equations represented in terms of exterior differential forms as follows:

$$\mathrm{d}\Delta = 0 \quad \mathrm{d}B = 0 \qquad (1)$$
$$\mathrm{d}H = \frac{\partial \Delta}{\partial t} \quad \mathrm{d}E = -\frac{\partial B}{\partial t},$$

with constraints

$$\Delta = \epsilon^* E, \quad B = \mu^* H, \qquad (2)$$

where E and H are 1-forms, Δ and B are 2-forms, asterisk in the equations above is related to the 1-forms and stands for the 3-dimensional asterisk conjugation because in our approach, time is not included as the fourth dimension. It is convenient to represent the quartet of two strengths and two inductions as components of single 2-form on the space-time

$$\Psi = \mathrm{d}t \wedge (E + \imath H) + (B - \imath\Delta). \qquad (3)$$

In this representation the four source-free Maxwell equations (1) take the form of single equation

$$\mathrm{d}\Psi = 0. \qquad (4)$$

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Indeed, if the field satisfies the source-free Maxwell equations, the exterior derivative of this 2-form is zero:

$$\mathrm{d}\Psi = \mathrm{d}t\wedge\left(\frac{\partial B}{\partial t} - \mathrm{d}E\right) - \imath\,\mathrm{d}t\wedge\left(\frac{\partial \Delta}{\partial t} + \mathrm{d}H\right) + \mathrm{d}(B - \imath\Delta),$$

where exterior derivatives in the right-hand side are taken only in spatial directions. In vacuum this form is known to be self-dual (in sense of four-dimensional space-time), whereas in a non-uniform dielectric this condition violates with electric permittivity and magnetic permeability. This representation and the form of equations will be used below when constructing analytical solutions of the Maxwell equations in an exponential half-space.

In general, dielectrics have ϵ and μ which are not given by one and the same function in the space, so that the asterisk operation changes the form of Ψ completely and as the result, its conjugate $^*\Psi$ does not coincide with Ψ in form. Therefore in general, it is convenient to assume that the 1-form of potential is complex-valued, so that the four-dimensional asterisk operation is not used at all. The dielectric factors appear when applying the three-dimensional asterisk operation to its real and imaginary parts separately for electric and magnetic components. In this approach Maxwell equations are equivalent to the equations

$$\Delta = \epsilon^* E, \quad B = \mu^* H \tag{5}$$

where expressions for strengths and inductions are taken from exterior derivative of the complex-valued 1-form of the potential:

$$\begin{aligned} A = \phi\,\mathrm{d}t + f_i\ \mathrm{d}x^i,\ \phi = \phi_1 + \imath\phi_2,\ f_i = f_{1i} + \imath f_{2i} \quad (6)\\ E = \mathrm{d}\phi_1 + \frac{\partial f_{1i}}{\partial t}\,\mathrm{d}x^i,\ B = \mathrm{d}f_{1i}\wedge\mathrm{d}x^i,\\ H = \mathrm{d}\phi_2 + \frac{\partial f_{2i}}{\partial t}\,\mathrm{d}x^i,\ \Delta = \mathrm{d}f_{2i}\wedge\mathrm{d}x^i. \end{aligned}$$

Combination of this representation and the equations (5) is equivalent to the Maxwell equations, therefore instead of Maxwell equations we solve this combination. Note that in this case complex potentials produce only real-valued strengths and inductions, so, it essentially differs from complex vector potentials commonly used when solving Maxwell equations in Cartesian coordinates. Now, taking the constraints (2) into account provides complete plane wave solutions of Maxwell equations in the medium. The goal of this work is to show that Maxwell equations in a medium with ε and μ given as functions of single Cartesian coordinate, reduce to ordinary differential equation and obtain complete solution in particular case exponential half-space ($z \geq 0$) with these functions specified explicitly as $\epsilon(z) = e^{-\kappa z}$ and $\mu(z) = c^{-2}e^{-\lambda z}$. Though we try to consider more or less realistic situation, in which these functions are, at least, bounded, our solutions have mainly illustrative character to demonstrate possibility of describing the phenomenon of electromagnetic wave propagation in terms of pure states and orthogonal expansions in the half-space $z \geq 0$.

III. COMPLEX POTENTIALS AND THE FIELD IN CARTESIAN COORDINATES

The electromagnetic potential can be taken in the form

$$A = f\ \mathrm{d}x + g\ \mathrm{d}y + h\ \mathrm{d}z \tag{7}$$

whose coefficients f, g and h depend on four coordinates. Here we do not include the time component $\phi\ \mathrm{d}t$ because number of equations of the system (5) is six, consequently, we need only six unknowns. The 1-form A will be represented as $A = A_1 + \imath A_2$ where A_1 and A_2 are real-valued 1-forms specified by their components f_1, g_1, h_1 and f_2, g_2, h_2 correspondingly. To see what components of strengths and inductions the field has, we find out exterior derivative of the 1-form (7) (temporarily using the fourth-dimensional exterior differentiation):

$$\begin{aligned} \mathrm{d}A = \frac{\partial f}{\partial t}\,\mathrm{d}t\wedge\mathrm{d}x + \frac{\partial g}{\partial t}\,\mathrm{d}t\wedge\mathrm{d}y +\\ + \frac{\partial h}{\partial t}\,\mathrm{d}t\wedge\mathrm{d}z + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\mathrm{d}x\wedge\mathrm{d}y +\\ + \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\mathrm{d}y\wedge\mathrm{d}z + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\mathrm{d}z\wedge\mathrm{d}x. \end{aligned}$$

Now, as the 1-form A is complex, we have two strengths and two inductions

$$\begin{aligned} E = \frac{\partial f_1}{\partial t}\,\mathrm{d}x + \frac{\partial g_1}{\partial t}\,\mathrm{d}y + \frac{\partial h_1}{\partial t}\,\mathrm{d}z \qquad (8)\\ B = \left(\frac{\partial g_1}{\partial x} - \frac{\partial f_1}{\partial y}\right)\mathrm{d}x\wedge\mathrm{d}y +\\ + \left(\frac{\partial h_1}{\partial y} - \frac{\partial g_1}{\partial z}\right)\mathrm{d}y\wedge\mathrm{d}z + \left(\frac{\partial f_1}{\partial z} - \frac{\partial h_1}{\partial x}\right)\mathrm{d}z\wedge\mathrm{d}x\\ H = \frac{\partial f_2}{\partial t}\,\mathrm{d}x + \frac{\partial g_2}{\partial t}\,\mathrm{d}y + \frac{\partial h_2}{\partial t}\,\mathrm{d}z\\ \Delta = \left(\frac{\partial f_2}{\partial y} - \frac{\partial g_2}{\partial x}\right)\mathrm{d}x\wedge\mathrm{d}y +\\ + \left(\frac{\partial g_2}{\partial z} - \frac{\partial h_2}{\partial y}\right)\mathrm{d}y\wedge\mathrm{d}z + \left(\frac{\partial h_2}{\partial x} - \frac{\partial f_2}{\partial z}\right)\mathrm{d}z\wedge\mathrm{d}x. \end{aligned}$$

Substituting the components into the equations (5) yields explicit form of the equations. In this point we should make the following remark. On one hand, electric and magnetic strengths have different dimensions, on the other hand, they are real and imaginary parts of the time derivative of the co-vector potential. Therefore, we either have to admit that its real and imaginary part have different dimensions or to annul the difference putting speed of light equal to unity. Hereafter we do so, thus, put $c = 1$ and remove this coefficient from the Maxwell equations.

IV. INCLUSION OF CONSTRAINTS

Substituting the strengths and inductions (8) into the equations (2) yields the following system:

$$\varepsilon\frac{\partial f_1}{\partial t}=\frac{\partial g_2}{\partial z}-\frac{\partial h_2}{\partial y},\quad \mu\frac{\partial f_2}{\partial t}=\frac{\partial h_1}{\partial y}-\frac{\partial g_1}{\partial z}$$
$$\varepsilon\frac{\partial g_1}{\partial t}=\frac{\partial h_2}{\partial x}-\frac{\partial f_2}{\partial z},\quad \mu\frac{\partial g_2}{\partial t}=\frac{\partial f_1}{\partial z}-\frac{\partial h_1}{\partial x}$$
$$\varepsilon\frac{\partial h_1}{\partial t}=\frac{\partial f_2}{\partial y}-\frac{\partial g_2}{\partial x},\quad \mu\frac{\partial h_2}{\partial t}=\frac{\partial g_1}{\partial x}-\frac{\partial f_1}{\partial y}.$$

In this work we consider the simplest case of dispersive medium in which ε and μ depend on one of Cartesian coordinates, say, z, therefore, it suffices to obtain solutions which do not depend on another Cartesian coordinate, say, y. Then we can ignore derivatives on this variable that simplifies the system. Indeed, the system decays into two independent subsystems, one for the functions f_1, g_2 and h_1 and another for three other unknowns, which describe two different polarizations. As such, it suffices to restrict the scope with one of them. So, hereafter only functions f_1, g_2 and h_1 are non-zero, they do not depend on y and we omit the subscripts. The system becomes

$$\epsilon\frac{\partial f}{\partial t}=\frac{\partial g}{\partial z},\quad \epsilon\frac{\partial h}{\partial t}=-\frac{\partial g}{\partial x}$$
$$\mu\frac{\partial g}{\partial t}=\frac{\partial f}{\partial z}-\frac{\partial h}{\partial x}.$$

To solve this system assume that the strengths draw a wave which propagates in the x-direction as a plane wave. It is easy to find out what components of the potential are proportional to sine and what to cosine of arguments like $\omega t-px$. The result is

$$\begin{aligned} f&=F(z)\cos(\omega t-px),\\ g&=G(z)\sin(\omega t-px),\\ h&=H(z)\sin(\omega t-px). \end{aligned} \tag{9}$$

The system simplifies and yields ordinary differential equations

$$H=\frac{Gp}{\omega\epsilon},\quad G'=-\omega\epsilon F,\quad G=\frac{\omega\epsilon F'}{\omega^2\epsilon\mu-p^2}. \tag{10}$$

Here two first equations can be used as definitions of the functions F and H and the third one reduces to an ordinary differential equation. Now we introduce again the subscripts "1" and "2" which denote two different polarizations, thus, the subscript "1" corresponds to polarization specified by f_1, g_2 and h_1 and the subscript "2" – to f_2, g_1 and h_2:

$$\epsilon\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{1}{\epsilon}\frac{\mathrm{d}G_1}{\mathrm{d}z}\right)+\omega^2(\epsilon\mu-\sin^2\alpha)G_1=0 \tag{11}$$
$$\mu\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{1}{\mu}\frac{\mathrm{d}G_2}{\mathrm{d}z}\right)+\omega^2(\epsilon\mu-\sin^2\alpha)G_2=0,$$

where we have introduced incident angle α:

$$p=\omega\sin\alpha. \tag{12}$$

In case of interface of a uniform medium and exponential half-space α is exactly the incident angle indeed, therefore we called it so.

V. COMPLEX POTENTIALS AND THE FIELD IN COORDINATES OF ROUND CYLINDER

Propagation of electromagnetic wave in exponential half-space can also be described in coordinates of round cylinder $\{t,z,\rho,\varphi\}$, but in this case we include the time component of the electromagnetic potential:

$$A=\phi\ \mathrm{d}t+f\ \mathrm{d}z+g\ \mathrm{d}\rho+h\rho\ \mathrm{d}\varphi. \tag{13}$$

Its exterior derivative has the form

$$\mathrm{d}A=\left(\frac{\partial f}{\partial t}-\frac{\partial\phi}{\partial z}\right)\mathrm{d}t\wedge\mathrm{d}z+\left(\frac{\partial g}{\partial t}-\frac{\partial\phi}{\partial\rho}\right)\mathrm{d}t\wedge\mathrm{d}\rho+$$
$$+\left(\frac{\partial h}{\partial t}-\frac{1}{\rho}\frac{\partial\phi}{\partial\varphi}\right)\rho\ \mathrm{d}t\wedge\mathrm{d}\varphi+\left(\frac{\partial g}{\partial z}-\frac{\partial f}{\partial\rho}\right)\mathrm{d}z\wedge\mathrm{d}\rho+$$
$$+\left[\left(\frac{\partial}{\partial\rho}+\frac{1}{\rho}\right)h-\frac{1}{\rho}\frac{\partial g}{\partial\varphi}\right]\rho\ \mathrm{d}\rho\wedge\mathrm{d}\varphi+$$
$$\left(\frac{1}{\rho}\frac{\partial f}{\partial\varphi}-\frac{\partial h}{\partial z}\right)\rho\,\mathrm{d}\varphi\wedge\mathrm{d}z.$$

So, as ϕ_1, f_1, g_1 and h_1 specify the real-valued part of the potential and ϕ_2, f_2, g_2 h_2 do its imaginary part, then due to the equation (3) strengths and inductions of the field are

$$E = \left(\frac{\partial f_1}{\partial t} - \frac{\partial \phi_1}{\partial z}\right) \mathrm{d}z + \left(\frac{\partial g_1}{\partial t} - \frac{\partial \phi_1}{\partial \rho}\right) \mathrm{d}\rho + \left(\frac{\partial h_1}{\partial t} - \frac{1}{\rho}\frac{\partial \phi_1}{\partial \varphi}\right) \rho\, \mathrm{d}\varphi \tag{14}$$
$$B = \left(\frac{\partial g_1}{\partial z} - \frac{\partial f_1}{\partial \rho}\right) \mathrm{d}z \wedge \mathrm{d}\rho + \left[\left(\frac{\partial}{\partial \rho} + \frac{1}{\rho}\right) h_1 - \frac{1}{\rho}\frac{\partial g_1}{\partial \varphi}\right] \rho\, \mathrm{d}\rho \wedge \mathrm{d}\varphi + \left(\frac{1}{\rho}\frac{\partial f_1}{\partial \varphi} - \frac{\partial h_1}{\partial z}\right) \rho\, \mathrm{d}\varphi \wedge \mathrm{d}z$$
$$H = \left(\frac{\partial f_2}{\partial t} - \frac{\partial \phi_2}{\partial z}\right) \mathrm{d}z + \left(\frac{\partial g_2}{\partial t} - \frac{\partial \phi_2}{\partial \rho}\right) \mathrm{d}\rho + \left(\frac{\partial h_2}{\partial t} - \frac{1}{\rho}\frac{\partial \phi_2}{\partial \varphi}\right) \rho\, \mathrm{d}\varphi$$
$$\Delta = \left(\frac{\partial f_2}{\partial \rho} - \frac{\partial g_2}{\partial z}\right) \mathrm{d}z \wedge \mathrm{d}\rho + \left[\frac{1}{\rho}\frac{\partial g_2}{\partial \varphi} - \left(\frac{\partial}{\partial \rho} + \frac{1}{\rho}\right) h_2\right] \rho\, \mathrm{d}\rho \wedge \mathrm{d}\varphi + \left(\frac{\partial h_2}{\partial z} - \frac{1}{\rho}\frac{\partial f_2}{\partial \varphi}\right) \rho\, \mathrm{d}\varphi \wedge \mathrm{d}z.$$

Now the expressions (14) will be substituted into the equations (5) that yields their explicit form:

$$\begin{aligned}
\epsilon\left(\frac{\partial f_1}{\partial t} - \frac{\partial \phi_1}{\partial z}\right) &= \frac{1}{\rho}\frac{\partial g_2}{\partial \varphi} - \left(\frac{\partial}{\partial \rho} + \frac{1}{\rho}\right) h_2, & \mu\left(\frac{\partial f_2}{\partial t} - \frac{\partial \phi_2}{\partial z}\right) &= \left(\frac{\partial}{\partial \rho} + \frac{1}{\rho}\right) h_1 - \frac{1}{\rho}\frac{\partial g_1}{\partial \varphi}, \\
\epsilon\left(\frac{\partial g_1}{\partial t} - \frac{\partial \phi_1}{\partial \rho}\right) &= \frac{\partial h_2}{\partial z} - \frac{1}{\rho}\frac{\partial f_2}{\partial \varphi}, & \mu\left(\frac{\partial g_2}{\partial t} - \frac{\partial \phi_2}{\partial \rho}\right) &= \frac{1}{\rho}\frac{\partial f_1}{\partial \varphi} - \frac{\partial h_1}{\partial z}, \\
\epsilon\left(\frac{\partial h_1}{\partial t} - \frac{1}{\rho}\frac{\partial \phi_1}{\partial \varphi}\right) &= \frac{\partial f_2}{\partial \rho} - \frac{\partial g_2}{\partial z}, & \mu\left(\frac{\partial h_2}{\partial t} - \frac{1}{\rho}\frac{\partial \phi_2}{\partial \varphi}\right) &= \frac{\partial g_1}{\partial z} - \frac{\partial f_1}{\partial \rho}.
\end{aligned} \tag{15}$$

To separate variables in these equations we assume that the functions to be found have certain form of dependence on the coordinates t and φ:

$$\begin{aligned}
\phi_1 &= \phi_1(z,\rho)\sin(\omega t - m\varphi), & \phi_2 &= \phi_2(z,\rho)\sin(\omega t - m\varphi), \\
f_1 &= f_1(z,\rho)\cos(\omega t - m\varphi), & f_2 &= f_2(z,\rho)\cos(\omega t - m\varphi), \\
g_1 &= g_1(z,\rho)\cos(\omega t - m\varphi), & g_2 &= g_2(z,\rho)\cos(\omega t - m\varphi), \\
h_1 &= g_1(z,\rho)\sin(\omega t - m\varphi), & h_2 &= -g_2(z,\rho)\sin(\omega t - m\varphi).
\end{aligned} \tag{16}$$

Now, we replace the functions h_1 and h_2 of two variables with the functions g_1 and g_2 correspondingly and have six unknowns in six equations. The equations for the functions of two variables have the form

$$\begin{aligned}
-\epsilon\left(\omega f_1 + \frac{\partial \phi_1}{\partial z}\right) &= \left(\frac{\partial}{\partial \rho} + \frac{1+m}{\rho}\right) g_2, & -\mu\left(\omega f_2 + \frac{\partial \phi_2}{\partial z}\right) &= \left(\frac{\partial}{\partial \rho} + \frac{1-m}{\rho}\right) g_1, \\
-\epsilon\left(\omega g_1 + \frac{\partial \phi_1}{\partial \rho}\right) &= -\frac{\partial g_2}{\partial z} - \frac{m f_2}{\rho}, & -\mu\left(\omega g_2 - \frac{m \phi_2}{\rho}\right) &= \frac{\partial g_1}{\partial z} - \frac{\partial f_1}{\partial \rho}, \\
\epsilon\left(\omega g_1 + \frac{m \phi_1}{\rho}\right) &= \frac{\partial f_2}{\partial \rho} - \frac{\partial g_2}{\partial z}, & -\mu\left(\omega g_2 + \frac{\partial \phi_2}{\partial \rho}\right) &= \frac{m f_1}{\rho} - \frac{\partial g_1}{\partial z}.
\end{aligned}$$

Taking sums and differences of the equations on the second and third lines transforms the system as follows:

$$\epsilon\left(\omega f_1+\frac{\partial\phi_1}{\partial z}\right)=-\left(\frac{\partial}{\partial\rho}+\frac{1+m}{\rho}\right)g_2,\qquad \mu\left(\omega f_2+\frac{\partial\phi_2}{\partial z}\right)=-\left(\frac{\partial}{\partial\rho}+\frac{1-m}{\rho}\right)g_1,$$
$$\epsilon\left[2\omega g_1+\left(\frac{\partial}{\partial\rho}+\frac{m}{\rho}\right)\phi_1\right]=\left(\frac{\partial}{\partial\rho}+\frac{m}{\rho}\right)f_2,\qquad \mu\left[2\omega g_2+\left(\frac{\partial}{\partial\rho}-\frac{m}{\rho}\right)\phi_2\right]=\left(\frac{\partial}{\partial\rho}-\frac{m}{\rho}\right)f_1,$$
$$\epsilon\left(\frac{\partial}{\partial\rho}-\frac{m}{\rho}\right)\phi_1=2\frac{\partial g_2}{\partial z}-\left(\frac{\partial}{\partial\rho}-\frac{m}{\rho}\right)f_2,\qquad \mu\left(\frac{\partial}{\partial\rho}+\frac{m}{\rho}\right)\phi_2=2\frac{\partial g_1}{\partial z}-\left(\frac{\partial}{\partial\rho}+\frac{m}{\rho}\right)f_1.$$

It is seen that if ϵ and μ depend only on z, the functions to be found depend on the coordinate ρ as Bessel functions:

$$\begin{aligned}\phi_1&=\Phi_1(z)J_m(l\rho),\\ f_1&=F_1(z)J_m(l\rho),\\ g_1&=G_1(z)J_{m-1}(l\rho)\\ \phi_2&=\Phi_2(z)J_m(l\rho),\\ f_2&=F_2(z)J_m(l\rho),\\ g_2&=G_2(z)J_{m+1}(l\rho)\end{aligned}\tag{17}$$

where we have used the identities

$$\begin{aligned}\left(\frac{\mathrm{d}}{\mathrm{d}z}+\frac{\nu}{z}\right)J_\nu(z)&=J_{\nu-1}(z)\\ \left(\frac{\mathrm{d}}{\mathrm{d}z}-\frac{\nu}{z}\right)J_\nu(z)&=-J_{\nu+1}(z)\end{aligned}\tag{18}$$

(the second solutions $Y_l(l\rho)$ is also to be used. In general, modified Bessel functions $I_m(l\rho)$ and $K_m(l\rho)$ also can be used under some boundary conditions with account that their recurrent relations have different form

$$\begin{aligned}\left(\frac{\mathrm{d}}{\mathrm{d}z}+\frac{\nu}{z}\right)I_\nu(z)&=I_{\nu-1}(z)\\ \left(\frac{\mathrm{d}}{\mathrm{d}z}-\frac{\nu}{z}\right)I_\nu(z)&=I_{\nu+1}(z).\end{aligned}\tag{19}$$

Now it remains to solve ordinary differential equations:

$$\begin{aligned}\epsilon(\omega F_1+\Phi_1')&=-lG_2,\quad \mu(\omega F_2+\Phi_2')=lG_1\\ \epsilon(2\omega G_1+l\Phi_1)&=lF_2,\ \ \mu(2\omega G_2-l\Phi_2)=-lF_1\\ -l\epsilon\Phi_1&=2G_2'+lF_2,\ \ l\mu\Phi_2=2G_1'-lf_1.\end{aligned}$$

The equations of the second line simply express the functions f_1 and f_2 via others, therefore it remains to solve the system of four equations for four unknowns. Substituting these two functions leads to the following system:

$$\begin{aligned}\epsilon\left[\Phi_1'-\omega\mu\left(\tfrac{2\omega}{l}G_2-\Phi_2\right)\right]&=-lG_2,\\ \mu\left[\Phi_2'+\omega\epsilon\left(\tfrac{2\omega}{l}G_1+\Phi_1\right)\right]&=lG_1,\\ -l\epsilon\Phi_1&=2G_2'+\epsilon(\omega G_1+l\Phi_1),\\ l\mu\Phi_2&=2G_1'+\mu(2\omega G_2-l\Phi_2)\end{aligned}$$

and equations of the second line can be used as definitions of the functions ϕ_1 and ϕ_2, so that we obtain a system of two equations for two unknowns. In fact, the two last equations are independent because each of them contains only one function to be found, which coincides with the equations (11):

$$\begin{aligned}\epsilon\left(\frac{G_2'}{\epsilon}\right)'+(\epsilon\mu\omega^2-l^2)G_2&=0,\\ \mu\left(\frac{G_1'}{\mu}\right)'+(\epsilon\mu\omega^2-l^2)G_1&=0,\end{aligned}\tag{20}$$

and after it is solved the functions Φ_1 and Φ_2 can be found from the equalities

$$\Phi_1=-\frac{1}{l}\left(\frac{G_2'}{\epsilon}+\omega G_1\right),\quad \Phi_2=\frac{1}{l}\left(\frac{G_1'}{\mu}+\omega G_2\right),\tag{21}$$

and F_1 and F_2 – from the equalities

$$lF_1=G_1'-\omega\mu G_2,\quad lF_2=-(G_2'-\omega\epsilon G_1).\tag{22}$$

These functions are to be substituted to the equations (17) and the result – to the equations (9) for Cartesian coordinates and (16) for cylindric coordinates. The results provide amplitudes of electromagnetic wave in the medium as they appear in the equations (13).

The two equations (20) contain different cylindric functions, modified Bessel functions which grow unlimitedly either near the axis of under big values of ρ. Therefore usually the solution contains only one form of cylindric function which is $J_m(l\rho)$. When constructing wave fields in domains where the coordinate ρ definitely does not reach zero and infinity other cylindric functions can also be used, but this depends on explicit formulation of the problem. Certain problems of this sort will be considered in one of further sections and usage of all these functions will be demonstrated in action.

VI. EXPONENTIAL HALF-SPACE

In this section we consider propagation of wave in a medium with $\varepsilon(z)$ and $\mu(z)$ specified as certain functions of z that turns the equation (11) into an ordinary differential equation with known solutions. First, we specify $\varepsilon(z)$ and $\mu(z)$ by $\varepsilon(z)=e^{-\kappa z}$ and $\mu=c^{-2}e^{-\lambda z}$. The equations (11) take the form

$$\begin{aligned}G_1''+\kappa G'+\omega^2\left(e^{-(\kappa+\lambda)z}-\sin^2\alpha\right)G_1&=0\\ G_2''+\lambda G_2'+\omega^2\left(e^{-(\kappa+\lambda)z}-\sin^2\alpha\right)G_2&=0\end{aligned}$$

for both polarizations. Here, we introduce the parameter α by analogy with the incident angle (23):

$$l = \omega \sin \alpha. \tag{23}$$

The meaning of this parameter can be established in the case of uniform matter, when $\kappa = \lambda = 0$ the form of the functions G_1 and G_2 is evident, the parameter α reveals as the angle of focusing or divergence of the waves.

It is convenient to introduce a unit of length equal to $(\kappa + \lambda)^{-1}$ so that the equations in such a system of units simplify as follows:

$$\begin{aligned} G_1'' + \kappa G' + \omega^2(e^{-z} - \sin^2 \alpha)G_1 = 0 \\ G_2'' + \lambda G_2' + \omega^2(e^{-z} - \sin^2 \alpha)G_2 = 0. \end{aligned} \tag{24}$$

Solutions of these equations are found in the book[11] and have the form

$$\begin{aligned} G_1(z) &= e^{-\kappa z/2}\, Z_\nu(2\omega e^{z/2}), \\ \nu &= \sqrt{\kappa^2 - 4\omega^2 \sin^2 \alpha} \\ &\text{and} \\ G_2(z) &= e^{-\lambda z/2}\, Z_\nu(2\omega e^{z/2}), \\ \nu &= \sqrt{\lambda^2 - 4\omega^2 \sin^2 \alpha}, \end{aligned} \tag{25}$$

where $Z_\nu(x)$ is one of two cylindric functions $J_\nu(x)$ or $Y_\nu(x)$. Now, functions ϕ_i can be obtained from the equations (21) and the functions f_i from the equations (22). Subsequent substituting the results into the equations (17,16,7) and (13) provides complex potential from which strengths and inductions can be obtained due to the equations (8) and (14).

VII. CONCLUSION

The problem of waves propagation in an exponential half-space is solved by the method of variables separation applied to Maxwell equations in Cartesian and round cylinder coordinate systems. In Cartesian coordinates each solution corresponds to arbitrary frequency, one component of wave vector (represented as $\omega \sin \alpha$) and one of two linear polarizations, they constitute complete set. In coordinates of round cylinder each solution also is labeled with necessary number of characteristics, one of which is angular momentum about the axis $\rho = 0$. These solutions can be used for describing electromagnetic waves in media with dielectric factors having form of smooth monotonic functions of single Cartesian coordinate denoted z. As for the method, it allows to separate Maxwell equations in Cartesian and round cylinder coordinates and obtain complete sets of solutions for various media specified by their dielectric factors as functions of z. Thus, the theory of electromagnetic waves in a medium possessing planar symmetry takes the form of standard linear theory which describes the phenomenon in terms of orthogonal basis in the corresponding functional space.

ACKNOWLEDGMENTS

The author is grateful to A. Zakhidov for very helpful discussions. We also very much appreciate comments and advice on different points from A. Hooshyar and participants of the seminar on the subject organized by him. We acknowledge support under AFOSR program FA9550-08-1-0261, grant "Tunable and Nonlinear Negative Refraction in Inverted Opal Photonic Crystals for Visible and Infrared". Part of this paper was written during the authors visit to the NanoTech Institute in the University of Texas at Dallas.

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