RIEMANN ZEROS AND AN EXPONENTIAL POTENTIAL

Jose Javier Garcia Moreta
Graduate student of Physics at the UPV/EHU (University of Basque country)
In Solid State Physics
Addres: Practicantes Adan y Grijalba 2 5 G
P.O 644 48920 Portugalete Vizcaya (Spain)
Phone: (00) 34 685 77 16 53
E-mail: josegarc2002@yahoo.es

MSC: 34L05 , 34L15, 65F40 , 35Q40 , 81Q05 , 81Q50

ABSTRACT: We study a given exponential potential ae^{bx} on the Real half-line which is possible related to the imaginary part of the Riemann zeros. We extend alsostudy also our WKB method to recover the potential from the Eigenvalue Staircase for the Riemann zeros, this eigenvalue staircase includes the oscillatory and smooth part of the Number of Riemann zeros.

In this paper and for simplicity we use units so $2m = 1 = \hbar$

- *Keywords: =* Riemann Hypothesis, WKB semiclassical approximation, exponential potential.

1. Exponential potential and Riemann zeros:

For $T >> 1$, the number of Riemann zeros with imaginary part on the interval $[0,T]$ is given by [3]

$$N(T) = \frac{T}{2\pi}\ln\left(\frac{T}{2\pi e}\right) + \frac{7}{8} + O\left(\frac{1}{T}\right) + \frac{1}{\pi}\arg\zeta\left(\frac{1}{2} + iT\right) \qquad (1)$$

Here $\zeta(s) = \frac{1}{1-2^{1-s}}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^s}$ $\mathrm{Re}(s) > 0$ is the Riemann zeta function [2] , and the Branch of the logarithm is chosen, so the condition $N(0) = 0$ is satisfied

The Hilbert-Polya version for Riemann Hypothesis is the following ,can we find a Hamiltonian operador with positive and Real (since is a self-adjoint operador) so their Energies satisfy $E_n = \gamma_n^2$ with $\rho_n = \frac{1}{2} + i\gamma_n$ a non-trivial zero of the Riemann zeta function?.

For this Hamiltonian on the Real half-line $[0,\infty)$ in the form $H = p^2 + f(x)$,the potential should be positive $V(x) \ge 0$, so the energies would be also positive

$$E_n = \langle \Psi_n | H | \Psi_n \rangle = \langle p\Psi_n | p\Psi_n \rangle + \langle \Psi_n | V | \Psi_n \rangle \ge 0 \qquad (2)$$

In order to obtain a Hamiltonian we will use the Bohr-Sommerfeld quantization conditions [6] in the form

$$2\pi N(E) = 2\int_0^{a=a(E)} \sqrt{E_n - V(x)}dx = 2\int_0^E \sqrt{E_n - x}\frac{df^{-1}}{dx} = 2\sqrt{\pi}D_x^{-\frac{1}{2}}f^{-1}(x) \quad (3)$$

Here 'a' inside $V(a) = E$ is a turning point of the classical Hamiltonian $H = p^2 + f(x)$, inside (3) we have used the definition of the half-derivative and the half-integral [7]

$$\frac{d^{\frac{1}{2}}f(x)}{dx^{\frac{1}{2}}} = \frac{1}{\Gamma(1/2)}\frac{d}{dx}\int_0^x \frac{dtf(t)}{\sqrt{x-t}} \qquad \frac{d^{-\frac{1}{2}}f(x)}{dx^{-\frac{1}{2}}} = \frac{1}{\Gamma(1/2)}\int_0^x dt\frac{f(t)}{\sqrt{x-t}} \quad (4)$$

Also for our Hamiltonian we have imposed boundary conditions on the half line $[0,\infty)$ so the Eigenfunctions $Hy_n(x) = E_n y(x)$ satisfy the boundary conditions $y_n(0) = 0 = y_n(\infty)$.

From (3) we obtain that the inverse of the potencial can be described implicitly in terms of the half-derivative of the Eigenvalue staircase (the smooth par) function $N(E) = \sum_{n=0}^{\infty} H(E - \gamma_n^2)$ with $H(x) = \begin{cases} 1 & \text{x>0} \\ 0 & \text{x<0} \end{cases}$ the Heaviside's step function $f^{-1}(x) = 2\sqrt{\pi}\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}n(x)$. For the case of the square of the Riemann zeros, then the smooth part is given approximately by $N_{smooth}(E) = \frac{\sqrt{E}}{2\pi}\ln\left(\frac{\sqrt{E}}{2\pi e}\right)$

To compute the half-derivative we use the representation for the logarithm $\ln(x) \approx \frac{x^{\varepsilon} - 1}{\varepsilon}$ $\varepsilon \to 0$, $e = \sum_{n=0}^{\infty}\frac{1}{n!}$ in this case we get

$$f^{-1}(x) \approx \frac{\left(4\pi^2 e^2\right)^{-\varepsilon/2} A(\varepsilon)x^{\varepsilon/2} - B}{\sqrt{\pi}\varepsilon} \qquad f(x) \approx 4\pi^2 e^2\left(\frac{\varepsilon\sqrt{\pi}x + B}{A(\varepsilon)}\right)^{\frac{2}{\varepsilon}} \quad (5)$$

The constants are $A(\varepsilon) = \frac{\Gamma\left(\frac{3+\varepsilon}{2}\right)}{\Gamma\left(1+\frac{\varepsilon}{2}\right)}$ and $B = \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$ and we have used the property of the half-derivative of powers of 'x' $\frac{d^{\frac{1}{2}}x^n}{dx^{\frac{1}{2}}} = \frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)}x^{n-\frac{1}{2}}$.

The last expression inside (5) is equal to $4\pi^2 \exp\left(2-\frac{2}{\sqrt{\pi}}\frac{\partial F(s)}{\partial s}|_{s=0}\right)e^{4x} = f(x)$

$F(s)=\frac{\Gamma\left(\frac{3}{2}+s\right)}{\Gamma(1+s)}$ .So our toy model or approximate model for the Riemann zeros is given by the Hamiltonian on the half line

$$E_n y(x) = -\frac{d^2 y(x)}{dx^2}+f(x)y(x) \quad y(0)=0=y(\infty) \quad E_n \approx \gamma_n^2 \quad \zeta\left(\frac{1}{2}+i\sqrt{E_n}\right)=0 \quad (6)$$

$f(x)=\begin{cases} 4\pi^2 \exp\left(2-\frac{2}{\sqrt{\pi}}\frac{\partial F(s)}{\partial s}|_{s=0}\right)e^{4x} & \text{x >0} \\ \infty & x \le 0 \end{cases}$. the properties of (6) are

- The potential inside (6) tends to ∞ in the limit $x \to \pm\infty$, so (6) has a discrete spectrum
- The potential inside (6) is always positive so the Energies will be always positive $\langle H \rangle = E_n > 0$
- The spectrum is approximately given by the imaginary part of the Riemann Zeros, Hamiltonian (6) reproduces approximately the imaginary part for the Riemann zeros
- The Bohr-sommerfeld conditions for the exponential potential inside (6) reproduces the smooth part of the spectral staircase for the square of the imaginary zeros $2\int_0^{a=a(E)} \sqrt{E_n - ce^{bx}}dx \approx N_{smooth}(E)=\frac{\sqrt{E}}{2\pi}\ln\left(\frac{\sqrt{E}}{2\pi e}\right)$
- The factor $\frac{7}{8}$ may be viewed as a Maslov index inside the Bohr-Sommerfeld quantization conditions $\pi\left(n(E)+\frac{7}{8}\right)=\int_0^{a=a(E)} p(x)dx$ with $p=\sqrt{E-ae^{bx}}$ the momentum of the particle inside the potential.
- The exponential potential for an Schrödinger equation can be solved analytically $V(x)=ae^{bx}$
- The square root of the Energies satisfy that $\left|\sqrt{E_{n+1}}-\sqrt{E_n}\right| \to 0$ in the limit of big quantum numbers $n \to \infty$.
- Berry and Keating [3] get a similar smooth density of states for their Hamiltonian $-i\left(x\frac{d}{dx}+\frac{1}{2}\right)\Psi(x)=E_n\Psi(x)$ however they do not know what boundary conditions to impose in order to get a discrete spectrum ,which is equal to the imaginary part of the zeros

Equation (6) can be solved, we have used the Mathematica Wolfram alpha software [13]

$$y(x)=C_1 e^{\frac{\pi\sqrt{E_n}}{4}}\Gamma\left(1-i\frac{\sqrt{E_n}}{2}\right)I_{\mu}\left(\frac{\sqrt{\lambda}}{2}e^{2x}\right)+C_2 e^{\frac{-\pi\sqrt{E_n}}{4}}\Gamma\left(1+i\frac{\sqrt{E_n}}{2}\right)I_{-\mu}\left(\frac{\sqrt{\lambda}}{2}e^{2x}\right) \quad (7)$$

Where $\mu=i\frac{\sqrt{E_n}}{2}$ defines the Energy , $\Gamma(x)$ is the Gamma function and $\lambda=4\pi^2\exp\left(2-\frac{2}{\sqrt{\pi}}\frac{\partial F(s)}{\partial s}|_{s=0}\right)=157.9136704...=16\pi^2$, the constants are C_1, C_2 can be chosen from the normalization condition $\int_0^\infty dx\,|\,y_n(x)\,|^2=1$, the function $I_n(x)=\left(\frac{x}{2}\right)^n\sum_{k=0}^{\infty}\frac{x^{2k}}{2^{2k}}\frac{1}{k!\Gamma(k+n+1)}$ is the modified Bessel function of first kind.

The exact quantization condition (not the one coming from the Bohr-Sommerfeld rules) is then determined by the boundary condition on the half real line $[0,\infty)$

$$I_{\mu}\left(\frac{\sqrt{\lambda}}{2}\right)=0=I_{\mu}(2\pi) \qquad \lambda=4\pi^2\exp\left(2-\frac{2}{\sqrt{\pi}}\frac{\partial F(s)}{\partial s}|_{s=0}\right) \qquad \mu=\frac{i\sqrt{E_n}}{2} \quad (8)$$

Unfortunately , there is no exact analytic method to solve the equation (8) to obtain the energies of the Hamiltonian so we can only solve (8) by numerical methods, an aproxímate method to obtain the Energies for big values of the Quantum number n is to use the semiclassical method $N_{smooth}(E)=\frac{\sqrt{E}}{2\pi}\ln\left(\frac{\sqrt{E}}{2\pi e}\right)=n+\frac{1}{2}\approx n$, this equation can be inverted to get the energies in term of the Lambert W-function

$$E_n\approx\frac{4\pi^2 n^2}{W^2(ne^{-1})} \qquad W(x)e^{W(x)}=x \qquad W(x)=\sum_{n=1}^{\infty}(-n)^{n-1}\frac{x^n}{n!} \quad (9)$$

If we use the asymptotic property for the Lambert W-function $\lim_{x\to\infty}\frac{W(x)}{\ln x}=1$ and take the positive square root we find $\sqrt{E_n}=k_n\approx\frac{2\pi n}{\ln n}$, this is precisely the imaginary part of the Riemann zeros in the limit $n\to\infty$

The Quantum condition for the energies inside (8) can be generalized to the half line $[u_0,\infty)$ in the form $I_{\mu}\left(\frac{\sqrt{\lambda}}{2}e^{2u_0}\right)=0$, one of our conjecture is that ,depending on the value of u_0, we should have different approximations to the Riemann Xi-function $\xi(s)=\frac{s(s-1)}{2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$.

Another quantization condition on $[0,\infty)$ with $C_1=-kC_2$ and $y(0)=0=y(\infty)$ is

$$e^{\frac{\pi\sqrt{E_n}}{4}}\Gamma\left(1-i\frac{\sqrt{E_n}}{2}\right)I_{\mu}\left(\frac{\sqrt{\lambda}}{2}\right)=ke^{\frac{-\pi\sqrt{E_n}}{4}}\Gamma\left(1+i\frac{\sqrt{E_n}}{2}\right)I_{-\mu}\left(\frac{\sqrt{\lambda}}{2}\right) \quad (10)$$

This expresión (10) is very similar to the functional equatio for the Riemann zeta in the variable $s=\frac{1}{2}+i\sqrt{E_n}$

$$\Gamma\left(\frac{s}{2}\right)\pi^{-s/2}\zeta(s)=\Gamma\left(\frac{1-s}{2}\right)\pi^{-1/2+s/2}\zeta(1-s) \qquad \zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s} \qquad \mathrm{Re}(s)>1 \quad (11)$$

Now , we can use the Gelfand-Yaglom theorem [4] to define the even entire function

$$\frac{D(s)}{D(0)}=\frac{\det\left(-\frac{d^2}{dx^2}+16\pi^2e^{4x}-s^2\right)}{\det\left(-\frac{d^2}{dx^2}+16\pi^2e^{4x}\right)}=\prod_{n=0}^{\infty}\left(1-\frac{s^2}{E_n}\right)=\frac{\Psi(s,L)}{\Psi(0,L)} \qquad L\to\infty \quad (12)$$

Where $\Psi(s,L)$ solves the initial value problem

$$-\frac{d^2\Psi(s,x)}{dx^2}+16\pi^2e^{4x}\Psi(s,x)=s^2\Psi(s,x) \qquad \Psi(s,0)=0 \quad \frac{d\Psi(s,0)}{dx}=1 \quad (13)$$

The solution to this inicial value problem is given in (7) , the roots of $D(s)$ are all real and are given by the square root of the Energies of the Hamiltonian (6) $s=\pm\sqrt{E_n}$, the number of roots of $D(s)$ on the interval $(0,T)$ as $T\to\infty$ is given by $N(T)_{smooth}=\frac{T}{2\pi}\ln\left(\frac{T}{2\pi e}\right)$, which on average agrees with the Number of zeros of the Riemann Xi-function $\xi(s)$ on the critical line, but for the function D(s) ALL the zeros are real.

2. An implicit equation for the potential $f^{-1}(x)$ on the real line $[0,\infty)$,Beyond the smooth part of the Riemann zeros

The main problem with our Hamiltonian operator (6) is that we have simpli ignored the contribution of the sum of the primes to the eigenvalue staircase for the Riemann zeros defined by

$$N_{osc}(E)=-\frac{1}{\pi}\sum_{p}\sum_{n=1}^{\infty}\frac{1}{n}\frac{1}{p^{\frac{n}{2}}}\sin\left(n\sqrt{E}\ln p\right)=\frac{1}{\pi}\arg\zeta\left(\frac{1}{2}+i\sqrt{E}\right) \quad (14)$$

The EXACT equation for the potential (defined implicitly) is the following

$$f^{-1}(x)=2\sum_{n=0}^{\infty}\frac{H(x-\gamma_n^2)}{\sqrt{x-\gamma_n^2}}=\frac{2}{\sqrt{\pi}}\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\arg\xi\left(\frac{1}{2}+i\sqrt{E}\right)\quad \xi(s)=\frac{s(s-1)}{2}\pi^{\frac{-s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) \quad (15)$$

The sum $\sum_{n=0}^{\infty}\frac{H(x-\gamma_n^2)}{\sqrt{x-\gamma_n^2}}$ is made over the imaginary part of the Riemann zeros on the upper complex plane $\Im m(s)>0$, this sum over zeros can be turned into a sum over primes and prime powers with the aid of the Riemann-Weil explicit formula [11]

$$\sum_{\gamma}h(\gamma)=2h\left(\frac{i}{2}\right)-g(0)\ln\pi-2\sum_{n=1}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}g(\ln n)+\frac{1}{2\pi}\int_{-\infty}^{\infty}ds\,h(s)\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{is}{2}\right) \quad (16)$$

Here, $g(k)=\frac{1}{2\pi}\int_{0}^{\infty}dx\,h(x)\cos(kx)=g(-k)$ h(x) and g(x) are test functions which form a Fourier transform pair and $\Lambda(n)=\begin{cases}\ln p & n=p^k\\ 0 & \text{otherwise}\end{cases}$ is the Mangoldt function., see [2].

If we insert the expression $h(x,r)=\frac{H(x-r^2)}{\sqrt{x-r^2}}$ inside (12) and use the identity for the Bessel function $\frac{1}{\pi}\int_{0}^{x}\frac{dt\cos(ut)}{\sqrt{x^2-t^2}}=\frac{J_0(ux)}{2}$ then the expression for the potential of our Hamiltonian (2) on the real half-line $[0,\infty)$ becomes

$$f^{-1}(x)=\frac{4H\left(x+\frac{1}{4}\right)}{\sqrt{4x+1}}+\frac{1}{2\pi}\int_{-\sqrt{x}}^{\sqrt{x}}\frac{dr}{\sqrt{x-r^2}}\left(\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{ir}{2}\right)-\ln\pi\right)-\sum_{n=1}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}J_0\left(\sqrt{x}\ln n\right) \quad (17)$$

The last sum over primes and prime powers can be interpreted in terms of the half derivative of the argument of the Riemann zeta function on the critical line

$$\frac{2}{\sqrt{\pi}}\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\arg\zeta\left(\frac{1}{2}+i\sqrt{x}\right)=-\sum_{n=1}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}J_0\left(\sqrt{x}\ln n\right) \quad (18)$$

For x>>>1 equation (16) becomes $\varepsilon\to 0$.

$$f^{-1}(x)\approx\frac{\left(4\pi^2e^2\right)^{-\varepsilon/2}A(\varepsilon)x^{\varepsilon/2}-B}{\sqrt{\pi}\varepsilon}-\frac{2}{\sqrt[4]{x}}\sum_{n=2}^{\infty}\frac{\Lambda(n)}{\sqrt{2\pi n\ln n}}\cos\left(\sqrt{x}\ln n-\frac{\pi}{4}\right) \quad (19)$$

Unfortunately the sum $\sum_{n=1}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}J_0\left(\sqrt{x}\ln n\right)$ is DIVERGENT so we must truncate it , for example by summing only up to some finite number of primes and their prime powers to get some corrections to the exponential potential deduced for the Hamiltonian inside (6).

Equations (15) and (16) can be improved a little more if we could prove (conjecture) that the quantity

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\arg\zeta\left(\frac{1}{2}+i\sqrt{x}\right)\to 0 \qquad x\to\infty \qquad (20)$$

In this case the potential $f(x)=4\pi^2\exp\left(2-\frac{2}{\sqrt{\pi}}\frac{\partial F(s)}{\partial s}|_{s=0}\right)e^{4x}$ would be almost the exact potential, and the quantization condition $I_{\frac{i}{2}\sqrt{E_n}}\left(\frac{\sqrt{\lambda}}{2}\right)=0$ would give the imaginary part for the Riemann zeros for big quantum number $n>>>1$ with $E_n=\gamma_n^2$. In this approach the imaginary part of the Rieman zeros on the critical line are not energies but allowed values of the quantizied momenta for the exponential potential λe^{4x} so $I_{\frac{i\gamma_n}{2}}\left(\frac{\sqrt{\lambda}}{2}\right)=0$ with $\zeta\left(\frac{1}{2}+i\gamma_n\right)=0$.

From the WKB method and the properties of the half-derivative operator we know for our model that the potential is related to the density of states by

$$f^{-1}(x)=2\sqrt{\pi}\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\sum_{n=0}^{\infty}H(x-E_n) \qquad \frac{1}{2\sqrt{\pi}}\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}f^{-1}(x)=\sum_{n=0}^{\infty}\delta(x-E_n) \quad (21)$$

Since $\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\left(\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\right)f(x)=\frac{df(x)}{dx}$, if we take the half derivative inside (14) and use the identities for the Bessel function and the Dirac delta function

$$\sqrt{\pi}\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}J_0\left(a\sqrt{x}\right)=\frac{\cos\left(a\sqrt{x}\right)}{\sqrt{x}} \qquad \delta\left(f(x)\right)=\sum_n\frac{\delta\left(x-x_n\right)}{|f'(x_n)|} \qquad (22)$$

We obtain the distributional Riemann-Weil trace formula, so the density of states of our Hamiltonian , with the potential defind implicitly inside (17) is just the Riemann-Weil trace formula

$$\sum_{n=0}^{\infty}\delta\left(x-\gamma_n\right)+\sum_{n=0}^{\infty}\delta\left(x+\gamma_n\right)=\frac{1}{2\pi}\frac{\zeta}{\zeta}\left(\frac{1}{2}+ix\right)+\frac{1}{2\pi}\frac{\zeta'}{\zeta}\left(\frac{1}{2}-ix\right)-\frac{\ln\pi}{2\pi}$$
$$+\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+i\frac{x}{2}\right)\frac{1}{4\pi}+\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}-i\frac{x}{2}\right)\frac{1}{4\pi}+\frac{1}{\pi}\delta\left(x-\frac{i}{2}\right)+\frac{1}{\pi}\delta\left(x+\frac{i}{2}\right) \quad (23)$$

If we take the integral inside (22) with respect to 'x' we obtain the eigenvalue staircase $N(x)=\sum_{n}H\left(x-E_n\right)=\frac{1}{\pi}\arg\xi\left(\frac{1}{2}+i\sqrt{x}\right)$ the spectral eigenvalue staircase for the Riemann zeros.

In general for small 'x' we can evaluate the inverse of the potential numerically by computing the sum $\sum_{n=0}^{\infty}\frac{H(x-\gamma_n^2)}{\sqrt{x-\gamma_n^2}}$, from the properties of the Heaviside step function this sum is finite and easy to evaluate with a computer for $\gamma_n^2\le 10^4$,for big 'x' we can use the asymptotics x>>>1

$$f^{-1}(x)\approx 2\sqrt{\pi}\,\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\left(\frac{\sqrt{x}}{2\pi}\ln\left(\frac{\sqrt{x}}{2\pi e}\right)\right)+\frac{2}{\sqrt{\pi}}\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\arg\zeta\left(\frac{1}{2}+i\sqrt{x}\right)+O\left(\frac{1}{\sqrt{x}}\right) \quad (24)$$

Perhaps , equation (24) is a good candidate to give a proof of RH, at least this is better thant the simple and childish model from Berry and Keating iwth the operator $H_{B-k}=-i\left(x\frac{d}{dx}+\frac{1}{2}\right)$, however many refeeres give the cheap excuse that the implicit equation may (they do not give any proof of course) have no inverse on the interval $[0,\infty)$, however , we have proved how for $x\to\infty$ the smooth part of the potential giving the square of the imaginary part of the Riemann zeros is just an exponential potential .

References

[1] Amore P. , Fermandez F.M "Accurate calculation of the complex eigenvalues of the Schrödinger equation with an exponential potencial" avaliable as an e-print at http://arxiv.org/abs/0712.3375

[2] Apostol Tom "*Introduction to Analytic Number theory" ED: Springuer ",* (1976)

[3] Berry M. Keating J.P " The Riemann Zeros and Eigenvalue Asymptotics " Siam Review archive Vol. 41 Issue 2, June 1999 pages: 236-266

[4] Bolte J: "*Semiclassical trace formulae and eigenvalue statistics in quantum*

chaos", Open Sys. & Information Dyn.**6** (1999) 167-226 , avaliable at: http://arxiv.org/abs/chao-dyn/9702003

[5] Dunne G. And Min H. “A comment on the Gelfand–Yaglom theorem, zeta functions and heat kernels for PT-symmetric Hamiltonians

[6] Griffiths, David J. (2004). “*Introduction to Quantum Mechanics”* Prentice Hall. ISBN 0-13-111892-7.

[7] Lagarias J. “The Schrödinger operator with Morse potential on the right half line “Avaliable as e-print at http://arxiv.org/abs/0712.3238

[8] Nishimoto, K. “*Fractional Calculus”.* New Haven, CT: University of New Haven Press, 1989.

[9] Odlyzko A. “ Tables of Zeros of Riemann Zeta functions” ,webpage http://www.dtc.umn.edu/~odlyzko/zeta_tables/

[10] Polyanin A. “http://eqworld.ipmnet.ru/en/solutions/ode/ode-toc2.htm

[11] Sierra G. “A physics pathway to the Riemann hypothesis” avaliable at Arxiv.org http://arxiv.org/abs/1012.4264

[12] Voros A. “ *Exercises in exact quantization”* e-print avaliable at http://arxiv.org/pdf/math-ph/0005029v2.pdf

[13] Wolfram Alpha computational knowledge Engine wepage http://www.wolframalpha.com/input/?i=-y%27%27%28x%29%2Baexp%28bx%29y%28x%29%3Dcy%28x%29