Characterization of Partitions for Solid Graphs

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Abstract: - The paper deals with the problems of characterization of simple graphical partitions belonging to the solid graphs, i.e. graphs, in which there are no four of vertices such that it is possible some shift of edges incidental to them and with characterization of the one class of steady graphs too. The necessary and sufficient conditions for the partition belonging to the solid graph have been established.

Key-Words: - simple partition, edges shift, steady graph, solid graph, planar graph.

1 Introduction

Study of the properties of simple graphs linked to their partitions is one of interesting and perspective directions of the graph theory [1]. The present work is devoted to characterization of the one class of simple graphic partitions. Here we use concepts of simple graphic partitions [1], shifts of edges of the graph [2], and besides, two definitions are introduced.

1) If in the graph there are no four of vertices such that some shift of edges incidental to them is possible, that we name such graph as solid.

2) The graph which has simple partition we name steady.

2 Characterization of Solid Graphs

Theorem 1. If $G=(X_1,U_1)$ and $H=(X_2,U_2)$ - two graphs with identical partitions then it is possible to obtain H from G by means of finite number of shifts of graph edges.

Proof. The proof is similar to the proof of the theorem of "demi-degrees" for the oriented graphs [2]. From theorem 1 it follows that solid graphs are steady.

Let us consider in detail the structure of solid graphs. Lemma 1 follows from the first definition directly.

Lemma 1. Graph G is solid if and only if the sub-graph formed by any of its four vertices and edges connecting them, contains either a triangle, or three vertices not adjacent in pairs.

Lemma 2. The solid graph contains no more than one connected component.

Proof. Indeed, if the graph contains two connected components, that, taking in each of them in twos adjacent vertices, we will obtain the four of vertices forbidden by Lemma 1. This contradiction proves Lemma 2.

Lemma 3. If $G=(X,U)$ is the solid graph without isolated vertices then it is connected graph, and also the greatest of degrees of its vertices is $\Delta=|X|-1$.

Proof. Connectivity of the graph follows from Lemma 2. The second statement will be proved by contradiction. We will assume that maximum degree $\Delta<|X|-1$. Let v_1 be the vertex with maximum degree, i.e. $\deg v_1=\Delta$, and $v_2,\ldots,v_{\Delta+1}$ are vertices, adjacent to v_1. Then, as the graph is connected, there is the vertex w such that $(w,v_1)\notin U$ and $(w,v_k)\in U$ for some vertex v_k such that $(v_1,v_k)\in U$. Applying Lemma 1 to the four of vertices of graph $G: w,v_1,v_k,v_i$ where v_i is any of vertices, adjacent to v_1, but not coinciding with v_k , we obtain $(v_k,v_i)\in U$. From here it follows $\deg v_k\geq\Delta+1$ that is impossible, since Δ is the greatest degree. This contradiction proves Lemma 3.

Lemma 4 follows from Lemma 1.

Lemma 4. After removing any vertex together with edges incidental to it from the solid graph we obtain the graph that also will be solid.

Let us use further two forms of graphic partitions:
1) Nonincreasing sequence of degrees of vertices $\Pi=d_1,d_2,\ldots,d_n$;

2) The form $\tilde{\Pi} = a_1, a_2, \ldots, a_{n-1}$, where a_i is the number of vertices having degree i.

Solid graphs and their partitions are characterized by the following theorem.

Theorem 2. The graphic partition $\Pi(\tilde{\Pi})$ is a partition of the connected solid graph $G = (X, U)$ if and only if for all $d_j \geq j$ the following relations are fulfilled:

$$\left.\begin{array}{l} d_1 = |X| - 1, \\ \underset{i \neq 1}{d_i} = d_{i-1} - a_{i-1} \end{array}\right\}.$$

Proof. *Necessity.* In the connected solid graph we have $d_1 = |X| - 1$ (lemma 3). Let us remove from G the vertex v_1 of degree d_1 and all edges incidental to it. Thus we will receive the solid graph G' (lemma 4) with a_1 isolated vertices. If it contains also non-trivial component, then maximum degree of its vertices is $\Delta' = d_1 - a_1 - 1$ (lemma 3). Returning to graph G, we have $d = \Delta' + 1 = d_1 - a_1$. Deleting, thus, the vertices of degrees $d_1, d_2, \ldots, d_j$ from graph G until the graph consisting of isolated vertices will turn up, we receive at each stage equalities $d_i = d_{i-1} - a_{i-1}$ for all $d_j \geq j$.

Sufficiency. Let $\Pi(\tilde{\Pi})$ be the partition satisfying to conditions (1). The algorithm for constructing the graph belonging to this partition consists of the following steps.

1. We build the star with the partition $\Pi = |X| - 1, 1, \ldots, 1$.
2. We choose any of vertices of degree 1 and connect it with $(|X| - a_1 - 2)$ vertices of the same degree. Then we repeat this procedure with vertices of degree 2 etc., backward to how it was done at the proof of the necessity of condition (1), yet we will receive the vertex of degree $d_j \geq j$ such that $d_{j+1} \leq j$. The constructed graph belongs to the set partition evidently.

Sample 1. Graphic partition $\Pi = 9,7,5,4,4,3,2,2,1,1$ is given. Let us show that it is a partition of the solid graph. We check performance of conditions (1): $d_1 = |X| - 1 = 9; d_2 = d_1 - a_1 = 7; d_3 = d_2 - a_2 = 5;$ $d_4 = d_3 - a_3 = 4.$

The graph is depicted on Fig. 1

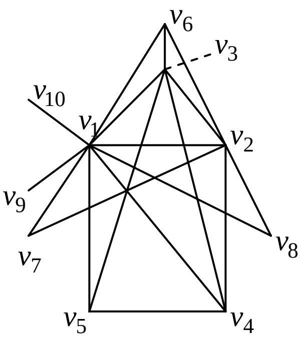

Fig.2- The sample of solid graph

3 One Class of Steady Graphs

The problem of characterization of steady graphs (and simple graphic partitions) can be formulated in the matrix form [3, 4].

Let A be a matrix of contiguities of some graph $G = (X, U)$. We form the sum $\sum_{i=1}^{|X|} C_i A^2 C_i$, where C_i is the square matrix of order $|X|$ in which the element c_{ii} is equal to 1, and other elements are equal to 0. Let (Π) be the matrix in which the diagonal elements are degrees of vertices of the graph, and other elements are equal to zero. Then it is clear that at corresponding enumerating of vertices of the graph we obtain

$$\sum_{i=1}^{|X|} C_i A^2 C_i = (\Pi).$$

If A_1 and A_2 are matrixes of contiguities of two isomorphic graphs then they are connected among themselves by relations of the type:

$$A_2 = I_{i_1 j_1} \ldots I_{i_k j_k} A_1 I_{i_k j_k} \ldots I_{i_1 j_1} = \pi(A_1),$$

where I_{ij} is the matrix obtained from a single matrix by the permutation of i-th and j-th lines [3]. As $I_{ij}^2 = I$ then $A_2^2 = \pi(A_1^2)$.

If now we designate $A^2 = Y$ and will consider expression (2) as the matrix equation at the given Π then its solution can be given by matrixes of steady graphs in following two cases.

1. There exists unique solution $Y = A^2$ of equation (2), where Y is the square of a symmetric matrix of order $|X|$ with a zero diagonal.

2. All solutions of equation (2) are connected among themselves by relations (3), but $A_2^2 = \pi\left(A_1^2\right) \neq A_1^2$.

It is obvious that any transformation of type (3) of the matrix of contiguities of graph G, keeping equality (2), will be equivalent to remarks of vertices of the graph G, consisting of cycles of the vertices having equal degrees.

Let us investigate case 1 in detail.

Theorem 3. If A is the matrix of contiguities of solid graph G, and $\pi(A)$ is the remark of type (3) keeping relation (2), then $\pi\left(A^2\right) = A^2$.

Proof. Let the vertices v_i and v_j have equal degrees $d_i = d_j = d$ in the solid graph $G = (X, U)$. Further, let M_i be the set of vertices, adjacent to v_i, and let M_j be the set of vertices, adjacent to v_j. Then there exist two different vertices $v_k; v_l$ of the graph G such that $v_k \in M_i / M_j$ and $v_l \in M_j / M_i$. If $(v_i, v_j) \notin U$ then v_k does not coincide with v_j, and v_l - with v_i.

However, in this case the four-in-hand of vertices v_i, v_j, v_k, v_l does not satisfy the conditions of lemma 1 and, consequently, it cannot belong to the solid graph. From here it follows that in the solid graph for any two vertices having equal degrees, one of the following two statements is correct:

a) $M_i \equiv M_j \;\&\; (v_i, v_j) \notin U$;

b) $v_j \in M_i / M_j \;\&\; v_i \in M_j / M_i$.

Extending our reasoning to some set $N_i = \{v_{i_1}, \ldots v_{i_l}\}$ of vertices having equal degrees in the graph G, we will receive that for this set N_i one of the following systems of relations must be correct:

$$\left. \underset{r=1,\ldots,l}{M_{i_r}} \middle/ \bigcap_{s=1}^{l} M_{i_s} \right. = \varnothing;$$

$$\left. \underset{r=1,\ldots,l}{M_{i_r}} \middle/ \bigcap_{s=1}^{l} M_{i_s} \right. = \underset{r=1,\ldots,l}{N_i / v_{i_r}}.$$

Thus, in the solid graph each sub-graph formed by vertices with equal degrees, is either the complete graph, or completely unconnected.

As each element $a_{lm}^{(2)}$ of the matrix A^2 is equal to number of ways of length 2 from the vertex v_l to the vertex v_m, and remarks of type (3) consist of cycles of vertices with equal degrees, the theorem statement is easily deduced from conditions (4) and (5).

From the proved theorem it follows that if Π is the partition of the solid graph, then the equation $\sum_i C_i Y C_i = (\Pi)$ where $Y = A^2$ has the unique solution Y.

Let us assume now that equation (2) has the unique solution Y, but Π is not the partition of the solid graph. Let A_1 be the matrix of contiguities of the graph G_1 belonging to the partition Π, and A_2 - the matrix of contiguities of the graph G_2 obtained from G_1 by shifting any pair edges and having the same partition Π.

Let us admit, for example, that such shift of edges is made: $(v_i, v_j) \to (v_i, v_l)$; $(v_k, v_l) \to (v_l, v_j)$. We will put for definiteness that $i > j > k > l$.

As a result of this shift the matrix of contiguities will change:

$$A_1 = \begin{pmatrix} \cdot & & & & & & \\ & \cdot & & & & & \\ & & \cdot & & & & \\ & & & 0 & 1 & \delta_1 & 0 \\ & & & 1 & 0 & 0 & \delta_2 \\ & & & \delta_1 & 0 & 0 & 1 \\ & & & 0 & \delta_2 & 1 & 0 \end{pmatrix} \to$$

$$A_2 = \begin{pmatrix} \cdot & & & & & & \\ & \cdot & & & & & \\ & & \cdot & & & & \\ & & & 0 & 0 & \delta_1 & 1 \\ & & & 0 & 0 & 1 & \delta_2 \\ & & & \delta_1 & 1 & 0 & 0 \\ & & & 1 & \delta_2 & 0 & 0 \end{pmatrix}$$

Let us investigate, to what requirements the elements of matrixes A_1 and A_2 should satisfy that condition $A_1^2 = A_2^2$ was met.

1. As $a_{ij}^{(2)}\left(A_2^2\right) = A_{ij}^{(2)}\left(A_1^2\right) + \delta_1 + \delta_2$ then equalities $\delta_1 = 0; \delta_2 = 0$ should be fulfilled.

2. For any $m \neq j$ we get

$a_{im}^{(2)}(A_1) = \Sigma + a_{jm}(A_1) \;\&\; a_{im}^{(2)}(A_2) = \Sigma + a_{lm}(A_1)$.

From this it follows that $a_{jm}(A_1) = a_{lm}(A_1)$.

It is similarly proved that

$a_{im}(A_1) = a_{km}(A_1) \;\&\; a_{jm}(A_1) = a_{km}(A_1)$.

Here we consider the shift

$(v_i, v_j,) \to (v_i, v_k)$ & $(v_k, v_l,) \to (v_j, v_l)$ which is possible, since $\delta_1 = \delta_2 = 0$.

3. If M'_i, M'_j, M'_k, M'_l are the sets of vertices, adjacent to vertices v_i, v_j, v_k, v_l accordingly, and these sets do not contain these vertices in themselves, then equalities $M'_i \equiv M'_j \equiv M'_k \equiv M'_l$ follow from the previous consideration.

If $\left|M'_{ijkl}\right| \geq 2$, and for some r, s we have $v_r \in M'_{ijkl}$ & $v_s \in M'_{ijkl}$, then we get

$a_{js} = a_{rk} = a_{ks} = 1$ & $a_{jk} = 0$.

As $a_{jk}(A_1) = 0$ then at $a_{rs}(A_1) = 0$ the shift $(v_k, v_r,) \to (v_k, v_j)$ & $(v_j, v_s,) \to (v_r, v_s)$ is possible in the graph G_1. But from here we will receive $a_{ks} = 0$ by repeating point 1. The received contradiction proves that $a_{rs}(A_1) = 1$, i.e. the sub-graph formed by set of vertices M'_{ijkl}, is complete.

4. We will consider now any edge (v_p, v_q) of the graph G_1. The following is obvious:

a) if the shift of edges (v_p, v_q) and $(v_{i;k}, v_{j;l})$ is impossible, then at least one of vertices- v_p or v_q - belongs to M'_{ijkl};

b) if the shift (v_p, v_q) and $(v_{i;k}, v_{j;l})$ is possible, then $M'_p \equiv M'_q \equiv M'_{ijkl}$.

5. From point 4 it follows that if (v_e, v_f) is such edge of the graph G_1 that $v_e \notin M'_{ijkl}$; $v_f \notin M'_{ijkl}$; v_e does not coincide with $v_i \vee v_j$, and v_f - with $v_i \vee v_j$ also, then the shift of edges (v_e, v_f) and (v_i, v_j) is possible, and consequently these equalities are correct: $M'_e \equiv M'_f \equiv M'_{ij}$.

6. If u and w are two edges, each of which is incidental at least to one vertex from M'_{ijkl} then the shift of edges u and w is impossible. It follows from points 1 and 3.

7. From points 1-6 it follows that graph G_1 can be realized in the form of superposition of three graphs. The first graph G'_1 is formed by a subset of edges of the graph G_1 in which each pair of edges is allowed for shift. This graph consists of components of type K_2.

Removing from the graph G_1 all vertices and edges of the graph G'_1, and edges incidental to vertices of G'_1 too, we obtain the second graph - G''_1 which is solid.

The third graph G'''_1 is formed by the edges connecting each vertex of the graph G'_1 with all vertices of some complete sub-graph $\tilde{G}$ of the graph G''_1; and other vertices of G''_1 form a trivial sub-graph. (As appears from the proof of theorem 2, in the solid graph all vertices of degrees $d_i \geq i$ form the complete sub-graph, and all vertices of degrees $d_i < i$ - the totally unconnected sub-graph).

It is easy to show now that the constructed graph G_1 is steady. From the reasoning spent in points 1-7, and theorems 2 and 3 we obtain the following theorem.

Theorem 4. If G is the steady graph such that for any remark π of its vertices keeping equality (2), the square of the matrix of contiguities of G does not change, then partitions $\Pi(\tilde{\Pi})$ of this graph satisfy to following conditions:

1) $d_1 = n - 1$;

2) for all $d_i > i$ it is true: $d_i = d_{i-1} - a_{i-1}$;

3) if there exists the term $d_i = i$ then the subset consisting of even number of terms of the partition such that $d_i = d_{i+1} = \ldots = d_{i+2s-1} = i$ exists also. Thus the changed partition

$\Pi' = d_1 - 2s, d_2 - 2s, \ldots, d_{i-1} - 2s, d_{i+2s}, \ldots, d_n$,

consisting of $(n - 2s)$ terms, is the partition of the solid graph.

The return to Theorem 4 statement is also correct.

Sample 2. The graphic partition

$\Pi = 12,11,10,9,5,5,5,5,4,4,3,2,1$ is given. We check the first and second conditions of Theorem 4:

$d_1 = n - 1 = 12; d_2 = d_1 - a_1 = 11;$

$d_3 = d_2 - a_2 = 10; d_4 = d_3 - a_3 = 9$.

In the partition Π there exists the subset consisting of four terms: $d_5 = d_6 = d_7 = d_8 = 5$. Changed partition $\Pi' = 8,7,6,5,4,4,3,2,1$ is the partition of the solid graph, as it is easy to check up. From this it follows that partition Π is simple, and for steady graph belonging to it by any π the equality $\pi(A^2) = A^2$ is carried out. This graph is represented on Fig. 2. Two edges, allowed for shift, are led round.

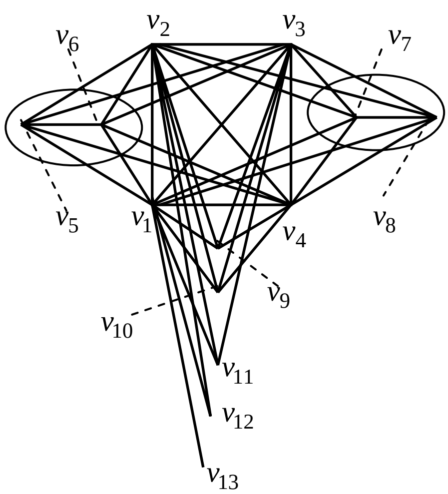

Fig. 2- The sample of steady graph

4 Interesting Problem

The following problem is very interesting and important in my opinion [5, 6, 7, 8]. What is the criterion (or algorithm) for defining graphic partitions such that graphs belonging to them:

1) are planar without fail (strongly planar);
2) are non-planar without fail (strongly non-planar);
3) can be planar or non-planar.

For example, Kuratowski's graph K_5 (4,4,4,4,4) is strongly non-planar; however Kuratowski's graph $K_{3,3}$ (3,3,3,3,3,3) is neither strongly non-planar nor strongly planar. Samples of the first case are obvious.

We know the work of Chvátal [9], where the conditions for planarity of graphs belonging to the given partitions were found. But we do not know if the mentioned above problem is solved.

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