Geometric Algebra of Quarks*

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Abstract

Quarks are described mathematically by (3 x 3) matrices. To include these quarkonian mathematical structures into Geometric algebra it is helpful to restate Geometric algebra in the mathematical language of (3 x 3) matrices. It will be shown in this paper how (3 x 3) permutation matrices can be interpreted as unit vectors. And as S_3 permutation symmetry is flavour symmetry a unified flavour picture of Geometric algebra will emerge.

1 Warning

In this version of Geometric algebra of quarks negative numbers are avoided. There will be only imaginary units i, positive scalars as multiples of 1, and matrices. This paper lives in a positive, but yet complex world.

Of course it is possible to include minus signs into Geometric algebra of quarks as it is no crime against mathematics to write equations like

$$\begin{pmatrix} 0 & 0 & i \\ 0 & i & 0 \\ i & 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1}$$

But for ontological reasons (see [7, sec. 8]) I prefer to write this equation without a minus sign as

$$\begin{aligned} \begin{pmatrix} 0 & 0 & i \\ 0 & i & 0 \\ i & 0 & 0 \end{pmatrix}^2 &\simeq \begin{pmatrix} 1 & 1 & 1+i \\ 1 & 1+i & 1 \\ 1+i & 1 & 1 \end{pmatrix}^2 \\ &= \begin{pmatrix} 2+2i & 3+2i & 3+2i \\ 3+2i & 2+2i & 3+2i \\ 3+2i & 3+2i & 2+2i \end{pmatrix} \simeq \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \end{aligned} \tag{2}$$

2 Introduction

It is well known that generators of the symmetric group S_3, which is isomorphic to the dihedral group of order 6, can be represented by positive (3 x 3) matrices. In Geometric algebra it is possible to consider these matrices as geometric objects with a clear geometric meaning.

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And it is well known that permutation symmetry S_3 closely resembles flavour symmetry [10]. To prepare the scene for a unified Geometric algebra picture of quarks (which will be constructed one day) permutation symmetry S_3 will be used in the following to restate Geometric algebra in the language of (3 x 3) matrices. As Gell-Mann matrices are (3 x 3) matrices, a unification of Geometric algebra and Gell-Mann matrix algebra (which will be found one day) is surely made easier to construct.

One possible way to identify (3 x 3) permutation matrices with geometric objects is presented in [7], where special emphasis is given to the fact that a purely positive world without negative and without complex numbers can be formulated. In the present paper a more direct relation to matrix representations of the symmetric group S_3 is drawn, now using imaginary units to describe directions perpendicular to the S_3-plane of [7]. For this reason imaginary numbers are included, while the representation of the null matrix (or nihilation matrix) is still applied by analogy to [7].

3 Ugliness in Geometric Group Theory

Geometric group theory [12] discusses among other things matrix representations of permutations. The six different permutations of three objects or positions or flavour families a, b, and c are represented by the six positive (3 x 3) matrices [13, p. 356] as operators of S_3. For example the second and third positions of a column vector are exchanged in [12, p. 180] by:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ c \\ b \end{pmatrix} \tag{3}$$

This can be considered as geometric operation in three-dimensional space when we interprete the numbers a, b, and c as coordinates:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ z \\ y \end{pmatrix} \tag{4}$$

From a Geometric algebra perspective this is a very, very ugly equation. The operator is represented by a (3 x 3) matrix while the operand is represented by a column vector or (1 x 3) matrix. Compared with the (3 x 3) matrix, a column vector is a totally different mathematical object. Thus we have an algebra of two different mathematical worlds: the world of (3 x 3) matrices and the world of (1 x 3) matrices.

In Geometric algebra we have a more ambitious dream. Vianna, Trindade and Fernandes [15, p. 962] state this dream in the following way: "We share with many authors the idea that operators and operands should be elements of the same space." To fulfill this dream and to find an algebra which shows a "proper conformity of the parts to one another and to the whole" (as Heisenberg [5] and Chandrasekhar [1] characterise mathematical beauty) it is tried in the following to use only (3 x 3) matrices to describe three-dimensional objects, operations, or geometrical situations.

It is surely more beautiful to represent all parts of a mathematical system by the same mathematical structures. And if it is not considered as more beautiful by aesthetically desillusioned pragmatists, it should at least be considered as more consistent, practical or convenient.

4 Interpreting (3 x 3) Matrices

The (3 x 3) matrix of eq. (4) exchanges the y- and z-coordinates of three-dimensional Euclidean space. Therefore this (3 x 3) matrix acts like a reflection at a plane which is spanned by the x-axis and the diagonal line between the y- and z-axis (see figure 1(b)). In a first approach it can be checked whether it is possible to identify this matrix with the corresponding plane in Geometric algebra.

In the following the (3 x 3) matrix representation will be given at the left side of the double sided arrow, while the standard Pauli matrix representation of Geometric algebra can be found at the right side of the double sided arrow:

$$e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \overset{??}{\longleftrightarrow} \frac{1}{\sqrt{2}}(\sigma_y + \sigma_z)\sigma_x = \frac{1}{\sqrt{2}}(\sigma_z\sigma_x - \sigma_x\sigma_y) \tag{5}$$

Please note the question marks, because a problem arises. The square of bivectors or of linear combinations of bivectors in three-dimensional space of Geometric algebra is negative. An evaluation of the right side of the double sided arrow of eq, (5) consequently gives

$$\left(2^{-0.5}(\sigma_z\sigma_x - \sigma_x\sigma_y)\right)^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{6}$$

while the square of matrix e_1 at the left side of the double sided arrow is positive:

$$e_1^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \neq \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \tag{7}$$

This leads to the conclusion that we have to identify the dual of e_1 (which will be called E_1) with the considered plane of eq. (5):

$$E_1 = ie_1 = \begin{pmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix} \longleftrightarrow \frac{1}{\sqrt{2}}(\sigma_y + \sigma_z)\sigma_x = \frac{1}{\sqrt{2}}(\sigma_z\sigma_x - \sigma_x\sigma_y) \tag{8}$$

In this way we have identified a (3 x 3) matrix on the left side of the double sided arrow with a (2 x 2) matrix on the right side of the double sided arrow. It will be shown later that this (3 x 3) matrix E_1 indeed acts in the same way on a vector $r = x\sigma_x + y\sigma_y + z\sigma_z$ like the standard Geometric algebra reflection matrix of eq. (8), which exchanges the y-and z-coordinates:

$$r' = \frac{1}{\sqrt{2}}(\sigma_y + \sigma_z)\sigma_x(x\sigma_x + y\sigma_y + z\sigma_z)\frac{1}{\sqrt{2}}(\sigma_y + \sigma_z)\sigma_x = x\sigma_x + z\sigma_y + y\sigma_z \tag{9}$$

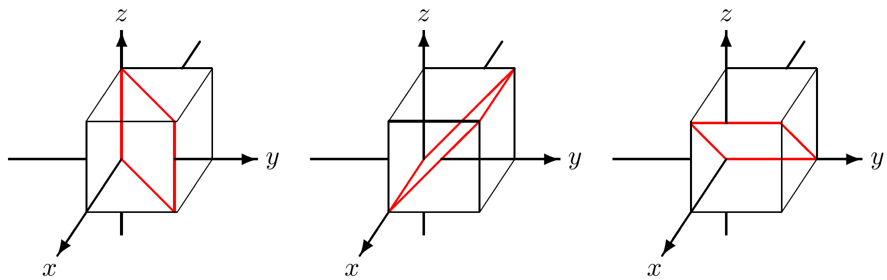
But first we have to find the (3 x 3) matrix equivalent of vector r.

The two other (3 x 3) matrices E_2 and E_3 which exchange two other coordinates in each case can be interpreted in a similar way. The (3 x 3) matrix E_2 exchanges the x- and z-coordinates. Therefore it can be identified with a plane which is spanned by the y-axis and the diagonal line between the x- and z-axis (see figure 1(c)):

$$E_2 = ie_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & i & 0 \\ i & 0 & 0 \end{pmatrix} \longleftrightarrow \frac{1}{\sqrt{2}}(\sigma_x + \sigma_z)\sigma_y \tag{10}$$

And the (3 x 3) matrix E_3 can be identified with a plane which is spanned by the z-axis and the diagonal line between the x- and y-axis (see figure 1(a)):

$$E_3 = ie_3 = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & i \end{pmatrix} \longleftrightarrow \frac{1}{\sqrt{2}}(\sigma_x + \sigma_y)\sigma_z \tag{11}$$



(a) $\sqrt{2}\,E_3$ (b) $\sqrt{2}\,E_1$ (c) $\sqrt{2}\,E_2$

Figure 1: Imaginary permutation matrices E_1, E_2, and E_3 represent planes.

And it is clear that the red area elements of figure 1 have surface areas of $\sqrt{2}$ times the unit area.

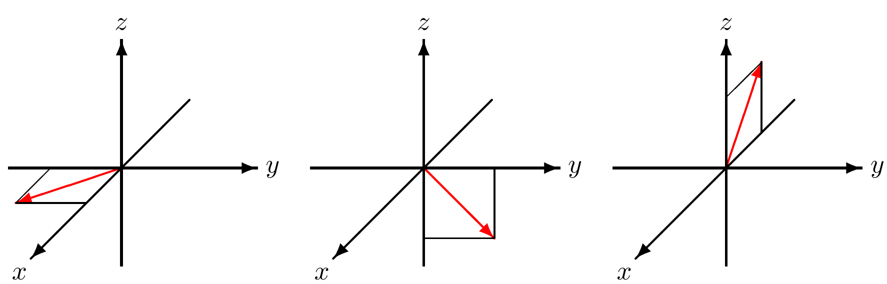
A multiplication by the imaginary unit i in matrix algebra can be considered as a multiplication by the volume element $\sigma_x\sigma_y\sigma_z$ in Geometric algebra. The three unit vectors e_1, e_2, and e_3 of Geometric algebra of quarks can thus be identified with the following standard Geometric algebra vectors:

$$e_2 = i^3 E_2 = \frac{E_2}{i} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \longleftrightarrow -\frac{1}{\sqrt{2}}(\sigma_x+\sigma_z)\sigma_y\,\sigma_x\sigma_y\sigma_z = \frac{1}{\sqrt{2}}(\sigma_z-\sigma_x) \tag{12}$$

$$e_3 = \frac{E_3}{i} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \longleftrightarrow \frac{1}{\sqrt{2}}(\sigma_x - \sigma_y) \tag{13}$$

$$e_1 = \frac{E_1}{i} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \longleftrightarrow \frac{1}{\sqrt{2}}(\sigma_y - \sigma_z) \tag{14}$$

In this way (3 x 3) matrices can be identified with vectors. This is an important message: S_3 **permutation matrices represent vectors.** These three vectors are shown in figure 2.



(a) $\sqrt{2}\,e_3$ (b) $\sqrt{2}\,e_1$ (c) $\sqrt{2}\,e_2$

Figure 2: Permutation matrices e_1, e_2, and e_3 represent vectors.

5 Nihilation Matrix and Identity

The three permutation vectors e_1, e_2, and e_3 are unit vectors because they square to one[1]. They are furthermore coplanar as e_1, e_2, and e_3 are located in the same plane. This has been tried to visualise in figure 3.

But figure 3 shows another important feature: The sum of the three vectors $e_1 + e_2 + e_3$ (see left picture of figure 3) or the double sum $2e_1 + 2e_2 + 2e_3$ (see right picture of figure 3) or every other multiple sum like $3(e_1 + e_2 + e_3)$ (see middle picture of figure 3) results in a vector of length zero. That is why we should identify the matrix of ones N geometrically **and** algebraically with the zero matrix O:

$$N = e_1 + e_2 + e_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \simeq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = O \tag{15}$$

This identification of N with zero is also justified when we compare the sum $e_1 + e_2 + e_3$ with its Geometric algebra counterpart by adding eq. (12), (13) and (14).

$$e_1 + e_2 + e_3 \simeq O \quad \longleftrightarrow \quad \frac{1}{\sqrt{2}}(\sigma_z - \sigma_x + \sigma_x - \sigma_y + \sigma_y - \sigma_z) = 0 \tag{16}$$

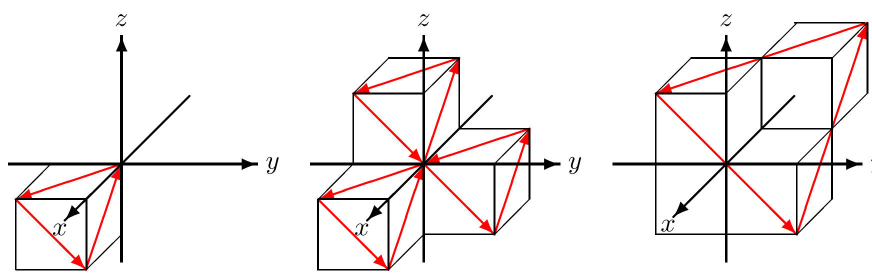


Figure 3: Some attempts to visualise that the vectors e_1, e_2, and e_3 lie in the same plane.

In the same way the sum of all three imaginary permutation matrices E_1, E_2, and E_3 which represent unit area elements has to be identified with nothingness, nihilation, null or zero.

$$\begin{aligned} E_1 + E_2 + E_3 = \begin{pmatrix} i & i & i \\ i & i & i \\ i & i & i \end{pmatrix} = iO \simeq N \\ \longleftrightarrow \quad \frac{1}{\sqrt{2}}(\sigma_x\sigma_y + \sigma_z\sigma_y + \sigma_y\sigma_z + \sigma_x\sigma_z + \sigma_z\sigma_x + \sigma_y\sigma_x) = 0 \end{aligned} \tag{17}$$

In the literature the matrix of ones is sometimes called unit matrix (see eg. [16]), which is rather confusing. The matrix of ones is not the identity matrix. And sometimes the matrix of ones is called democratic matrix (see eg. [2]), which seems even more confusing and hides the structural meaning of N. If a (3 x 3) matrix is multiplied with N in Geometric algebra of quarks,

[1] In a world with positive numbers only, it makes sense to call them base vectors, because they form a minimal set of vectors spanning a plane, see [7]. It is not possible to reach every point of a plane when there are just two base vectors with only positive coordinates.

it will be nihilated und becomes zero. Thus we have indefinitely many representations of matrices meaning zero. For example there are (with $r \in \mathbb{C}$):

$$N = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \simeq \begin{pmatrix} i & i & i \\ i & i & i \\ i & i & i \end{pmatrix} \simeq \begin{pmatrix} r & r & r \\ r & r & r \\ r & r & r \end{pmatrix} \simeq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = O \tag{18}$$

Hence every other (3 x 3) matrix possess indefinitely many representations too. The matrix Z_2' given by Dev, Gautam & Singh in [2, eq. (16)]

$$Z_2' = \frac{1}{3}\begin{pmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} -2 & -2 & -2 \\ -2 & -2 & -2 \\ -2 & -2 & -2 \end{pmatrix} \simeq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{19}$$

is just another representation of the identity matrix. Therefore it is obvious that every mathematical structure should be invariant under Z_2' in Geometric algebra of quarks.

And every vector r can be written in different ways:

$$\begin{aligned} r = x_1e_1 + x_2e_2 + x_3e_3 &\simeq (x_1 - x_3)e_1 + (x_2 - x_3)e_2 \\ &\simeq (x_2 - x_1)e_2 + (x_3 - x_1)e_3 \\ &\simeq (x_3 - x_2)e_3 + (x_1 - x_2)e_1 \\ &\longleftrightarrow \quad \frac{1}{\sqrt{2}}\left[(x_2 - x_1)\sigma_x + (x_3 - x_2)\sigma_y + (x_1 - x_3)\sigma_z\right] \end{aligned} \tag{20}$$

Therefore it is always possible to find a fundamental expression of vector r with only two unit vectors e_i and purely positive coefficients. For example, if $x_3 \geq x_2 \geq x_1$ then it would make sense to use the second line of eq. (20) as the two coefficients are greater than or equal to zero.

Although it seems that we are living in a three-dimensional space with x-, y- and z-axes as shown in the previous figures, till now we are not able to reach points outside the plane indicated in figure 3. We are frozen in this plane. Every point we can reach till now is considered to be a linear combination of the three vectors e_1, e_2, and e_3. To reach points outside this plane it is crucial to identify a vector perpendicular to the S_3-plane.

6 Products of Permutation Matrices

The following products of permutation matrices exist:

$$e_0 := e_1^2 = e_2^2 = e_3^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \longleftrightarrow \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \tag{21}$$

$$\ominus := E_1^2 = E_2^2 = E_3^2 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad \longleftrightarrow \quad -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -1 \tag{22}$$

$$e_{12} := e_1e_2 = e_2e_3 = e_3e_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \longleftrightarrow \quad \frac{1}{2}(-1 + \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x) \tag{23}$$

$$e_{21} := e_2e_1 = e_3e_2 = e_1e_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \longleftrightarrow \quad \frac{1}{2}(-1 - \sigma_x\sigma_y - \sigma_y\sigma_z - \sigma_z\sigma_x) \tag{24}$$

$$E_{12} := E_1E_2 = E_2E_3 = E_3E_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \longleftrightarrow \frac{1}{2}(1 - \sigma_x\sigma_y - \sigma_y\sigma_z - \sigma_z\sigma_x) \tag{25}$$

$$E_{21} := E_2E_1 = E_3E_2 = E_1E_3 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \longleftrightarrow \frac{1}{2}(1 + \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x) \tag{26}$$

These matrix products are geometric products. They thus bear geometrical meaning. The entities of eq. (21) and (22) can be identified with scalars. The entities of eq. (23) to (26) can be identified as linear combinations of a scalar and bivectors. The trivector or pseudoscalar can be found by the following permutation matrix multiplications:

$$I := E_1e_1 = E_2e_2 = E_3e_3 = \begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{pmatrix} \longleftrightarrow \sigma_x\sigma_y\sigma_z \tag{27}$$

If we restrict ourselves to the plane of figure 3, we can do everything in this plane using (3 x 3) matrices of Geometric algebra of quarks what we are able to do with (2 x 2) Pauli matrices in conventional Geoetric algebra in a plane. This is important! **(2 x 2) matrices can be thought as and seen as (3 x 3) permutation matrices**. So it is no mathematical question, which system we use, it is instead a didactical question.

7 The Philosophy of Negative Numbers

As indicated in section 1 minus signs are avoided in this paper. Instead of this algebraic sign "–" the geometric entity

$$e_{12} + e_{21} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \longleftrightarrow -1 \tag{28}$$

gives us a (3 x 3) matrix which does all that a minus sign usually does. Algebraically $e_{12} + e_{21}$ reduces every scalar ke_0 by one unit:

$$ke_0 + e_{12} + e_{21} = (k-1)e_0 + \underbrace{e_0 + e_{12} + e_{21}}_{N} \simeq (k-1)e_0 \tag{29}$$

But at the same time the matrix $e_{12} + e_{21}$ has a clear geometric meaning: It reverses the direction of vectors:

$$(e_{12} + e_{21})(x_1e_1 + x_2e_2 + x_3e_3) = x_1(e_2 + e_3) + x_2(e_3 + e_1) + x_3(e_1 + e_2) \tag{30}$$

This is indeed a complete reversion as for example the unit vector $e_2 + e_3$ is parallel to the unit vector e_1, but shows into the opposite direction.

Therefore this matrix $(e_{12} + e_{21})$ is called $\ominus$ in this paper, using the \ominus symbol of Latex like it is done in eq. (22). So $\ominus\, e_0$ is no multiplication of a negative sign with the scalar 1, but a matrix multiplication resulting in $\ominus\, e_0 = \ominus$. This avoidance of minus signs indicates that we might live in a mathematically purely positive world.

In this world negative entities do not exist. We just reverse directions. And sometimes we do not totally reverse a direction but change the direction only a little bit. This might have epistemological and ontological consequences for our physical world too. Do we really measure negative entities anywhere in physics? Or do we only measure positive entities in different or sometimes in opposite directions? The possibility of avoiding the minus sign might indicate that we not only might live in a mathematically positive world, but that we might live in a world which can be decribed in physics as a purely positive world too.

And as we actually speak about something like "reality" it is even possible that the world of physics not only can be but even must be described as purley positve, to understand it conceptully as "The Road to Reality" (see discussion in [11, §3.5]) is a mathematical road[2].

But whatever our ontological and epistemological positions are: We have reached here the true heart of Geometric algebra: $\ominus$ can be interpreted as algebraic and as well as geometric operation. Algebra and geometry are deeply connected now. We live in both worlds: in the world of algebra and in the world of geometry. And as we can transfer from algebra to geometry and back to algebra at every moment, these worlds coalesce structurally.

8 Constructing e_4

After having found an appropriate entity to describe negativities in geometric algebra of quarks we are able to split up the geometric product of two vectors $r_1 r_2$ into a dot product and a wedge product.

$$r_1 r_2 = r_1 \cdot r_2 + r_1 \wedge r_2 \tag{31}$$

The dot product results in a scalar

$$r_1 \cdot r_2 = \frac{1}{2}(r_1 r_2 + r_2 r_1) \tag{32}$$

and is connected with the cosine of the angle α between the two vectors r_1 and r_2:

$$\cos\alpha = \hat{r}_1 \cdot \hat{r}_2 = \frac{1}{2}\left(\hat{r}_1 \hat{r}_2 + \hat{r}_2 \hat{r}_1\right) \tag{33}$$

where $\hat{r}$ is the unit vector of $r = x_1 e_1 + x_2 e_2 + x_3 e_3$:

$$\hat{r} = \frac{r}{|r|} \tag{34}$$

with

$$|r| = \sqrt{r^2} = \sqrt{x_1^2 + x_2^2 + x_3^2 - x_1 x_2 - x_2 x_3 - x_3 x_1} \tag{35}$$

As indicated in figure 3 the angles between all unit vectors e_1, e_2, and e_3 indeed equal $2\pi/3$:

$$\cos\alpha = e_1 \cdot e_2 = e_2 \cdot e_3 = e_3 \cdot e_1 = \frac{1}{2}(e_1 e_2 + e_2 e_1) = \ominus\frac{1}{2}e_0 \quad\longleftrightarrow\quad -\frac{1}{2} \tag{36}$$

It surely makes sense to identify the arccosine of this expression with

$$\alpha = \arccos\left(\ominus\frac{1}{2}\right) \simeq \frac{2\pi}{3} = 120^\circ \tag{37}$$

Now the wedge product is defined as:

$$r_1 \wedge r_2 = \frac{1}{2}(r_1 r_2 + \ominus r_2 r_1) \tag{38}$$

[2] At the beginning of this discussion Penrose writes: "I think that it is clear that, unlike the natural numbers, there is no evident physical content to the notion of a negative number of physical objects" [11, p. 65]. But he later on revises this position slightly.

Thus we get an expression for a bivector representing the plane A_{S_3} in which the unit vectors e_1, e_2, and e_3 are situated:

$$\begin{aligned} A_{S_3} :&= e_1 \wedge e_2 = e_2 \wedge e_3 = e_3 \wedge e_1 = \frac{1}{2}(e_1 e_2 + \ominus e_2 e_1) \\ &= \frac{1}{2}e_0 + e_{12} = \begin{pmatrix} \frac{1}{2} & 0 & 1 \\ 1 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} \end{pmatrix} \quad \longleftrightarrow \quad \frac{1}{2}(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x) \end{aligned} \tag{39}$$

The magnitude of this area element is

$$\begin{aligned} |A_{S_3}| =& \sqrt{\ominus\left(e_{12} + \frac{1}{2}e_0\right)^2} = \sqrt{\ominus\left(\frac{1}{4}e_0 + e_{12} + e_{21}\right)} \\ \simeq& \sqrt{\ominus\frac{3}{4}\left(e_{12} + e_{21}\right)} = \sqrt{\ominus^2\frac{3}{4}} \simeq \sqrt{\frac{3}{4}e_0} = \frac{1}{2}\sqrt{3} \end{aligned} \tag{40}$$

Therefore the unit area element E_4 which is perpendicular to the wanted unit vector e_4 equals

$$E_4 = \frac{A_{S_3}}{|A_{S_3}|} = \frac{1}{\sqrt{3}}(e_0 + 2e_{12}) = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \quad \longleftrightarrow \quad \frac{1}{\sqrt{3}}(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x) \tag{41}$$

By analogy to eq. (12), (13), or (14) the unit vector e_4 perpendicular to all other unit vectors e_1, e_2, and e_4 can be found as

$$e_4 = \ominus i E_4 = \frac{1}{\sqrt{3}}i(e_0 + 2e_{21}) = \frac{1}{\sqrt{3}}\begin{pmatrix} i & 2i & 0 \\ 0 & i & 2i \\ 2i & 0 & i \end{pmatrix} \quad \longleftrightarrow \quad \frac{1}{\sqrt{3}}(\sigma_x + \sigma_y + \sigma_z) \tag{42}$$

As expected e_4 is a unit vector:

$$\begin{aligned} e_4^2 &= \frac{1}{3}\begin{pmatrix} i & 2i & 0 \\ 0 & i & 2i \\ 2i & 0 & i \end{pmatrix}^2 \simeq \frac{1}{3}\begin{pmatrix} i+2 & 2i+2 & 2 \\ 2 & i+2 & 2i+2 \\ 2i+2 & 2 & i+2 \end{pmatrix}^2 \\ &= \frac{1}{3}\begin{pmatrix} 12i+11 & 12i+8 & 12i+8 \\ 12i+8 & 12i+11 & 12i+8 \\ 12i+8 & 12i+8 & 12i+11 \end{pmatrix} \simeq \frac{1}{3}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned} \tag{43}$$

Or written in vector notation instead of matrix notation:

$$\begin{aligned} e_4^2 = \left(\frac{1}{\sqrt{3}}i(e_0 + 2e_{21})\right)^2 &= \frac{1}{3}(e_{12} + e_{21})(3e_{12} + 3e_{21}) \simeq e_0 \\ \longleftrightarrow \quad &\left(\frac{1}{\sqrt{3}}(\sigma_x + \sigma_y + \sigma_z)\right)^2 = 1 \end{aligned} \tag{44}$$

And e_4 is perpendicular to all other unit vectors:

$$\begin{aligned} \cos\alpha_{14} &= e_1 \cdot e_4 = e_2 \cdot e_4 = e_3 \cdot e_4 = \frac{1}{2}(e_1 e_4 + e_4 e_1) \\ &= \frac{1}{2\sqrt{3}}i\left(2e_1 + 2e_2 + 2e_3\right) = \frac{1}{\sqrt{3}}iN \simeq 0 \quad \longleftrightarrow \quad \cos\alpha_{14} = 0 \end{aligned} \tag{45}$$

$$\Rightarrow \qquad \alpha_{14} = \frac{\pi}{2} = 90^\circ \tag{46}$$

Therefore the products of e_4 with the other unit vectors are:

$$
\begin{aligned}
e_{14} := e_1 e_4 = \frac{1}{\sqrt{3}} i(e_1 + 2e_3) = \frac{1}{\sqrt{3}} \begin{pmatrix} i & 2i & 0 \\ 2i & 0 & i \\ 0 & i & 2i \end{pmatrix} = \ominus\, e_4 e_1 \\
\longleftrightarrow \quad \frac{1}{\sqrt{6}}(-\sigma_x\sigma_y + 2\sigma_y\sigma_z - \sigma_z\sigma_x)
\end{aligned}
\tag{47}
$$

$$
\begin{aligned}
e_{41} := e_4 e_1 = \frac{1}{\sqrt{3}} i(e_1 + 2e_2) = \frac{1}{\sqrt{3}} \begin{pmatrix} i & 0 & 2i \\ 0 & 2i & i \\ 2i & i & 0 \end{pmatrix} = \ominus\, e_1 e_4 \\
\longleftrightarrow \quad \frac{1}{\sqrt{6}}(\sigma_x\sigma_y - 2\sigma_y\sigma_z + \sigma_z\sigma_x)
\end{aligned}
\tag{48}
$$

$$
\begin{aligned}
e_{24} := e_2 e_4 = \frac{1}{\sqrt{3}} i(e_2 + 2e_1) = \frac{1}{\sqrt{3}} \begin{pmatrix} 2i & 0 & i \\ 0 & i & 2i \\ i & 2i & 0 \end{pmatrix} = \ominus\, e_4 e_2 \\
\longleftrightarrow \quad \frac{1}{\sqrt{6}}(-\sigma_x\sigma_y - \sigma_y\sigma_z + 2\sigma_z\sigma_x)
\end{aligned}
\tag{49}
$$

$$
\begin{aligned}
e_{42} := e_4 e_2 = \frac{1}{\sqrt{3}} i(e_2 + 2e_3) = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 2i & i \\ 2i & i & 0 \\ i & 0 & 2i \end{pmatrix} = \ominus\, e_2 e_4 \\
\longleftrightarrow \quad \frac{1}{\sqrt{6}}(\sigma_x\sigma_y + \sigma_y\sigma_z - 2\sigma_z\sigma_x)
\end{aligned}
\tag{50}
$$

$$
\begin{aligned}
e_{34} := e_3 e_4 = \frac{1}{\sqrt{3}} i(e_3 + 2e_2) = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & i & 2i \\ i & 2i & 0 \\ 2i & 0 & i \end{pmatrix} = \ominus\, e_4 e_3 \\
\longleftrightarrow \quad \frac{1}{\sqrt{6}}(2\sigma_x\sigma_y - \sigma_y\sigma_z - \sigma_z\sigma_x)
\end{aligned}
\tag{51}
$$

$$
\begin{aligned}
e_{43} := e_4 e_3 = \frac{1}{\sqrt{3}} i(e_3 + 2e_1) = \frac{1}{\sqrt{3}} \begin{pmatrix} 2i & i & 0 \\ i & 0 & 2i \\ 0 & 2i & i \end{pmatrix} = \ominus\, e_3 e_4 \\
\longleftrightarrow \quad \frac{1}{\sqrt{6}}(-2\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x)
\end{aligned}
\tag{52}
$$

Reflecting the unit vectors e_1, e_2, e_3 or e_4 at each other then results in:

$$e_1e_1e_1 = e_1 \qquad e_2e_1e_2 = e_3 \qquad e_3e_1e_3 = e_2 \qquad e_4e_1e_4 = e_2 + e_3 \tag{53}$$
$$e_1e_2e_1 = e_3 \qquad e_2e_2e_2 = e_2 \qquad e_3e_2e_3 = e_1 \qquad e_4e_2e_4 = e_3 + e_1 \tag{54}$$
$$e_1e_3e_1 = e_2 \qquad e_2e_3e_2 = e_1 \qquad e_3e_3e_3 = e_3 \qquad e_4e_3e_4 = e_1 + e_2 \tag{55}$$
$$e_1e_4e_1 = \ominus e_4 \qquad e_2e_4e_2 = \ominus e_4 \qquad e_3e_4e_3 = \ominus e_4 \qquad e_4e_4e_4 = e_4 \tag{56}$$

9 Pauli Matrices

In a last step to reach a full identification of Pauli matrices with (3 x 3)-matrices explicit formulae for them can be found using eq. (12), (13), (14), and (42):

$$e_x = \frac{1}{3}\left(\sqrt{2}(e_1 + 2e_3) + \sqrt{3}e_4\right) \quad \longleftrightarrow \quad \sigma_x \tag{57}$$

$$e_y = \frac{1}{3}\left(\sqrt{2}(e_2 + 2e_1) + \sqrt{3}e_4\right) \quad \longleftrightarrow \quad \sigma_y \tag{58}$$

$$e_z = \frac{1}{3}\left(\sqrt{2}(e_3 + 2e_2) + \sqrt{3}e_4\right) \quad \longleftrightarrow \quad \sigma_z \tag{59}$$

Reflections of these (3 x 3) Pauli vectors e_x, e_y, and e_z at unit vector e_1 then are:

$$e_1 e_x e_1 = \frac{1}{3}\left(\sqrt{2}(e_1 + 2e_2) + \ominus\sqrt{3}e_4\right) = \ominus e_x \quad \longleftrightarrow \quad -\sigma_x \tag{60}$$

$$e_1 e_y e_1 = \frac{1}{3}\left(\sqrt{2}(e_3 + 2e_1) + \ominus\sqrt{3}e_4\right) = \ominus e_z \quad \longleftrightarrow \quad -\sigma_z \tag{61}$$

$$e_1 e_z e_1 = \frac{1}{3}\left(\sqrt{2}(e_2 + 2e_3) + \ominus\sqrt{3}e_4\right) = \ominus e_y \quad \longleftrightarrow \quad -\sigma_y \tag{62}$$

or

$$\ominus e_1 e_x e_1 = \frac{1}{3}\left(\sqrt{2}(e_2 + e_3 + 2e_1 + 2e_3) + \sqrt{3}e_4\right) \tag{63}$$

$$\simeq \frac{1}{3}\left(\sqrt{2}(e_1 + 2e_3) + \sqrt{3}e_4\right) = e_x \quad \longleftrightarrow \quad \sigma_x \tag{64}$$

$$\ominus e_1 e_y e_1 = \frac{1}{3}\left(\sqrt{2}(e_1 + e_2 + 2e_2 + 2e_3) + \sqrt{3}e_4\right) \tag{65}$$

$$\simeq \frac{1}{3}\left(\sqrt{2}(e_3 + 2e_2) + \sqrt{3}e_4\right) = e_z \quad \longleftrightarrow \quad \sigma_z \tag{66}$$

$$\ominus e_1 e_z e_1 = \frac{1}{3}\left(\sqrt{2}(e_1 + e_3 + 2e_1 + 2e_2) + \sqrt{3}e_4\right) \tag{67}$$

$$\simeq \frac{1}{3}\left(\sqrt{2}(e_2 + 2e_1) + \sqrt{3}e_4\right) = e_y \quad \longleftrightarrow \quad \sigma_y \tag{68}$$

Now we can construct a (3 x 3) matrix expression which is equivalent to eq. (9). A reflection of vector

$$r = xe_x + ye_y + ze_z \quad \longleftrightarrow \quad x\sigma_x + y\sigma_y + z\sigma_z \tag{69}$$

at plane $E_1 = ie_1$ is given by

$$\begin{aligned} r' &= ie_1(xe_x + ye_y + ze_z)ie_1 \\ &= \ominus x\, e_1 e_x e_1 + \ominus y\, e_1 e_y e_1 + \ominus z\, e_1 e_z e_1 \\ &= x\, e_x + z\, e_y + y\, e_z \quad \longleftrightarrow \quad x\sigma_x + z\sigma_y + y\sigma_z \end{aligned} \tag{70}$$

and exchangs indeed the y- and z-coordinates. This and similar equations for reflections at plane E_2 and E_3 show the inherent linkage between a Geometric algebra of (3 x 3) matrices and permutation symmetry S_3. And as S_3 seems to describe important features of flavour symmetry I hope that this will indeed help us to understand quarks and neutrions one day in a geometrically convincing manner.

And again: we are able do everything in three-dimensional Euclidean space using (3 x 3) matrices of Geometric algebra of quarks what we are able to do with (2 x 2) Pauli matrices in conventional Geometric algebra. It is no mathematical question, which system we use, it is a didactical one.

10 Epilog

This AGACSE paper has been reviewed by two reviewers whom I wish to thank for their very constructive and helpful remarks. But there are two comments I do not agree with, and I think this should be discussed openly.

First of all one of the reviewers wrote: "Please avoid the use of words like crime, ugly and ugliness in a scientific papers." I want to clarify why I didn't follow this proposition. As I am a physics teacher and a physics education researcher my daily work is to analyse the process of physics and mathematics learning. Categories like 'beauty' or 'ugliness' consciously and unconciously influence learning processes [9]. Not only students but we all evaluate ideas and concepts according to such categories – even if we do not speak about them.

But not to speak about them does not mean that theses categories are not there. Scientific research is learning too: We learn something new about nature, and we do that on the basis of our preconceptions. These preconsecptions about whether an idea in physics or mathematics is beautiful or ugly are important features of our understanding of a subject. The more we learn about a subject the more we care about its inherent beauty or bother about its inherent lack of beauty.

This care about beauty even is a sign of professionality, and Dirac once explained: "With increasing knowledge of a subject, when one has a great deal of support to work from, one can go over more and more towards the mathematical procedure. One then has as one's underlying motivation the striving for mathematical beauty. Theoretical physicists accept the need for mathematical beauty as an act of faith. There is no compelling reason for it, but it has proved a very profitable objective in the past" [3, p. 21]. Therefore the statement that the equation

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a & c & b \\ c & b & a \\ b & a & c \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix} \tag{71}$$

is much more beautiful than equation (3) is important for me. And it directly adresses the main point of this paper, as another comment of the reviewers shows.

This comment was: "Note that the (3 x 3) matrices 'describing' quarks are the 8 infinitesimal generators of SU(3). One representation are the Gell-Mann matrices. They act on columns of 3 spinors, the spinors represent quarks." This describes the standard procedure given in standard physics books like [4, p. 51/52].

To find a unified Geometric algebra picture of quarks it might be helpful to use a representation of space by (3 x 3) matrices instead of (2 x 2) Pauli matrices. But in addition to that it seems inevitable to get rid of these columns of spinors used today (which are in my eyes as ugly as eq. (3)) and to construct (3 x 3) matrices of spinors similar to the second matrix of eq. (71).

There exist more ideas how to construct a Geometric algebra picture of quarks in the literature. For example Hestenes [6], Schmeikal [14] or Keller [8, chap. 4.6] present some of these. But it seems that all these ideas go into more or less different directions, and we are still in need of a really unified Geometric algebra picture of quarks.

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Submission History of This Paper

This is the paper of my poster presentation at the 5th conference on Applied Geometric Algebras in Computer Science and Engineering (AGACSE), which took place from July 2nd till July 4th, 2012 at the University of La Rochelle in France. It was part of the USB flash drive distributed at this conference by the organizers, see http://agacse2012.univ-lr.fr

At October 2nd, 2012 I uploaded this paper at the preprint archive of Cornell University (www.arxiv.org). The next day, October 3rd, 2012 the arxiv administration rejected it and told me via email:

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Therefore I decided to upload this paper at www.vixra.org.

M.E.H.