Entanglement Dynamics : Application to Quantum Systems

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Dedicated to Marie-Louise Nykamp

Abstract

For the first time in known literature, one studies *entanglement dynamics* which is the way the complexity of entanglement may change in time, for instance, in the solution of a Schr/"odinger equation giving the state of a composite quantum system. The paper is a preliminary study which gives the rigorous definition of the respective general mathematical model. Applications to effective Schr/"odinger equations are given in a subsequent paper.

"History is written with the feet ..."

Ex-Chairman Mao, of the Long March fame ...

"Of all things, good sense is the most fairly distributed : everyone thinks he is so well supplied with it that even those who are the hardest to satisfy in every other respect never desire more of it than they already have." :-) :-) :-)

R Descartes, Discourse de la Méthode

"Creativity often consists of finding hidden assumptions. And removing those assumptions can open up a new set of possibilities ..."

Henry R Sturman

"Science is not done scientifically, since it is mostly done by non-scientists ..."

Anonymous

"Science is nowadays not done scientifically, since it is mostly done by ... scientists ..."

Anonymous

"Physics is too important to be left only to physicists ..."

Anonymous

"Is the claim about the validity of the so called 'physical intuition' but a present day version of medieval claims about the sacro-sanct validity of theological revelations ?"

Anonymous

"A physical understanding is a completely unmathematical, imprecise, and inexact thing, but absolutely necessary for a physicist ..."

R. Feynman

1. Preliminaries

Recently, [1, 2], a non-negative integer valued *grading* function was considered on tensor products in order to distinguish between non-entangled and entangled elements. The essential property of this grading function is that it gives the *minimally* entangled expression for all entangled elements in a tensor product. A main interest in such a minimal entanglement is in the study of the *variation* of that minimum when the respective elements are *time dependent*, like for instance, when they evolve according to a corresponding Schrödinger equation.

In [2], a brief mention of such a *dynamics of entanglement* was made, based on earlier unpublished work of the present author. Here, some of the related details are now presented.

For convenience, first we recall here briefly the way this grading function classifies entangled elements. Namely, the larger the grade of such an element, the higher the extent to which it is entangled, and of course, the other way round. In essence, this is done as follows. Let X and Y be two vector spaces over a field $\mathbb{K}$, then we define

(1.1) $\quad gr : X \bigotimes Y \longrightarrow \mathbb{N}$

where for $u \in X \bigotimes Y$, we have

(1.2) $\quad gr(u) = \min\{\, n \mid u = \sum_{i=1}^{n} x_i \otimes y_i, \quad x_i \in X, \quad y_i \in Y \,\}$

with the convention that $gr(0 \otimes 0) = 0$.

One of the relevant results is that, given $u = \sum_{i=1}^{n} x_i \otimes y_i \in X \bigotimes Y$, then

(1.3) $\quad gr(u) = \min\{\, k, h \,\}$

where k and h are, respectively, the dimensions of the linear span of $\{x_1, \ldots, x_n\}$ in X, and of $\{y_1, \ldots, y_n\}$ in Y.

In particular, $u \in X \bigotimes Y$ is not entangled, if and only if $gr(u) \leq 1$.

Clearly, $gr(u)$ can be computed by well known methods in linear algebra, for instance, methods which give the rank of a matrix.

Also, if X and Y are finite dimensional, then for $u \in X \bigotimes Y$, we have

(1.4) $\quad gr(u) \leq \min\{dim\, X, dim\, Y\}$

A specific feature of the grade function (1.1) - (1.3) is that it is defined exclusively in terms of the respective tensor product $X \bigotimes Y$.

As for obtaining for a given

$$u = \sum_{i=1}^{n} x_i \otimes y_i \in X \bigotimes Y$$

a corresponding *minimum* representation

$$u = \sum_{j=1}^{m} u_j \otimes v_j \in X \bigotimes Y$$

where $m = gr(u) \leq n$, we have the following result, see [1].

Proposition 1.1.

Let X and Y be two vector spaces over a field $\mathbb{K}$, and let $u = \sum_{i=1}^{n} x_i \otimes y_i \in X \bigotimes Y$. If

(1.5) $\quad gr(u) = m < n$,

(1.6) $\quad$ the dimension of the linear span of $\{x_1, \ldots, x_n\}$ is m, and it is less or equal with the dimension of the linear span of $\{y_1, \ldots, y_n\}$,

(1.7) $\quad \{x_1, \ldots, x_m\}$ are linearly independent

then

(1.8) $\quad u = \sum_{i=1}^{m} x_i \otimes v_i$

where

(1.9) $\quad \{v_1, \ldots, v_m\}$ is linearly independent, and it is contained in the linear span of $\{y_1, \ldots, y_n\}$

Furthermore, as seen next in the Proof, one can obtain an *explicit* expression for the linearly independent vectors $\{v_1, \ldots, v_m\}$, as seen in (1.10) below.

Proof.

In view of (1.6), (1.7), we have

$$x_j = \sum_{i=1}^{m} \mu_{j,i} \, x_i, \qquad m < j \leq n$$

where $\mu_{j,i} \in \mathbb{K}$. Hence

$$u = \sum_{i=1}^{m} x_i \otimes y_i + \sum_{j=m+1}^{n} \sum_{i=1}^{m} \mu_{j,i} \, x_i \otimes y_j =$$

$$= \sum_{i=1}^{m} x_i \otimes y_i + \sum_{i=1}^{m} \sum_{j=m+1}^{n} \mu_{j,\,i}\, x_i \otimes y_j =$$

$$= \sum_{i=1}^{m} x_i \otimes (y_i + \sum_{j=m+1}^{n} \mu_{j,\,i}\, y_j)$$

Consequently

$$(1.10) \qquad v_i = y_i + \sum_{j=m+1}^{n} \mu_{j,\,i}\, y_j, \qquad 1 \leq i \leq m$$

and $\{v_1, \ldots, v_m\}$ must be linearly independent in view of (1.8), (1.5). □

In this paper the above grading function will be applied to the study of the dynamics of composite quantum systems. Namely, let X, Y be complex Hilbert spaces and let S be a quantum system with the state space $X \bigotimes Y$. Then its evolution is given by a one parameter family of *unitary* operators $U(t)$, with $t \in [0, \infty)$, where

$$(1.11) \qquad X \bigotimes Y \ni |\, \psi > \longmapsto U(t)(\, |\, \psi >\,) \in X \bigotimes Y$$

Namely, given any preparation $|\, \psi_0 >$ of the system S at time $t = 0$, then the state of the system at a time moment $t \geq 0$ will be

$$(1.12) \qquad |\, \psi_t > \;=\; U(t)(\, |\, \psi_0 >\,)$$

The *problem* under study in this paper is as follows. We obviously have

$$(1.13) \qquad |\, \psi_0 > \;=\; \sum_{i=1}^{n(0)} x_i(0) \otimes y_i(0) \in X \bigotimes Y$$

while, for $t \geq 0$, we shall have

$$(1.14) \qquad |\, \psi_t > \;=\; U(t)(\, |\, \psi_0 >\,) = \sum_{i=1}^{n(t)} x_i(t) \otimes y_i(t) \in X \bigotimes Y$$

Thus in general

- the state $|\, \psi_t >$ of the composite system S at any moment of time $t \geq 0$ may be *entangled*
- the extent of the entanglement may *vary* from one moment of time to another

We therefore intend to study this variation in the extent of entanglement, and do so with the help of the grading function gr.

2. An Simple Instance of Possible Entanglement Dynamics

We recall that the evolution of quantum systems which are not subject to measurement is supposed to take place according to the Schrödinger equation. In other words, the state $|\psi>$ of a quantum system - a state which is a vector in a suitable Hilbert space H, and which is a square integrable function on a corresponding configuration space given by a finite dimensional Euclidian space E - satisfies a linear partial differential equation, namely the Schrödinger equation, in which the independent variables are the time $t \in \mathbb{R}$, and the coordinates $x \in E$ of that configuration space.

Our interest here being in *entanglement dynamics*, see its definition at the end of this section, we focus on composite quantum systems which, therefore, have their state space given by suitable tensor products.

In view of the above, it will help first to have a look at the following entanglement dynamics. Let $(X, || \ ||), (Y, || \ ||)$ be two Banach spaces over a field $\mathbb{K}$. In particular, they can be finite dimensional Euclidean spaces. We consider ODEs of the form

$$(2.1) \qquad dF(t)/dt = A(F(t)), \quad t \in [0, \infty)$$

where

$$(2.2) \qquad [0, \infty) \ni t \longmapsto F(t) \in X \bigotimes Y$$

while

$$(2.3) \qquad A : X \bigotimes Y \longrightarrow X \bigotimes Y$$

The problem is that, in terms of X and Y, the solution of (2.1) - (2.3) will in general be of the form

(2.4) $\quad F(t) = x_1(t) \otimes y_1(t) + \ldots + x_{n(t)}(t) \otimes y_{n(t)}(t)$

And it is quite likely that $x_i(t) \in X,\ y_i(t) \in Y$ and $n(t) \in \mathbb{N}$ do indeed depend on t. Thus the situation is of considerable difficulty, since (2.4) means that the ODE in (2.1) - (2.3), when considered in terms of X and Y, will have a *variable* number of unknowns and equations. Furthermore, the representation of the solution $F(t)$ in (2.4) is not unique.

Of course, when instead of (2.1) - (2.4), we have the classical case of

(2.5) $\quad [0, \infty) \ni t \longmapsto F(t) \in X \times Y$

then instead of (2.4) we have the trivial form of solution, namely

(2.6) $\quad F(t) = (x(t), y(t)) \in X \times Y$

and thus we simply have a usual system of two ODEs in $X \times Y$.

In view of the above, it is natural to introduce

Definition 2.1.

We call *entanglement dynamics* the situation when given a regular enough, for instance, continuous mapping

(2.7) $\quad \mathbb{R} \ni t \longmapsto F(t) = x_1(t) \otimes y_1(t) + \ldots + x_{n(t)}(t) \otimes y_{n(t)}(t) \in X \bigotimes Y$

where

(2.8) $\quad gr(F(t)) = n(t), \qquad t \in \mathbb{R}$

there may occur a variation in $n(t)$, as t ranges over $\mathbb{R}$.

3. An Example

Let us consider a simple example of (2.1) - (2.4). Let X, Y be Euclidean spaces. Given $a, b \in X \bigotimes Y$, we define the *infinite straight* *line* between a and b, namely

$$(3.1) \qquad \mathbb{R} \ni t \longmapsto F(t) \in X \bigotimes Y$$

by

$$(3.2) \qquad F(t) = (1-t)a + tb, \qquad t \in \mathbb{R}$$

thus

$$(3.3) \qquad F(0) = a, \qquad F(1) = b$$

and $F(t)$ obviously satisfies the following ODE in $X \bigotimes Y$, namely

$$(3.4) \qquad dF(t)/dt = A(F(t)), \quad t \in \mathbb{R}$$

where A is the constant mapping

$$(3.5) \qquad A : X \bigotimes Y \ni u \longmapsto A(u) = b - a \in X \bigotimes Y$$

Let us assume now that

$$(3.6) \qquad a = \sum_{i=1}^{n} x_i \otimes y_i, \qquad b = \sum_{j=1}^{m} w_j \otimes z_j$$

then (3.2) gives

$$(3.7) \qquad F(t) = \sum_{i=1}^{n} (1-t)x_i \otimes y_i + \sum_{j=1}^{m} tw_j \otimes z_j, \qquad t \in \mathbb{R}$$

thus in view of (1.3) we have

$$(3.8) \qquad F(t) = \sum_{q=1}^{p(t)} c_q(t) \otimes d_q(t), \qquad t \in \mathbb{R}$$

where

$$(3.9) \qquad p(t) = min\{\, k(t), h \,\}$$

with $k(t)$ and h being, respectively, the dimension of the linear span of $\{(1-t)x_1, \ldots, (1-t)x_n\} \cup \{tw_1, \ldots, tw_m\}$ in X, and of $\{y_1, \ldots, y_n\} \cup$

$\{z_1, \ldots, z_m\}$ in Y.

Let us further refine the result in (3.7) - (3.9) above. In this regard, we make use of the following Lemma whose proof is in the Appendix

Lemma 3.1.

Let

(3.10) $\quad t \in \mathbb{R},\ t \neq 0,\ t \neq 1$

then

1) To any linearly independent subset $\{x_{i_1}, \ldots, x_{i_r}\} \cup \{w_{j_1}, \ldots, w_{j_s}\}$ in $\{x_1, \ldots, x_n\} \cup \{w_1, \ldots, w_m\}$ corresponds the linearly independent subset $\{(1-t)x_{i_1}, \ldots, (1-t)x_{i_r}\} \cup \{tw_{j_1}, \ldots, tw_{j_s}\}$ in $\{(1-t)x_1, \ldots, (1-t)x_n\} \cup \{tw_1, \ldots, tw_m\}$, and conversely.

2) The linear span of $\{x_{i_1}, \ldots, x_{i_r}\} \cup \{w_{j_1}, \ldots, w_{j_s}\}$ and of $\{(1-t)x_{i_1}, \ldots, (1-t)x_{i_r}\} \cup \{tw_{j_1}, \ldots, tw_{j_s}\}$ are equal.

□

Let us assume that

(3.11) the dimension of the linear span of $\{(1-t)x_1, \ldots, (1-t)x_n\} \cup \{tw_1, \ldots, tw_m\} = k \leq$ the dimension of the linear span of $\{y_1, \ldots, y_n\} \cup \{z_1, \ldots, z_m\}$

then (1.8), (1.9) hold, therefore we have, see (3.8)

(3.12) $\quad F(t) = \sum_{q=1}^{p} \lambda_q(t) c_q \otimes d_q, \qquad t \in \mathbb{R},\ t \neq 0,\ t \neq 1$

where $\lambda_q(t) \in \mathbb{R}$, and we have the inclusion $\{c_1, \ldots, c_p\} \subseteq \{x_1, \ldots, x_n\} \cup \{w_1, \ldots, w_m\}$, with $\{c_1, \ldots, c_p\}$ being linearly independent.

In case instead of (3.11), we have

(3.13) the dimension of the linear span of

$\{(1-t)x_1, \ldots, (1-t)x_n\} \cup \{tw_1, \ldots, tw_m\} = k \geq$
the dimension of the linear span of $\{y_1, \ldots, y_n\} \cup \{z_1, \ldots, z_m\}$

then we note that (3.7) gives

$$(3.14) \qquad F(t) = \textstyle\sum_{i=1}^{n} x_i \otimes ((1-t)y_i) + \sum_{j=1}^{m} w_j \otimes (tz_j), \qquad t \in \mathbb{R}$$

and the above argument leading to (3.12) can be applied, with the difference in the result that this time we have the inclusion $\{d_1, \ldots, d_p\} \subseteq \{y_1, \ldots, y_n\} \cup \{z_1, \ldots, z_m\}$, with $\{d_1, \ldots, d_p\}$ being linearly independent.

Let us return to the situation in (3.12) which implies that

$$(3.15) \qquad gr(F(t)) = f(t) \leq p, \qquad t \in \mathbb{R},\ t \neq 0,\ t \neq 1$$

and let us now suppose that in (3.6) we have

$$(3.16) \qquad gr(a) = n \neq m = gr(b)$$

Then obviously

$$(3.17) \qquad f(0) = n \neq m = f(1)$$

thus the non-negative integer valued function

$$(3.18) \qquad f : [0,1] \ni t \longrightarrow f(t) \in \{0, 1, 2, \ldots\}$$

is *not* constant. Consequently, in terms of Definition 2.1., we obtain

Proposition 3.1.

The solution (3.1) - (3.3), (3.7) - (3.9) of the system of ODEs (3.4), (3.5) exhibits *entanglement dynamics*.

4. Another Example

Let $X = \mathbb{R}^2$, $Y = \mathbb{R}^3$, then $X \bigotimes Y = \mathbb{R}^6$. Further, let, see (2.3), $A : \mathbb{R}^6 \longrightarrow \mathbb{R}^6$ be the identity operator I_6 on $\mathbb{R}^6$, while $a = x_1 \otimes y_1 + x_2 \otimes y_2 \in X \bigotimes Y$. We consider the ODE, see (2.1)

$$(4.1) \qquad dF(t)/dt = A(F(t)), \quad t \in [0, \infty)$$

where

$$(4.2) \qquad [0, \infty) \ni t \longmapsto F(t) \in \mathbb{R}^6$$

with the initial condition

$$(4.3) \qquad F(0) = a = x_1 \otimes y_1 + x_2 \otimes y_2 \in X \bigotimes Y$$

and assume that the general form of the solution is, see (2.4)

$$(4.4) \qquad F(t) = x_1(t) \otimes y_1(t) + x_2(t) \otimes y_2(t) \in X \bigotimes Y$$

where this time $x_1(t), x_2(t) \in \mathbb{R}^2$, $y_1(t), y_2(t) \in \mathbb{R}^3$.

Remark 4.1.

1) Clearly, in view of (1.4), we can assume that every element $c \in X \bigotimes Y$ can be written as $c = u_1 \otimes v_1 + u_2 \otimes v_2$. Therefore, there is no loss of generality in the above choice of $a \in X \bigotimes Y$ in (4.3), or in the expression of the solution $F(t)$ in (4.4).
Furthermore, in view of (4.3), (4.4), we assume that

$$(4.5) \qquad x_1(0) = x_1, \quad y_1(0) = y_1, \quad x_2(0) = x_2, \quad y_2(0) = y_2$$

2) A specific feature of the system of ODEs in (4.1) - (4.4) is that (4.1) contains 6 linear equations in 6 unknown functions from $\mathbb{R}$ to $\mathbb{R}$, namely

$$(4.6) \qquad F(t) = (F_1(t), F_2(t), F_3(t), F_4(t), F_5(t), F_6(t)) \in \mathbb{R}^6, \qquad t \in \mathbb{R}$$

while in (4.4) we have

(4.7) $\quad (x_1(t), y_1(t), x_2(t), y_2(t)) \in \mathbb{R}^{2+3+2+3} = \mathbb{R}^{10}, \quad t \in \mathbb{R}$

therefore, there are 10 unknown functions from $\mathbb{R}$ to $\mathbb{R}$.

Regarding this discrepancy we note that, even when (4.4) would be a representation of $F(t)$ with the minimum number of terms, that representation need not be unique.

□

Now, as is well known, the solution of (4.1) - (4.3) is given by

(4.8) $\quad F(t) = \exp(tI_6)\, a = \exp(t) x_1 \otimes y_1 + \exp(t) x_2 \otimes y_2 \in \mathbb{R}^6, \quad t \in \mathbb{R}$

and clearly, given any $t \in \mathbb{R}$, we have

$$\left(\begin{array}{l} x_1, x_2 \\ \text{linearly independent} \end{array} \right) \Longleftrightarrow \left(\begin{array}{l} \exp(t)x_1, \exp(t)x_2 \\ \text{linearly independent} \end{array} \right)$$

as well as

$$\left(\begin{array}{l} y_1, y_2 \\ \text{linearly independent} \end{array} \right) \Longleftrightarrow \left(\begin{array}{l} \exp(t)y_1, \exp(t)y_2 \\ \text{linearly independent} \end{array} \right)$$

It follows that, similar with section 3, here again there is no entanglement dynamics.

5. One More Example

Let us consider a linear system of ODEs which is neither autonomous, nor homogeneous. We can again take $X = \mathbb{R}^2$ and $Y = \mathbb{R}^3$, and then $X \bigotimes Y = \mathbb{R}^6$. This time the linear system of ODEs is given by

(5.1) $\quad dF(t)/dt = A(t)(F(t)) + b(t), \quad t \in [0, \infty)$

where

(5.2) $\quad [0, \infty) \ni t \longmapsto F(t) \in \mathbb{R}^6$

it the solution with the initial condition

$$(5.3) \qquad F(0) = a = x_1 \otimes y_1 + x_2 \otimes y_2 \in X \bigotimes Y$$

Here we assume that $[0, \infty) \ni t \longmapsto A(t) : \mathbb{R}^6 \longmapsto \mathbb{R}^6$ and $[0, \infty) \ni t \longmapsto b(t) : \mathbb{R}^6$ are continuous.

Further, we can assume that the general form of the solution of (5.2) is

$$(5.4) \qquad F(t) = x_1(t) \otimes y_1(t) + x_2(t) \otimes y_2(t) \in X \bigotimes Y, \qquad t \in [0, \infty)$$

where $x_1(t), x_2(t) \in \mathbb{R}^2,\ y_1(t), y_2(t) \in \mathbb{R}^3$.

Under the above, we obviously have

$$(5.5) \qquad 0 \leq gr(F(t)) \leq 2$$

thus *entanglement dynamics* arises when, instead of (5.5), we may have

$$(5.6) \qquad 0 \leq gr(F(t)) \leq 1$$

Now as is well known, the solution of (5.1) - (5.3) is given by

$$(5.7) \qquad F(t) = F_0(t) + \Phi(t) \int_0^t \Phi^{-1}(s)\, b(s)\, ds, \qquad t \in [0, \infty)$$

where $[0, \infty) \ni t \longmapsto F_0(t) \in \mathbb{R}^6$ is the solution of the homogeneous system of ODEs

$$(5.8) \qquad dF(t)/dt = A(t)(F(t)), \quad t \in [0, \infty)$$

with the initial condition

$$(5.9) \qquad F_0(0) = a$$

while $[0, \infty) \ni t \longmapsto \Phi(t) \in \mathbb{R}^{36}$ is a fundamental 6×6 matrix solution of (5.8).

Appendix

Proof of Lemma 3.1.

1) Let $\{x_{i_1},\dots,x_{i_r}\}\cup\{w_{j_1},\dots,w_{j_s}\}$ be linearly independent in $\{x_1,\dots,x_n\}\cup\{w_1,\dots,w_m\}$, then we show that $\{(1-t)x_{i_1},\dots,(1-t)x_{i_r}\}\cup\{tw_{j_1},\dots,tw_{j_s}\}$ is linearly independent in $\{(1-t)x_1,\dots,(1-t)x_n\}\cup\{tw_1,\dots,tw_m\}$. Indeed, assume that

$$\text{(A.1)}\qquad \lambda_{i_1}(1-t)x_{i_1}+\ldots+\lambda_{i_r}(1-t)x_{i_r}+\mu_{j_1}tw_{j_1}+\ldots+\mu_{j_s}tw_{j_s}=0$$

where $\lambda_{i_1},\dots,\lambda_{i_r},\mu_{j_1},\dots,\mu_{j_s}\in\mathbb{R}$ are not all zero.

But we have $1-t,\ t\neq 0$, hence not all $\lambda_{i_1}(1-t),\dots,(1-t)\lambda_{i_r},t\mu_{j_1},\dots,t\mu_{j_s}\in\mathbb{R}$ are zero either. And then (A.1) implies that $\{x_{i_1},\dots,x_{i_r}\}\cup\{w_{j_1},\dots,w_{j_s}\}$ are not linearly independent, which is contrary to the assumption.

Conversely, if $\{(1-t)x_{i_1},\dots,(1-t)x_{i_r}\}\cup\{tw_{j_1},\dots,tw_{j_s}\}$ is linearly independent in $\{(1-t)x_1,\dots,(1-t)x_n\}\cup\{tw_1,\dots,tw_m\}$, then $\{x_{i_1},\dots,x_{i_r}\}\cup\{w_{j_1},\dots,w_{j_s}\}$ is linearly independent in $\{x_1,\dots,x_n\}\cup\{w_1,\dots,w_m\}$. Assume indeed that

$$\text{(A.2)}\qquad \lambda_{i_1}x_{i_1}+\ldots+\lambda_{i_r}x_{i_r}+\mu_{j_1}w_{j_1}+\ldots+\mu_{j_s}w_{j_s}=0$$

where $\lambda_{i_1},\dots,\lambda_{i_r},\mu_{j_1},\dots,\mu_{j_s}\in\mathbb{R}$ are not all zero.

Since we have $1-t,\ t\neq 0$, it follows from (A.2) that

$$\text{(A.3)}\qquad \begin{aligned}&[\lambda_{i_1}/(1-t)](1-t)x_{i_1}+\ldots+[\lambda_{i_r}/(1-t)](1-t)x_{i_r}+\\ &\quad+[\mu_{j_1}/t]tw_{j_1}+\ldots+[\mu_{j_s}/t]tw_{j_s}=0\end{aligned}$$

and $\lambda_{i_1}/(1-t),\dots,\lambda_{i_r}/(1-t),\mu_{j_1}/t,\dots,\mu_{j_s}/t\in\mathbb{R}$ are not all zero. Thus (A.3) implies that $\{(1-t)x_{i_1},\dots,(1-t)x_{i_r}\}\cup\{tw_{j_1},\dots,tw_{j_s}\}$ is not linearly independent in $\{(1-t)x_1,\dots,(1-t)x_n\}\cup\{tw_1,\dots,tw_m\}$, which contradicts the assumption.

2) It is a direct consequence of 1).

References

[1] Khrennikov A, Rosinger E E, Van Zyl A : Graded Tensor Products and the Problem of Tensor Grade Computation and Reduction. http://hal.archives-ouvertes.fr/hal-00717662, http://vixra.org/abs/1207.0050

[2] Khrennikov A, Rosinger E E, Van Zyl A : Graded Tensor Products and Entanglement. Foundations of Propbability and Physics - 6 (Eds. M D'Ariano, et.al.), AIPI Conference Proceedings 1424, Vaxjo, Sweden, 14-16 June 2011, pp. 189-194