Van der Pauw sheet resistance-conductance and the Schwarzschild black hole

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Abstract

The entropy and the interior volume of the Schwarzschild black hole is considered in terms of Shannon's mathematical theory of communication and van der Pauw's theory of sheet resistance-conductance.

1 Discrete binary signal space

In Shannon's mathematical theory of communication, as presented in [1, 2], a discrete binary signal consists of a string of n binary samples ('off' or 'on' values). Each of the 2^n distinct signals form a distinct vector in an n dimensional flat space (signal space). Where $x_1, x_2, ..., x_n$ are the n binary samples in an individual signal, the length of the signal vector in the nD signal space is

$$d = \sqrt{\sum_{m=1}^{n} x_m^2}. \tag{1}$$

Note that the signal space is not quite the same as the familiar state space from quantum field theory: Although there is a one-to-one mapping between the 2^n discrete binary signals and the 2^n states (both are effectively defined by the same data – the samples – and as such are effectively the same thing), each state would instead be represented by a distinct *unit length* vector in a flat space of 2^n dimensions (the state vectors, altogether, would form the set of 2^n orthonormal basis vectors for the state space).

In this paper we will assume that 'on' ≡ '1'. As for the numerical value corresponding to 'off', we have to make a choice between '0' and '-1':

1. If 'off' ≡ '0', then the lengths of the signal vectors can be any one of $\sqrt{0}, \sqrt{1}, \sqrt{2}, ..., \sqrt{n}$. The distribution of the 2^n signal vector lengths is given by the binomial coefficient "n choose k", where k is the number of 'on' samples per signal (or equivalently, the number of 'off' samples per signal, due to a symmetry of the distribution). For large n, this signal vector length distribution can be reasonably approximated via the continuous normal distribution, which is much more computationally efficient.

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2. Otherwise, if 'off' $\equiv$ '-1', then the lengths of the signal vectors are always $d \equiv \sqrt{n}$. In this case, the tips of all of the 2^n signal vectors are constrained to a single $(n-1)$D shell in the nD signal space. Altogether, the 2^n signal vectors form a perfect binary tree in the nD signal space, where a branching in the signal tree occurs once per dimension.

In this paper we will assume that 'off' $\equiv$ '-1'.

In Shannon's theory, the value $n \equiv d^2$ is a measure of power. This paper will use geometrized units, where $c \equiv G \equiv \hbar \equiv k_\mathrm{b} \equiv 1$, and so measures of power are dimensionless (units of length / length). Incidentally, measures of electric potential are also dimensionless (units of length / length).

2 van der Pauw sheet resistance-conductance

The event horizon radius and event horizon area of a Schwarzschild black hole [3] is

$$R_\mathrm{s} = 2E_\mathrm{bh}, \tag{2}$$
$$A = 4\pi R_\mathrm{s}^2. \tag{3}$$

The binary entropy of a Schwarzschild black hole is

$$S = \frac{A}{4\hbar \ln 2} = \frac{\pi R_\mathrm{s}^2}{\hbar \ln 2} = \frac{4\pi E_\mathrm{bh}^2}{\hbar \ln 2}. \tag{4}$$

The Bekenstein-Hawking temperature of a Schwarzschild black hole is

$$T = \frac{\hbar}{8\pi E_\mathrm{bh}}. \tag{5}$$

Note that even though the value of $\hbar$ is set to unity, it is still a dimensionful constant (units of length^2) and so it has been explictly written out in the equations in order to make the dimension (or lack thereof) of each equation clear.

What may or may not be immediately obvious about the entropy and temperature equations is that:

1. The event horizon is a two-dimensional sheet.

2. The entropy is dimensionless (units of length^2 / length^2), which is dimensionally equivalent to resistance-conductance (both also have units of length^2 / length^2).

3. The factor $\pi/\ln 2$ is van der Pauw's constant from his theory of sheet resistance-conductance, and so the entropy is similar to resistance-conductance in terms of how to calculate the numerical magnitude.

As such, we have a choice to make as to whether we shall interpret the entropy to be a measure of sheet resistance or a measure of sheet conductance. Given that an increase in conductance is generally related to a decrease in temperature, it seems more likely that the entropy is a measure of sheet conductance.

Assuming that $n \equiv S$, the signal shell 'radius' $d \equiv \sqrt{n}$ is related to the event horizon radius and the van der Pauw constant by

$$d = R_{\mathrm{s}}\sqrt{\alpha/\hbar}. \tag{6}$$

An individual signal is related to the black hole energy and the van der Pauw constant by a dimensionless integer constant (units of length^3 / length^3)

$$\frac{\hbar(x_1^2 + x_2^2 + x_3^2 + ... + x_n^2)}{\alpha E_{\mathrm{bh}}^2} = 4, \tag{7}$$

which may or may not be related to the dimension of $(3+1)$D spacetime. As such, it is possible that the geometry of the signal space may be related to the geometry of spacetime, as least as far as the coordinate radius R_{s} is concerned, and at most as far as the dimension of spacetime itself is concerned.

Altogether, if the interpretation of the units and the numerical magnitude of the entropy given here is correct, then even a Schwarzschild black hole – with zero net electric charge – possesses an electrical property such as conductance.

Some discussion on black hole superconductivity and temperature can be found in [5, 6]. Some discussion of thermodynamic models of gravity can be found in [7, 8, 9, 10, 11]. Some discussion of the relation between Shannon's theory and quantum measurement/uncertainty can be found in [2, 12, 13], which were the direct inspiration for considering the possible relation between Shannon's theory, sheet resistance-conductance, and black holes.

3 Finite volume and volume derivative of the black hole interior

In Shannon's theory, it is often noted that as the dimension n of signal space increases, the majority of the volume of the space becomes more and more distributed toward the $(n-1)$D shell at 'radius' $d = \sqrt{n}$ in an exponentially increasing way. This can be stated generically in terms of the volume ratio

$$\frac{\mathrm{Vol}(n, r)}{\mathrm{Vol}(n, \frac{r}{2})} = 2^n. \tag{8}$$

For instance: Where $n = 2$, the 'volume' ratio for a 2D disk of radius $r = 1$ and a 2D disk of radius $r = 1/2$ is $2^2 = 4$. Where $n = 3$, the ratio is $2^3 = 8$. Where $n = 4$, the ratio is $2^4 = 16$. As such, as the dimension increases, the majority of the volume becomes more and more distributed toward the $(n-1)$D shell at radius $r = 1$ in an exponentially increasing way. This is also reflected by the fact that the volume equation is proportional to r^n, and that its derivative with respect to r is proportional to r^{n-1}. Where z is a positive integer, the factorial is

$$z! = \begin{cases} 1, & z = 0 \\ \prod_{m=1}^{z} m, & z > 0 \end{cases}. \tag{9}$$

For large z, the factorial can be reasonably approximated via Stirling's approximation, which is much more computationally efficient. The Gamma function is

$$\Gamma(z) = (z-1)!. \tag{10}$$

For half-integers, the Gamma function is

$$\Gamma_{1/2}(z+1/2) = \sqrt{\pi}\frac{(2z)!}{z!4^z}. \tag{11}$$

The volume of an zD ball of radius r is proportional to r^z

$$\mathrm{Vol}(z,r) = \begin{cases} \frac{\pi^{z/2} r^z}{\Gamma(z/2+1)}, & z \bmod 2 = 0 \\ \frac{\pi^{z/2} r^z}{\Gamma_{1/2}([z+1]/2)}, & z \bmod 2 = 1 \end{cases}, \tag{12}$$

and the derivative of the volume with respect to r is proportional to r^{z-1}

$$\frac{d\mathrm{Vol}(z,r)}{dr} = \mathrm{Vol}(z,r)\frac{z}{r}. \tag{13}$$

In the previous section, we highlighted the possibility that the coordinate radius R_s in spacetime is related to the coordinate 'radius' d in signal space. In this section, we will consider the possibility that the volume (and/or the derivative of the volume) of the black hole interior is also related to the signal space.

For instance, in the paper [14], an effectively time-independent total interior volume of $(4/3)\pi R_\mathrm{s}^3$ is calculated (also see [15]). If we wish to use this calculation as a starting point, then at most the signal space would only affect the derivative of the volume. For instance, where $z = n$ and $r \leq R_\mathrm{s}$

$$\mathrm{Vol}_\mathrm{bh}(r) = \frac{4}{3}\pi R_\mathrm{s}^3 \frac{\mathrm{Vol}(n, d[r/R_\mathrm{s}])}{\mathrm{Vol}(n,d)}. \tag{14}$$

For $n = 3$, the volume derivative would be the same as flat 3D space – the volume ratio from Eq. (8) holds as $\mathrm{Vol}_\mathrm{bh}(R_\mathrm{s})/\mathrm{Vol}_\mathrm{bh}(R_\mathrm{s}/2) = 2^3$. On the other hand, for $n = 1$ and $n = 2$ the volume would contract near the centre of the black hole – the volume ratio would be $\mathrm{Vol}_\mathrm{bh}(R_\mathrm{s})/\mathrm{Vol}_\mathrm{bh}(R_\mathrm{s}/2) < 2^3$. For $n \geq 4$ the volume would instead contract near the event horizon – the volume ratio would be $\mathrm{Vol}_\mathrm{bh}(R_\mathrm{s})/\mathrm{Vol}_\mathrm{bh}(R_\mathrm{s}/2) > 2^3$. As such, for $n \geq 4$, the contraction of space near the event horizon would accelerate anything within the black hole toward the event horizon, somewhat like how the contraction of space near the event horizon in the black hole exterior causes anything outside of the black hole to accelerate toward the event horizon. We take this to mean that for $n \geq 4$ that all of the black hole's 'matter' would not reside at a singularity at the centre of the black hole, but rather at the event horizon. As for $n \leq 3$, the 'matter' may very well reside at the centre of the black hole, but at least both the volume and the volume derivative are always finite – no central singularity arises. Similarly, we could consider the possibility that the *state* space affects the derivative of the volume. For instance, where $z = 2^n$ and $r \leq R_\mathrm{s}$

$$\mathrm{Vol}_\mathrm{bh}(r) = \frac{4}{3}\pi R_\mathrm{s}^3 \frac{\mathrm{Vol}(2^n, r/R_\mathrm{s})}{\mathrm{Vol}(2^n, 1)}, \tag{15}$$

which would simply serve to increase the degree of the contraction of space near the event horizon much more rapidly than the case where $z = n$.

If we do not wish to use the calculation from [14] as a starting point, then the signal space would affect not only the volume derivative, but also the volume itself. For instance, where $z = n$ and $r \leq R_{\mathrm{s}}$

$$\mathrm{Vol}_{\mathrm{bh}}(r) = \mathrm{Vol}(n, d[r/R_{\mathrm{s}}]). \tag{16}$$

Similarly, for *state* space, where $z = 2^n$ and $r \leq R_{\mathrm{s}}$

$$\mathrm{Vol}_{\mathrm{bh}}(r) = \mathrm{Vol}(2^n, r/R_{\mathrm{s}}). \tag{17}$$

In this case where $z = 2^n$, the total interior volume of the black hole all but vanishes as 2^n increases toward infinity (the black hole would be hollow, though not quite in the same way as presented in [16]).

In any case under consideration, both the volume and the volume derivative of the black hole interior are always finite – no central singularity arises. In most cases (where $z \geq 3$), it is found that the black hole's 'matter' would reside not at the centre of the black hole, but rather at the event horizon of the black hole. In any case, where z is taken to be an integer, we find that the volume (and/or the volume derivative) is quantized. At the very least, the possibility that the geometry of the signal (or state) space may be related to the geometry of spacetime, as far as the volume and the volume derivative of the black hole interior is concerned, leads to black hole interior solutions that do not suffer from the singularities that afflict standard general relativity.

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