The Transfer Operator of the Harmonic Sawtooth Map

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Abstract. The Frobenius-Perron transfer operator of the harmonic sawtooth map is investigated and some expressions for its eigenvalues are found.

Table of contents

1. The Riemann Zeta Function as the Mellin Transform of a Unit Interval Map

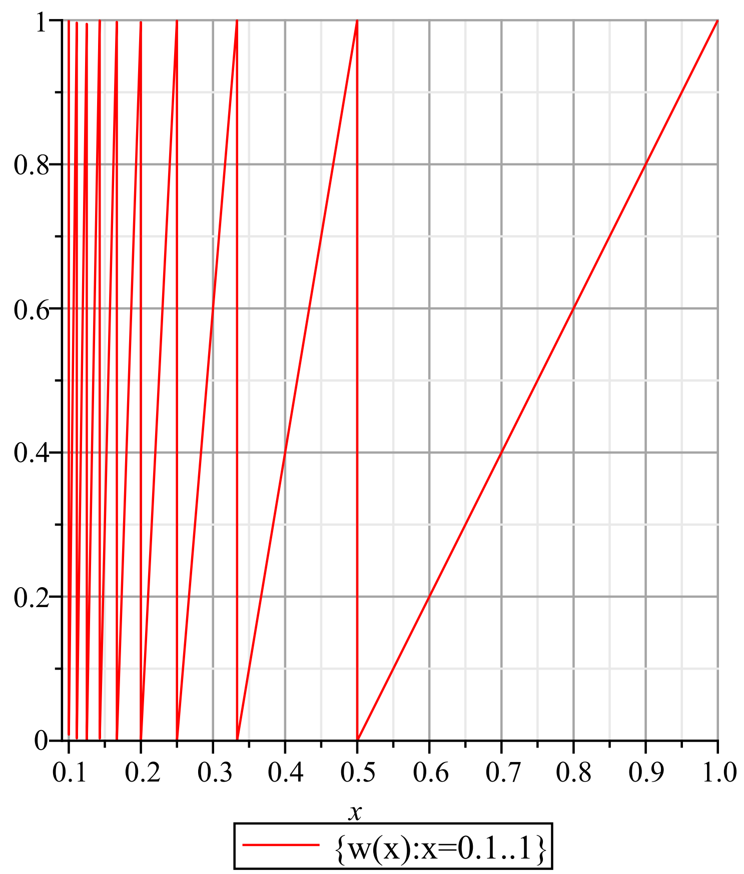

Figure 1. The Harmonic Sawtooth map

The Riemann zeta function can be written as the Mellin transform of the unit interval map multiplied by $s\frac{s+1}{s-1}$. [1, 2.3]

$$\begin{aligned} w(x) &= \lfloor x^{-1} \rfloor (x \lfloor x^{-1} \rfloor + x - 1) \\ &= \left\{ n - xn(n+1) \quad \frac{1}{n+1} < x \leqslant \frac{1}{n} \right. \end{aligned} \tag{1}$$

$$\begin{aligned} \zeta_w(s) &= \zeta(s) \forall -s \notin \mathbb{N}^* \\ &= s\frac{s+1}{s-1}\int_0^1 w(x) x^{s-1} \mathrm{d}x \\ &= s\frac{s+1}{s-1}\int_0^1 \lfloor x^{-1} \rfloor (x \lfloor x^{-1} \rfloor + x - 1) x^{s-1} \mathrm{d}x \\ &= s\frac{s+1}{s-1}\sum_{n=1}^{\infty} \int_{\frac{1}{n+1}}^{\frac{1}{n}} n(xn + x - 1) x^{s-1} \mathrm{d}x \\ &= \sum_{n=1}^{\infty} s\frac{s+1}{s-1}\left(-\frac{n^{1-s} - n(n+1)^{-s} - sn^{-s}}{s(s+1)} \right) \\ &= \sum_{n=1}^{\infty} \frac{n(n+1)^{-s} - n^{1-s} + sn^{-s}}{s-1} \\ &= \frac{1}{s-1}\sum_{n=1}^{\infty} n(n+1)^{-s} - n^{1-s} + sn^{-s} \end{aligned} \tag{2}$$

1.1. The Transfer Operator.

This article will focus on the Frobenius-Perron transfer operator of $w(x)$ [2, 3] defined by

$$[\mathcal{L}_w f](x) = \sum_{y: w(y) = x} \frac{f(y)}{|\mathrm{d}w(y)/\mathrm{d}y|} = \sum_{n=1}^{\infty} \frac{f\left(\frac{x+n}{n(n+1)} \right)}{n(n+1)} \tag{3}$$

The transfer operator maps densities to densities whereas $w(x)$ maps points to points.

1.1.1. Polynomial Eigenfunctions.

Following the paper of Vepstas, [2, 3] expanding $f(x)$ about $x = 0$ gives

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \tag{4}$$

and likewise

$$\begin{aligned} [\mathcal{L}_w f](x) &= \sum_{m=0}^{\infty} \frac{g^{(m)}(0)}{m!} x^m \\ &= \sum_{m=0}^{\infty} \frac{1}{n(n+1)} \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \left(\frac{x+n}{n(n+1)} \right)^k \end{aligned} \tag{5}$$

After rearranging sums and equating terms with the same power of x we get matrix elements $Z_{m,k}$ such that

$$\frac{g^{(m)(0)}}{m!} = \sum_{k=0}^{\infty} Z_{m,k} \frac{f^{(k)}(0)}{k!} \tag{6}$$

with

$$Z_{m,k} = \begin{cases} \sum_{n=1}^{\infty} n^{-k-1}(n+1)^{-m-1} & \text{for } k \geqslant m \\ 0 & \text{for } k < m \end{cases} \tag{7}$$

which satisfies

$$Z_{m,k} = Z_{m-1,k} - Z_{m,k-1} \tag{8}$$

with boundary conditions

$$
\begin{aligned}
Z_{0,0} &= 1\\
Z_{1,0} &= 2-\zeta(2)\\
Z_{m,0} &= Z_{m-1,0}-(\zeta(m+1)-1)\\
&= 1-\sum_{j=1}^{m}(\zeta(j+1)-1)\\
Z_{0,1} &= \zeta(2)-1\\
Z_{0,k} &= \zeta(k+1)-Z_{0,k-1}\\
&= (-1)^k\left(1+\sum_{j=1}^{k}(-1)^j\zeta(j+1)\right)
\end{aligned} \tag{9}
$$

The matrix elements $Z_{m,k}$ are finite sums of rationally weighted zeta functions evaluated at integer arguments.

$$
Z_{m,k}=\zeta(0)Y_{m,k,0}+\sum_{n=2}^{(m\wedge k)+1}\zeta(n)Y_{m,k,n} \tag{10}
$$

where $m\wedge k=\max(m,k)$. Note that $Y_{m,k,1}=0$ since the singular point $\zeta(1)$ never appears in the expression.

1.1.2. Diagonals.

Let us define two functions, one of which is just a shift of the other, which gives the coefficients of the diagonals of Z

$$
\alpha_{k,n}=\frac{2(-1)^n\Gamma(2n+2k)}{\Gamma(n+1)\Gamma(n+2k)} \tag{11}
$$

and

$$
\beta_{k,n}=\alpha_{k,n-2k}=\frac{2(-1)^{n-2k}\Gamma(2n-2k)}{\Gamma(n-2k+1)\Gamma(n)} \tag{12}
$$

then the eigenvalues of $\mathcal{L}_w$ are the diagonals of Z

$$
\begin{aligned}
\lambda_k &= Z_{k,k}\\
&= \sum_{n=1}^{\infty} n^{-k-1}(n+1)^{-k-1}\\
&= \sum_{n=0}^{\frac{k}{2}+\frac{1}{4}-\frac{(-1)^k}{4}}\zeta(2n)\beta_{k,n}\\
&= \sum_{n=0}^{\frac{k}{2}+\frac{1}{4}-\frac{(-1)^k}{4}}\zeta(2n)\frac{2(-1)^{k+1-2n}\Gamma(2k-2n+2)}{\Gamma(k+2-2n)\Gamma(k+1)}
\end{aligned} \tag{13}
$$

The coefficients $\alpha_{k,n}$ correspond to the generating functions

$$
\begin{aligned}
\sum_{n=0}^{\infty}\frac{\alpha_{k,n}x^n}{n!} &= {}_2F_2\left(\begin{matrix} k & k+\frac{1}{2}\\ 1 & 2k\end{matrix};-4x\right)2\\
\sum_{n=0}^{\infty}\frac{\alpha_{k,n}x^k}{k!} &= \frac{{}_2F_2\left(\begin{matrix} n & n+\frac{1}{2}\\ \frac{n}{2} & \frac{n}{2}+\frac{1}{2}\end{matrix};x\right)2(-1)^n\Gamma(2n)}{\Gamma(n)^2 n}
\end{aligned} \tag{14}
$$

The trace of $\mathcal{L}_w$ is given by

$$\begin{aligned}
\mathrm{Tr}(Z) &= \sum_{k=1}^{\infty} \lambda_k \\
&= \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} n^{-k-1}(n+1)^{-k-1} \\
&= \sum_{n=1}^{\infty} \frac{1}{n(n+1)-1} \\
&= 1 + \frac{\sqrt{5}\,\pi \tan\left(\frac{\pi\sqrt{5}}{2}\right)}{5} \\
&= 1.546250624110635744577890138 59...
\end{aligned} \tag{15}$$

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