Matrix Determinant as a verifier of a Path (cycle) in the Directed Hamiltonian Cycle Problem under two special conditions: a formal proof

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Abstract

In earlier work, the author conjectured that under two special conditions relating to theorems on the determinant of a matrix: the absence of a zero row (column) and the absence of similar rows (columns), a non-zero determinant value certifies the existence of a Directed Hamiltonian Path in an arbitrary adjacency matrix. Here, a formal proof is provided by means of deductive logic to establish that in an arbitrary adjacency matrix of size *n* (*n* rows and *n* columns), a non-zero determinant value verifies the existence of a Directed Hamiltonian Path in the adjacency matrix.

Keywords: P vs NP, Matrix Determinant, Directed Hamiltonian Path, proof, deductive logic, axioms.

1. INTRODUCTION

The decision version of the directed Hamiltonian cycle problem asks, "Given a graph G, does G have a Hamiltonian Cycle (Path)?" A directed graph G consists of a finite, non-empty set of vertices and edges – which are ordered pairs of vertices [1]. Graph G is said to have a Hamiltonian cycle or path if there exists a sequence of one-way edges across all vertices. This problem is known to be Non-Polynomial complete (NP-complete) [2] and as such not likely to be in class P – the class of problems with feasible algorithms i.e. polynomial-time algorithms [2]. An adjacency matrix – a square matrix of 0's and 1's, denoting the absence (0) or presence (1) of edges between vertices of a graph G is a common representation of a graph G [1]. An adjacency matrix contains a fixed zero diagonal from left to right. By this definition it is deducible that an adjacency matrix is a representation of combination of edges of a directed graph across the *n* rows and *n* columns of an adjacency matrix of size *n*.

The determinant of a square matrix A is a special scalar denoted by |A|. In an arbitrary square matrix of size *n* (*n* rows and *n* columns), under two special conditions relating to theorems on determinants with proofs given in [3]: the absence of a zero row (column) and the absence of similar rows (columns), the determinant of a matrix is non-zero. In previous work [3], the author conjectured that under these two conditions the non-zero determinant value of an adjacency matrix certifies the existence of a Directed Hamiltonian path in an arbitrary adjacency matrix – a representation of a directed graph G. Here, the author gives a formal proof by means of deductive logic that given an arbitrary adjacency matrix of size *n*, the absence of a zero row (column) and the absence of similar rows (columns) i.e. a non-zero determinant value certifies the existence of a directed Hamiltonian path.

2. Proof

A proof is provided by means of deductive logic where six preceding axioms are linked to reach a conclusion that is logically valid.

(1) An adjacency matrix is a square matrix representing the combinations of edges of a directed graph across *n* rows and *n* columns of adjacency matrix of size *n.* An adjacency matrix has a fixed zero diagonal from left to right.

(2). The rows (columns) of an adjacency matrix are listed as row (column) 1, 2, 3, …, (n-3), (n-2), (n-1), n in an adjacency matrix of size *n*.

(3).The presence of edges per row (column) of an adjacency matrix is denoted by 1's while their absence by 0's, so a combination of edges per row (column) of adjacency matrix size *n* is represented by a combination of 0's and 1's.

(4). Since the combination of edges per row (column) of an adjacency matrix is represented by a combination of 0's and 1's, in order to satisfy the dual conditions of the absence of a zero row (column) and the absence of similar rows (columns) across the *n* rows (columns) of adjacency matrix, each row (column) is assigned at least one edge (1) and a different combination of 0's and 1's is applied across row (column) 1, 2, 3, …, (n-3), (n-2), (n-1), n of adjacency matrix size *n.*

(5). Different combinations of 0's and 1's applied across row (column) 1, 2, 3, …, (n-3), (n-2), (n-1), n of an adjacency matrix size *n* implies different permutations or sequences of 0's and 1's (permutations of edges) across the respective rows (columns).

(6). In an arbitrary adjacency matrix, a directed Hamiltonian path is a unique sequence of one way edges across rows (columns) 1, 2, 3, …, (n-3), (n-2), (n-1), n. In effect a directed Hamiltonian Path is a unique permutation of edges across row (column) 1, 2, 3, …, (n-3), (n-2), (n-1), n.

(7). Therefore for an arbitrary adjacency matrix of size *n* satisfying the dual conditions of the absence of a zero row (column) and the absence of similar rows (columns) – implying a non-zero determinant value for the adjacency matrix, there exists a directed Hamiltonian path.

Proved

3. Discussion and Conclusion

This paper gives a formal proof using the axioms of deductive logic to establish that under two special conditions: the absence of a zero row (column) and the absence of similar rows (columns) – implying a non-zero determinant value, an arbitrary adjacency matrix, which is a common representation of a directed graph, encodes for a directed Hamiltonian path or cycle. In the axioms, precedence is given to the combination of 0's and 1's over the combination of edges because different combinations of 0's and 1's applied in the *n* rows (columns) automatically implies different combinations of edges in the *n* rows (columns) of the adjacency matrix but not vice-versa. Different combinations of edges in the *n* rows (columns) of an adjacency matrix could result in a similar combination of 0's and 1's, and hence similar rows (columns) of an adjacency matrix.

Matrix determinants can be computed in polynomial-time (matrices are an aspect of linear algebra), therefore given adjacency matrices satisfying the dual conditions above, a non-zero determinant value gives an efficient way of verifying the existence of a directed Hamiltonian path or cycle.

As this technique does not verify the existence of a directed Hamiltonian path for adjacency matrices that do not satisfy the two stated conditions, this solution does not imply that the decision version of the directed Hamiltonian path (cycle) problem is in class P. It does prove however that the decision version of the directed Hamiltonian path (cycle) problem which is an NP-complete problem hence in class NP, when expressed as an adjacency matrix, can be solved in polynomial-time when the distribution on its inputs satisfies the two stated conditions. This draws from Levin's theory of average-case completeness [4, 5], where it is relevant to ask whether every NP problem with a reasonable probability distribution on its inputs can be solved in polynomial-time on average.

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