# A New Conjecture On Prime Numbers

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**Abstract**

In this paper we propose a new conjecture about prime numbers called *Conjecture C*, and we prove that if this conjecture is true, then *Legendre's conjecture*, *Brocard's conjecture*, and *Andrica's conjecture* are all true. Moreover, we also prove that if Conjecture C is true, then there is at least one prime number in the interval $[n, n + 2\lfloor\sqrt{n}\rfloor - 1]$ for every positive integer $n$.

## 1 Introduction.

We start this paper by making the following conjecture:

**Conjecture 1.** *If $n$ is any positive integer and we take $n$ consecutive integers located between $n^2$ and $(n+1)^2$, then among those $n$ integers there is at least one prime number. In other words, if $a_1$, $a_2$, $a_3$, $a_4$, …, $a_n$ are $n$ consecutive integers such that $n^2 < a_1 < a_2 < a_3 < a_4 < ... < a_n < (n+1)^2$, then at least one of those $n$ integers is a prime number. This conjecture will be called **Conjecture C**.*

*Remark* 1. In this paper, whenever we say that a number $b$ is **between** a number $a$ and a number $c$, it means that $a < b < c$, which means that $b$ is never equal to $a$ or $c$. Moreover, the number $n$ that we use in this document is always a positive **integer**.

Let us see some cases in which Conjecture 1 is true:

- If we consider $n = 1$, we have $1^2 < 2 < 3 < (1+1)^2$, and we can see that the numbers 2 and 3 are both prime numbers.
- If we consider $n = 2$, we have $2^2 < 5 < 6 < 7 < 8 < (2+1)^2$. If we take any sequence of 2 consecutive integers greater than $2^2$ and smaller than $(2+1)^2$, then at least one of those 2 integers is a prime number. This is true because each of the sequences $\{5, 6\}$, $\{6, 7\}$, and $\{7, 8\}$ contains at least one prime number.
- If we consider the case where $n = 3$, we have $3^2 < 10 < 11 < 12 < 13 < 14 < 15 < (3+1)^2$. It is easy to verify that each of the sequences $\{10, 11, 12\}$, $\{11, 12, 13\}$, $\{12, 13, 14\}$, and $\{13, 14, 15\}$ contains at least one prime number.

We can easily prove that Conjecture 1 is also true for $n = 4$, $n = 5$, $n = 6$, and larger values of $n$.

## 2 Legendre's conjecture.

Legendre's conjecture [4] states that for every positive integer $n$ there exists at least one prime number $p$ such that $n^2 < p < (n+1)^2$.

It is easy to verify that the amount of integers located between $n^2$ and $(n+1)^2$ is equal to $2n$.

*Proof.*

$$\begin{aligned}
(n+1)^2 - n^2 &= 2n+1 \\
n^2 + 2n + 1 - n^2 &= 2n+1 \\
2n+1 &= 2n+1
\end{aligned}$$

We need to exclude the number $(n+1)^2$ because we are taking into consideration the integers that are greater than $n^2$ and smaller than $(n+1)^2$.

$$2n + 1 - 1 = 2n \qquad \square$$

According to this result, between $n^2$ and $(n+1)^2$ there are two groups of $n$ consecutive integers each which do not have any integer in common. Example for $n = 3$:

$$3^2 \quad \underbrace{\underbrace{10 \quad 11 \quad 12}_{\substack{\text{Group A}\\(n\text{ consecutive integers})}} \quad \underbrace{13 \quad 14 \quad 15}_{\substack{\text{Group B}\\(n\text{ consecutive integers})}}}_{2n\text{ consecutive integers}} \quad (3+1)^2$$

**Group A** and **Group B** do not have any integer in common. Now, according to Conjecture 1, Group A contains at least one prime number and Group B also contains at least one prime number, which implies that between $3^2$ and $(3+1)^2$ there are at least **two** prime numbers. This is true because the numbers 11 and 13 are both prime.

All this means that if Conjecture 1 is true, then there are at least **two** prime numbers between $n^2$ and $(n+1)^2$ for every positive integer $n$. **As a result, if Conjecture 1 is true, then Legendre's conjecture is also true.**

## 3 Brocard's conjecture.

Brocard's conjecture [3] states that if $p_n$ and $p_{n+1}$ are consecutive prime numbers greater than 2, then between $(p_n)^2$ and $(p_{n+1})^2$ there are at least four prime numbers.

Since $2 < p_n < p_{n+1}$, we have $p_{n+1} - p_n \geq 2$. This means that there is at least one positive integer $a$ such that $p_n < a < p_{n+1}$. As a result, there exists at least one positive integer $a$ such that $(p_n)^2 < a^2 < (p_{n+1})^2$.

Conjecture 1 states that between $(p_n)^2$ and $a^2$ there are at least two prime numbers and that between $a^2$ and $(p_{n+1})^2$ there are also at least two prime numbers. In other words, if Conjecture 1 is true, then there are at least four prime numbers between $(p_n)^2$ and $(p_{n+1})^2$. **As a consequence, if Conjecture 1 is true, then Brocard's conjecture is also true.**

# 4 Andrica's conjecture.

Andrica's conjecture [1, 2] states that $\sqrt{p_{n+1}} - \sqrt{p_n} < 1$ for every pair of consecutive prime numbers $p_n$ and $p_{n+1}$ (of course, $p_n < p_{n+1}$).

Obviously, every prime number is located between two consecutive perfect squares. Now, let us suppose that $p$ is any prime number and $q$ is the prime number immediately following $p$. If we take into account that $p$ is obviously located between $n^2$ and $(n+1)^2$ for some $n$, two things may happen:

**Case 1.** *The number $p$ is located among the first $n$ consecutive integers that are located between $n^2$ and $(n+1)^2$. These $n$ integers form what we call 'Group A,' and the following $n$ integers form what we call 'Group B.'*

Let us look at the following graphic.

$$n^2 \quad < \quad \underbrace{\underbrace{\bullet \quad \bullet \quad \ldots \quad \bullet \quad \bullet}_{\substack{\text{Group A} \\ (n \text{ consecutive integers})}} \quad \underbrace{\bullet \quad \bullet \quad \ldots \quad \bullet \quad \bullet}_{\substack{\text{Group B} \\ (n \text{ consecutive integers})}}}_{2n \text{ consecutive integers}} \quad < \quad (n+1)^2$$

If $p$ is located in Group A and Conjecture 1 is true, then $q$ is either located in Group A or in Group B. In both cases we have $\sqrt{q} - \sqrt{p} < 1$, since $\sqrt{(n+1)^2} - \sqrt{n^2} = 1$ and the numbers $\sqrt{q}$ and $\sqrt{p}$ are closer to each other than $\sqrt{(n+1)^2}$ in relation to $\sqrt{n^2}$.

**Case 2.** *The prime number $p$ is located in Group B.*

If $p$ is located in Group B and Conjecture 1 is true, it may happen that $q$ is also located in Group B. In this case, it is very easy to verify that $\sqrt{q} - \sqrt{p} < 1$, as explained before.

Otherwise, if $q$ is not located in Group B, then $q$ is located in 'Group C.' In this case, the largest value $q$ can have is $q = (n+1)^2 + n + 1 = n^2 + 2n + 1 + n + 1 = n^2 + 3n + 2$, while the smallest value $p$ can have is $p = n^2 + n + 1$ (in order to make the process easier, we are not taking into account the fact that in this case the numbers $p$ and $q$ have different parity, so they can not be both prime at the same time).

This means that the largest possible difference between $\sqrt{q}$ and $\sqrt{p}$ is $\sqrt{q} -$

$\sqrt{p} = \sqrt{n^2+3n+2} - \sqrt{n^2+n+1}$.

$$n^2 < \quad ... \quad \overbrace{\underbrace{\triangle \quad \bullet \quad ... \quad \bullet \quad \bullet}_{\substack{\text{Group B}\\ (n \text{ consecutive integers})}} < (n+1)^2 < \underbrace{\bullet \quad \bullet \quad ... \quad \bullet \quad \bullet \quad \square}_{\substack{\text{Group C}\\ (n+1 \text{ consecutive integers})}}}^{\text{maximum distance between } p \text{ and } q}$$

$\triangle = n^2 + n + 1 = p$
$\square = n^2 + 3n + 2 = q$

It is easy to prove that $\sqrt{n^2+3n+2} - \sqrt{n^2+n+1} < 1$.

*Proof.*

$$\begin{aligned}
\sqrt{n^2+3n+2} - \sqrt{n^2+n+1} &< 1 \\
\sqrt{n^2+3n+2} &< 1 + \sqrt{n^2+n+1} \\
n^2+3n+2 &< \left(1 + \sqrt{n^2+n+1}\right)^2 \\
n^2+3n+2 &< 1 + 2\sqrt{n^2+n+1} + n^2+n+1 \\
n^2+3n+2-n^2-n-1 &< 1 + 2\sqrt{n^2+n+1} \\
2n+1 &< 1 + 2\sqrt{n^2+n+1} \\
2n &< 2\sqrt{n^2+n+1} \\
n &< \frac{2\sqrt{n^2+n+1}}{2} \\
n &< \sqrt{n^2+n+1} \\
n^2 &< n^2+n+1,
\end{aligned}$$

which is true for every positive integer $n$. $\square$

*Remark* 2. In general, to prove that an inequality is correct, we can solve that inequality step by step. If we get a result which is obviously correct, then we can start with that correct result, 'work backwards from there' and prove that the initial statement is true.

We can see that even when the difference between $q$ and $p$ is the largest possible difference, we have $\sqrt{q} - \sqrt{p} < 1$. If the difference between $q$ and $p$ were smaller, then of course it would also happen that $\sqrt{q} - \sqrt{p} < 1$.

**According to Cases 1. and 2., if Conjecture 1 is true, then Andrica's conjecture is also true.**

**To conclude, if Conjecture 1 is true, then Legendre's conjecture, Brocard's conjecture, and Andrica's conjecture are all true.**

# 5 Possible new interval.

It is easy to verify that if Conjecture 1 is true, then in the interval $\left[n^2+n+1, n^2+3n+2\right]$ there are at least two prime numbers for every positive integer $n$.

The number $n^2 + n + 1$ is always an odd integer.

*Proof.*

- If $n$ is even, then $n^2$ is also even. Then we have

$$(even\ integer + even\ integer) + 1 = even\ integer + odd\ integer =$$

$$= odd\ integer.$$

- If $n$ is odd, then $n^2$ is also odd. Then we have

$$(odd\ integer + odd\ integer) + 1 = even\ integer + odd\ integer =$$

$$= odd\ integer. \qquad \square$$

Since the number $n^2 + n + 1$ is always an odd integer, then it may be prime or not.

Now, the number $n^2+3n+2$ can never be prime, since this number is always an even integer greater than 2.

*Proof.*

- If $n = 1$ (smallest value $n$ can have), then $n^2 + 3n + 2 = 1 + 3 + 2 = 6$.
- If $n$ is even, then $n^2$ and $3n$ are both even integers. The number 2 is also an even integer, and we know that

$$even\ integer + even\ integer + even\ integer = even\ integer.$$

- If $n$ is odd, then $n^2$ and $3n$ are both odd integers, and we know that

$$(odd\ integer + odd\ integer) + even\ integer =$$

$$= even\ integer + even\ integer = even\ integer. \qquad \square$$

From all this we deduce that if Conjecture 1 is true, then the maximum distance between two consecutive prime numbers is the one from the number $n^2+n+1$ to the number $n^2+3n+2-1 = n^2+3n+1$, which means that in the interval $[n^2+n+1, n^2+3n+1]$ there are at least two prime numbers. In other words, in the interval $[n^2+n+1, n^2+3n]$ there is at least one prime number.

The difference between the numbers $n^2 + n + 1$ and $n^2 + 3n$ is $n^2+3n-(n^2+n+1) = n^2+3n-n^2-n-1 = 2n-1$. In addition to this, $\left\lfloor\sqrt{n^2+n+1}\right\rfloor = n$. This means that in the interval $\left[n^2+n+1, n^2+n+1+2\left\lfloor\sqrt{n^2+n+1}\right\rfloor - 1\right]$ there is at least one prime number. In other words, if $a = n^2+n+1$, then the interval $[a, a+2\lfloor\sqrt{a}\rfloor - 1]$ contains at least one prime number.

*Remark* 3. The symbol $\lfloor\rfloor$ represents the *floor function*. The floor function of a given number is the largest integer that is not greater than that number. For example, $\lfloor 3.5 \rfloor = 3$.

Now, if Conjecture 1 is true, then the following statements are all true:

**Statement 1.** *If $a$ is a perfect square, then in the interval $[a, a + \lfloor\sqrt{a}\rfloor]$ there is at least one prime number.*

**Statement 2.** *If $a$ is an integer such that $n^2 < a \leq n^2 + n + 1 < (n+1)^2$, then in the interval $[a, a + \lfloor\sqrt{a}\rfloor - 1]$ there is at least one prime number.*

**Statement 3.** *If $a$ is an integer such that $n^2 < n^2 + n + 2 \leq a < (n+1)^2$, then in the interval $[a, a + 2\lfloor\sqrt{a}\rfloor - 1]$ there is at least one prime number.*

We know that $a + 2\lfloor\sqrt{a}\rfloor - 1 \geq a + \lfloor\sqrt{a}\rfloor$.

*Proof.*

$$\begin{aligned} a + 2\left\lfloor\sqrt{a}\right\rfloor - 1 &\geq a + \left\lfloor\sqrt{a}\right\rfloor \\ 2\left\lfloor\sqrt{a}\right\rfloor - 1 &\geq \left\lfloor\sqrt{a}\right\rfloor \\ 2\left\lfloor\sqrt{a}\right\rfloor &\geq \left\lfloor\sqrt{a}\right\rfloor + 1 \\ \left\lfloor\sqrt{a}\right\rfloor + \left\lfloor\sqrt{a}\right\rfloor &\geq \left\lfloor\sqrt{a}\right\rfloor + 1 \\ \left\lfloor\sqrt{a}\right\rfloor &\geq 1, \end{aligned}$$

which is true for every positive integer $a$. □

And we also know that $a + 2\lfloor\sqrt{a}\rfloor - 1 > a + \lfloor\sqrt{a}\rfloor - 1$.

*Proof.*

$$\begin{aligned} a + 2\left\lfloor\sqrt{a}\right\rfloor - 1 &> a + \left\lfloor\sqrt{a}\right\rfloor - 1 \\ 2\left\lfloor\sqrt{a}\right\rfloor &> \left\lfloor\sqrt{a}\right\rfloor, \end{aligned}$$

which is obviously true for every positive integer $a$. □

All this means that the interval $[a, a + 2\lfloor\sqrt{a}\rfloor - 1]$ can be applied to the number $a$ in Statement 1., to the number $a$ in Statement 2., and to the number $a$ in Statement 3.

**Therefore, if $n$ is any positive integer and Conjecture 1 is true, then in the interval $\left[n, n + 2\left\lfloor\sqrt{n}\right\rfloor - 1\right]$ there is at least one prime number (in order to provide more standardized notation, we are now replacing letter $a$ with letter $n$).** According to this, we can also say that if Conjecture 1 is true, then there is always a prime number in the interval $[n, n + 2\sqrt{n} - 1]$ for every positive integer $n$.

**Now... how can we prove (or disprove) Conjecture 1?**

## References

[1] Andrica, D. "Note on a Conjecture in Prime Number Theory." *Studia Univ. Babes-Bolyai Math.* 31, 44-48, 1986.

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