On the affine nonlinearity in circuit theory

Emanuel Gluskin

The Kinneret College on the Sea of Galilee;
and Ort Braude College (Carmiel), Israel.
gluskin@ee.bgu.ac.il; http://www.ee.bgu.ac.il/~gluskin/

ABSTRACT (SUMMARY) for the viXra posting …

According to the definition of the linear operator, as accepted in system theory, an affine dependence is a nonlinear one. This implies the nonlinearity of Thevenin's 1-port, while the battery itself is a strongly nonlinear element that in the 1-port's "passive mode" (when the 1-port is fed by a "stronger" circuit) can be replaced by a hardlimiter. For the theory, not the actual creation of the equivalent 1-port, but the selection of one of the ports of a (linear) many-port for interpreting the circuit as a 1-port, is important.

A practical example of the affine nonlinearity is given also in terms of waveforms of time functions. Emphasizing the importance of the affine nonlinearity, it is argued that even when straightening the curved characteristic of the solar cell, we retain the main part of the nonlinearity. Finally, the "fractal-potential" and "f-connection-analysis" of 1- ports, which are missed in classical theory, are mentioned.

And now, the original slides start:

No "confrontation" is expected here regarding the **definition**, originating from analytical geometry, of the **affine dependence**.

The question is whether we have ***affine linearity***, or ***affine nonlinearity***.

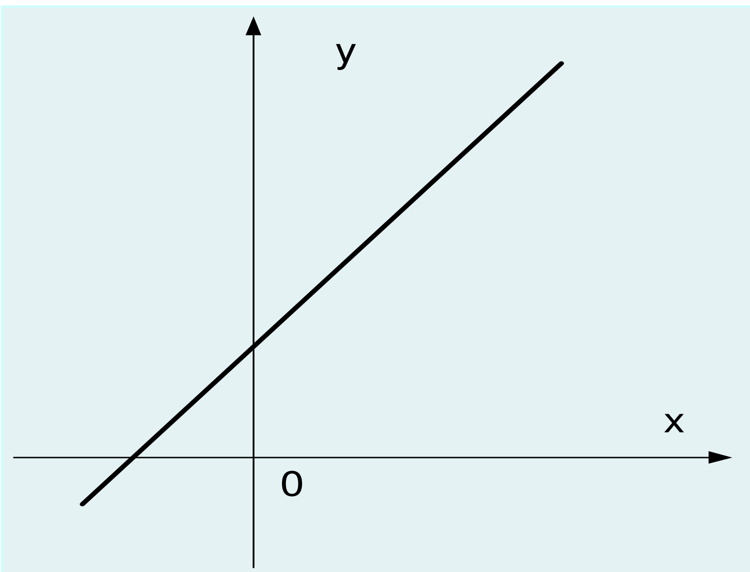
Though the latter possibility is somewhat "painful", because it readily means that "***Thevenin*** (*Norton, Helmholtz*) ***equivalent"*** is a ***nonlinear circuit**,* the answer may depend ***only*** on which **position** we are staying: that of **system theory**, or **that of the geometry, and one wishing to stay on the positions of system theory** should accept this news.

Planimetry introduces *affine dependence* as the straight line:

$$y = ax + b; \qquad (1)$$

a, b **--** constants

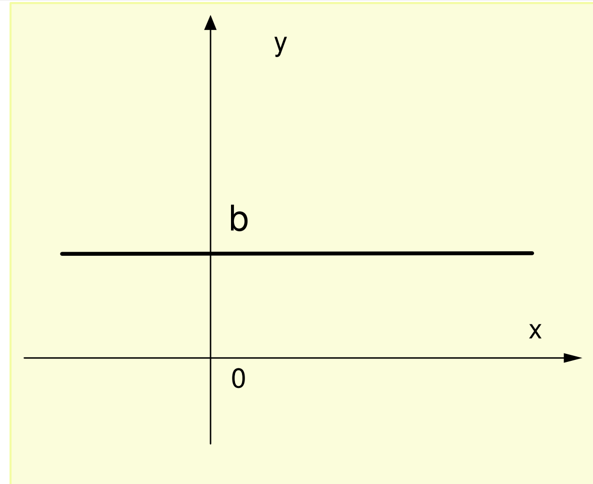
which is also seen as linear dependence:



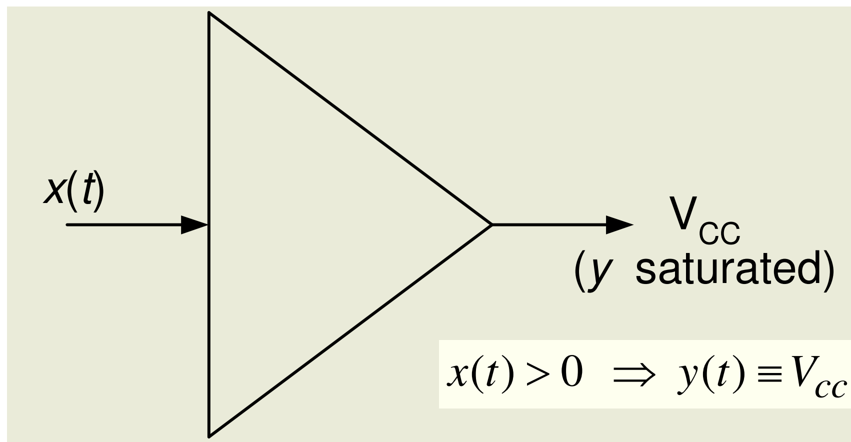
Linear dependence in planimetry.

For $x > 0$, the particular case of

$$a = 0, \text{ i.e. } y \equiv b$$



is a direct **analogy** to the following circuit:



The saturated (nonlinear) amplifier

In system theory (or functional analysis, or theory of operators) the following **definition of linearity** is used:

$$y\left(\sum_{p=1}^{n} k_p x_p\right) = \sum_{p=1}^{n} k_p y(x_p), \qquad (2)$$

$$n \geq 1, \quad \forall \{\{k_p\},\{x_p\}\}.$$

“superposition”

For $n = 1$ this is the linear scaling

$$y(kx) = k y(x).$$

As well, we have

$$y(x - x) = y(x) - y(x),$$

i.e. *any linear map* satisfies **0→0**.

Hence, affine map that does **not** satisfy **0→0**, is a nonlinear map. That is, in system theory affine dependence is “**affine nonlinearity**” (“*ANN*”).

!

A remark

Substituting

$$y = ax + b \qquad [\,(1)\,]$$

into

$$y\Big(\sum_{p=1}^{n} k_p x_p\Big) = \sum_{p=1}^{n} k_p y(x_p) \qquad [\,(2)\,]$$

we obtain

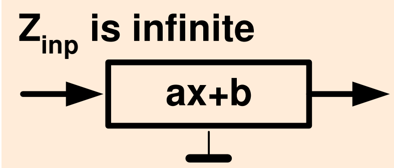
$$b = \sum_{m=1}^{n} k_p b \;.$$

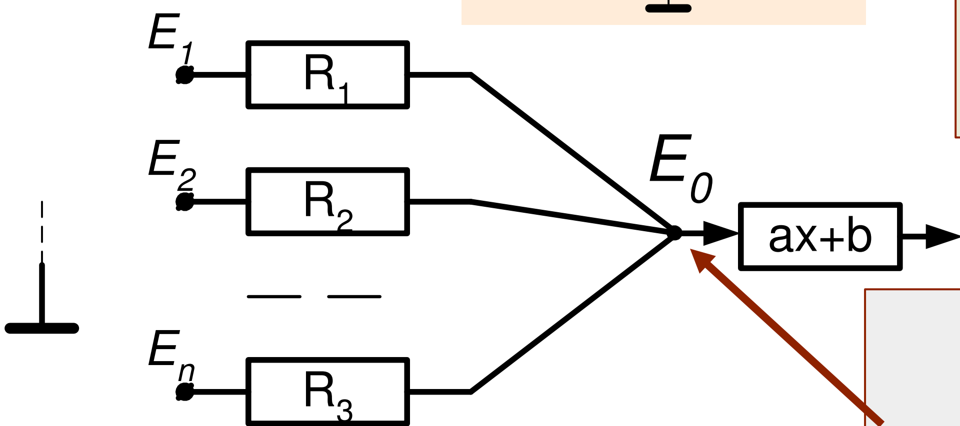
Thus, if

$$\sum_{p=1}^{n} k_p = 1\,, \qquad (*)$$

-- as if $\{k_p\}$ are some ***probabilities***, -- then (1) satisfies (2), and we have a *"filtration" of the nonlinear effect.*

Though, *a*s the point of principle, no constraints on $\{k_p\}$ are permitted for linearity of a system, let us give also a ***circuit*** realization for this specific case, observing how a structure can realize the "probability condition" **(*)**:





$$y\left(\sum_{p=1}^{n} k_p x_p\right) = \sum_{p=1}^{n} k_p y(x_p)$$

Here, $"x_p" \to E_p$:

$$E_o = \frac{\sum_{p=1}^{n} E_p \cdot R_p^{-1}}{\sum_{p=1}^{n} R_p^{-1}}$$

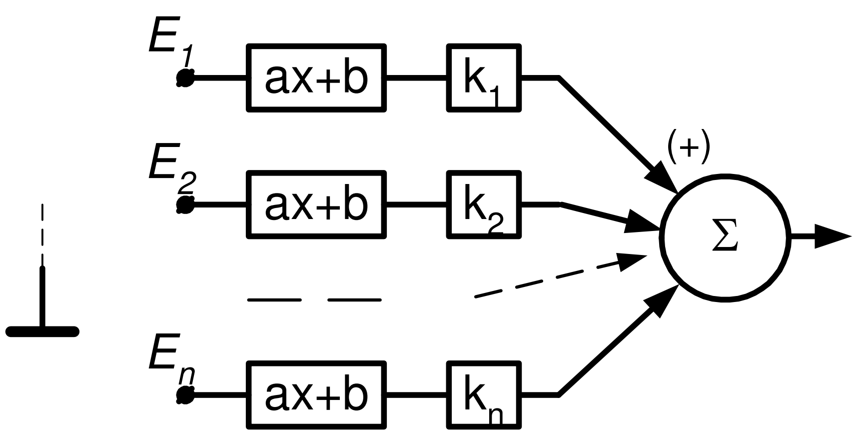
Kirchhoff's laws provide **(*)**.

$$= \sum_{p=1}^{n} k_p E_p , \quad k_p = \frac{R_p^{-1}}{\sum_{m=1}^{n} R_m^{-1}} ;$$

$$\sum_{p=1}^{n} k_p = 1 .$$

This is **(*)**.

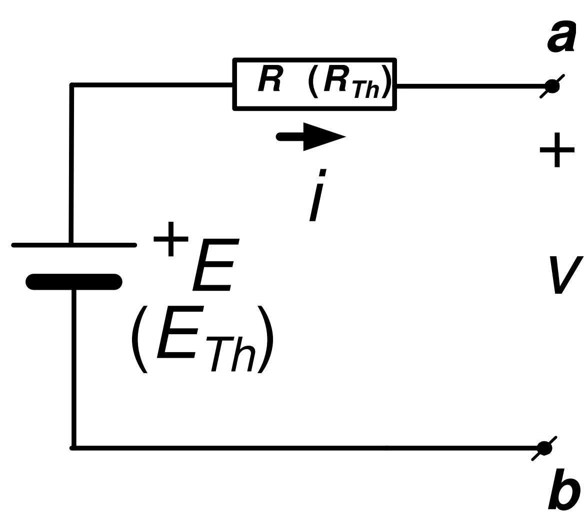
equivalent to:



Returning to the affine characteristic $y = ax + b$ by itself, we rewrite it in EE notions:

$$v = \mp Ri + E$$

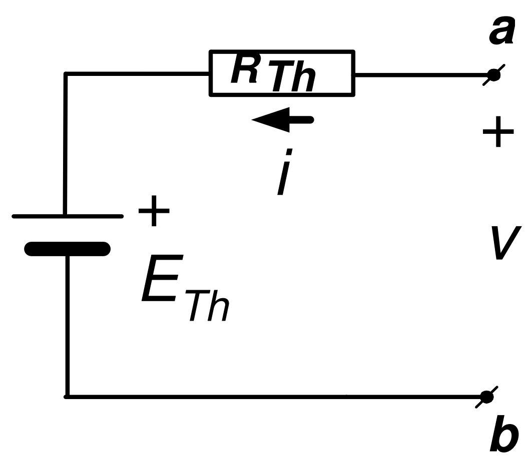
This is the port-characteristic of many circuits of which the simplest one is the Thevenin's equivalent:



$$v = -Ri + E$$

Notice the direction of $i(t)$.

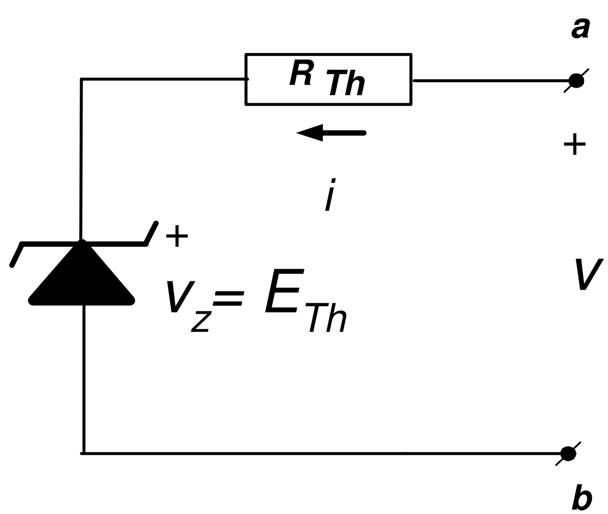
or:



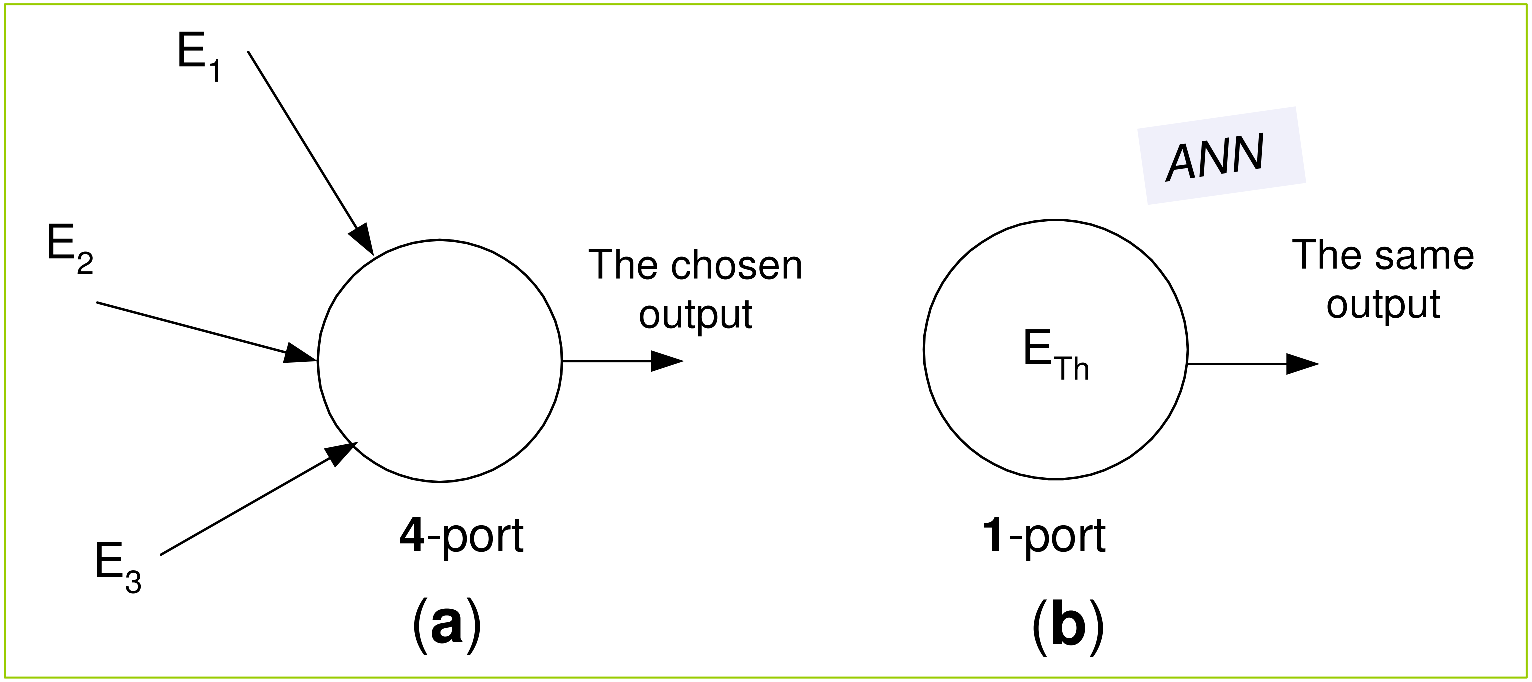
Notice the direction of $i(t)$.

$$v = Ri + E$$

This case of *i* entering the circuit (and sign[$i(t)$] = const) is equivalent to the following **obviously nonlinear** circuit:



This is the partial (not for any sign[*i*]), ***passive mode*** equivalence that is important for creation of nonlinear resistors $v(i)$, see Slide 11.



'(**a**)' is a *linear* 4-port *if we accept (recognize) all of the ports*. If we recognize only *one* port, i.e. start to see this whole circuit as a 1-port, then the circuit becomes *ANN*, in the sense of (1).

(**b**) is the (nonlinear) *Thevenin equivalent* of (**a**), *after* (**a**) *is recognized as a 1-port.*

***However, actual realization* of the equivalent version is NOT necessary for the point:** when a linear *N*-port (*N* >1) '(**a**)' is approached as a 1-port, it becomes a nonlinear (*ANN*) circuit.

The axiomatic aspect

That such *reduction of the number of the defined ports* is the reason for nonlinearity, is considered also in

E. Gluskin, "*An extended frame …*" CASS Newsletter, Dec. 2011;

and with more stress on the axiomatic side in:

E. Gluskin, "*An Application of Physical Units (Dimensional) Analysis to the Consideration of Nonlinearity in Electrical Switched Circuits*", Circuits Syst. Signal Process vol. 31, 737–752 (Apr. 2012).

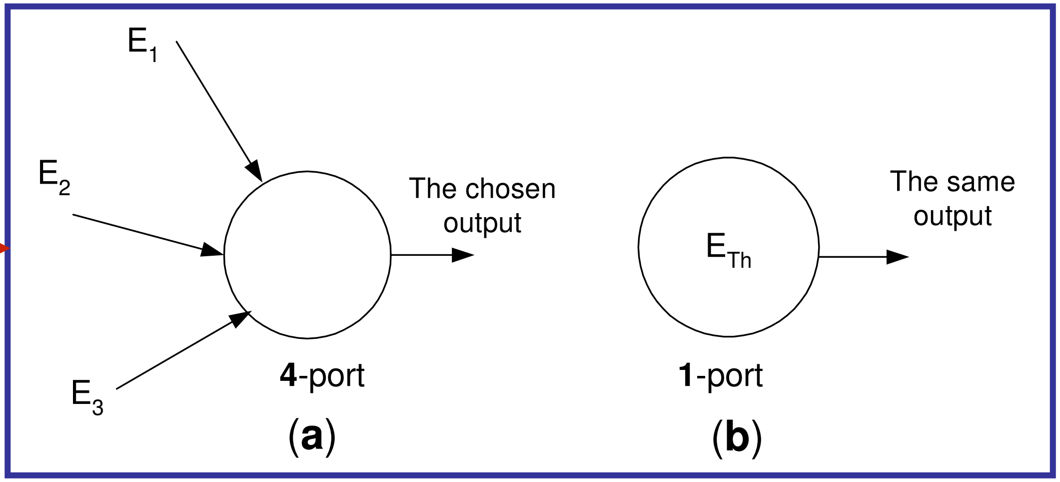
where also *LTV* systems are compared with *NL* systems.

!

Carefully *define* your system!

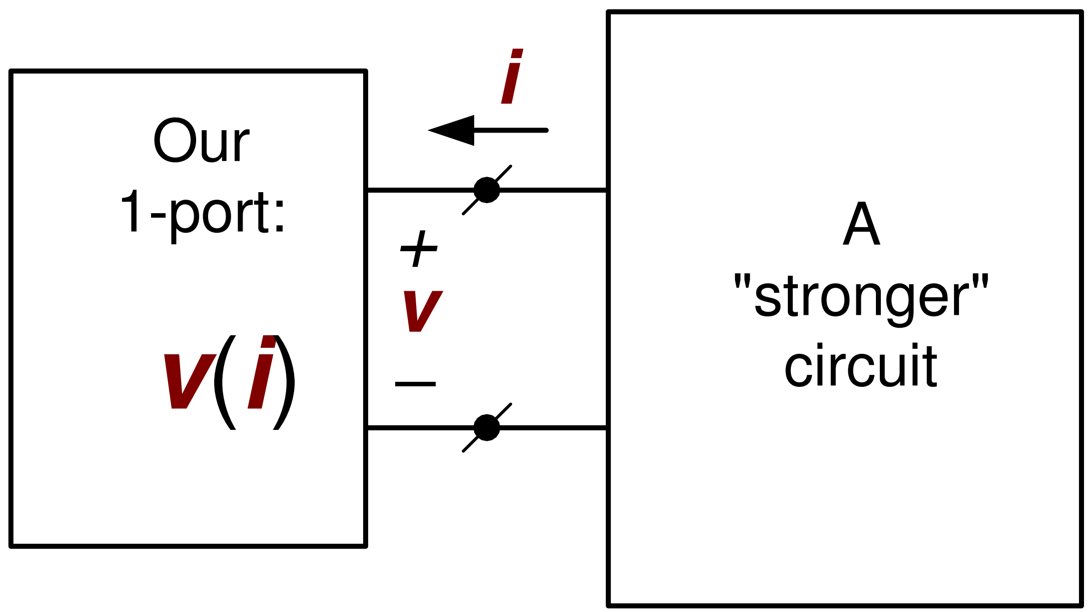
Where are the proposed sources, inside or outside?

If the inputs are defined so that the system appears to be active, as in case (**b**) (more generally, at least one of the ports is rejected as input), -- it is NL.



The very significant (and **unique** when compared to other numerous common textbooks on basic circuit theory) attention to the nonlinear resistive 1-ports in *Desoer&Kuh* (and in the known book by Professor L.O. Chua) should have, historically, the *ANN* of a 1-port as some background.

Indeed, it is reasonable to start the topic of **creation of the nonlinear resistive characteristics** $v(i)$, from the simple *ANN* characteristic:



A 1-port is used for creation of a nonlinear resistor

A dynamic version

$$y(t) = y(0) f(t) + (\hat{T}x)(t)$$

ZIR *ZSR*

Is it the case (**a**) of:

$$[y(0), x(t)]^T \to y(t)$$

i.e. the **linear one** (superposition),

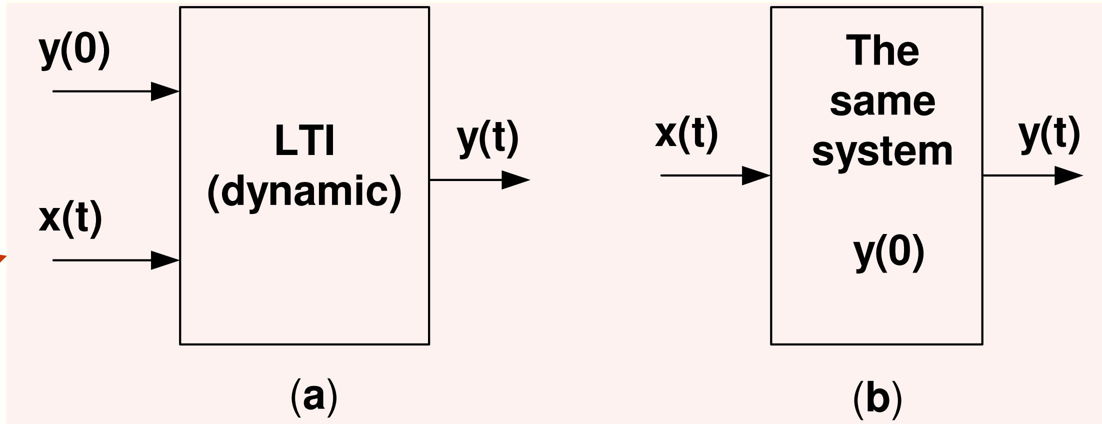
or the case (**b**) of:

$$x(t) \to y(t)$$

(with nonzero ZIR spoiling superposition) i.e. the **NL** one?

An outlook on the initial conditions:

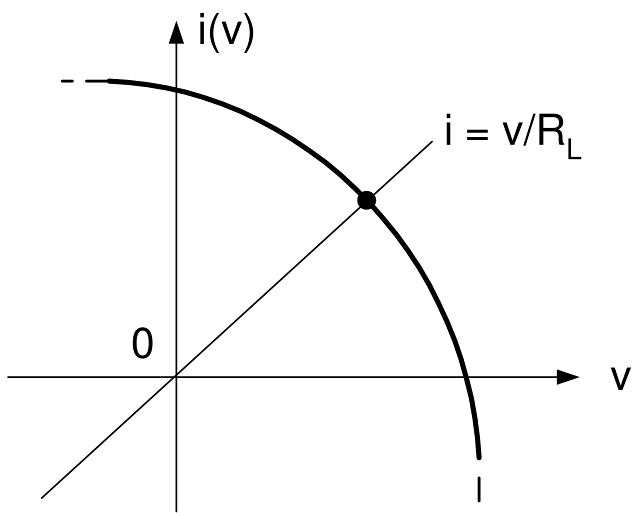
Since y(0) is given by us, it is also an input



Notice that in the domain of the Laplace variable, the interpretation of the initial values of state variables as "inputs" is even a standard one.

However, the ZIR + ZSR solution's structure exists also for LTV circuits.

Let us consider the solar cell characteristic:



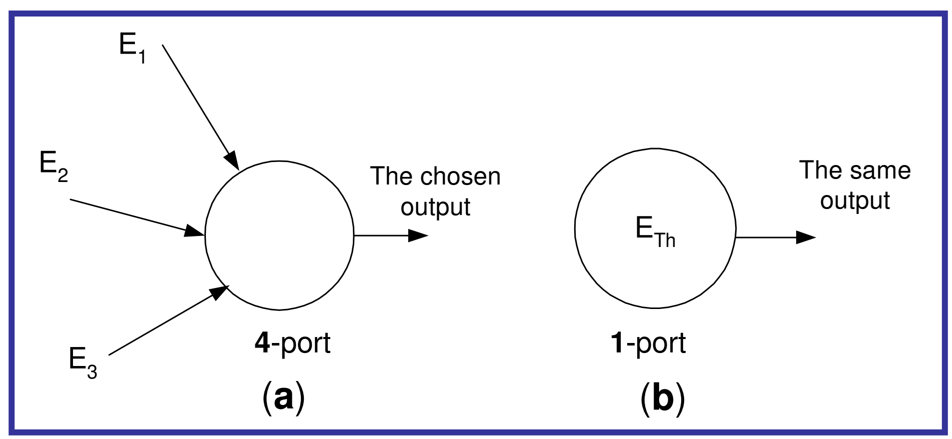
A power-supply unit

Writing

$$i(v) = i(0) + (i(v) - i(0))$$
$$= i(0) + f(v),$$
$$(where \;\; f(0) = 0) \, ,$$

one can see the “***affine kernel***” *i*(0) as the *main nonlinear* term, defining the power supply to the load.

Note that “$E_{1\text{-}3}$” can be physical inputs (here sun radiation), not necessarily batteries.



Consider now $x \to y$ given in the terms of time functions as

$$\hat{L}_1 y(t) = \hat{L}_2 x(t) + \psi(t) \quad (1a)$$

with some linear operators and ψ known. Obviously, it is the same *ANN* as (1).

Example for (1a)

Consider:

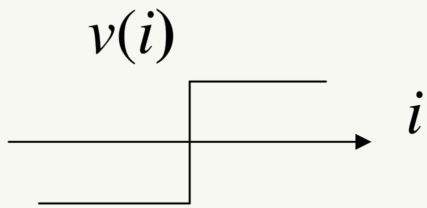
$$L\frac{di}{dt} + A\text{sign}[i(t)] + \frac{1}{C}\int i(t)\,dt \;=\; v(t) = U\xi(t) \quad (3)$$

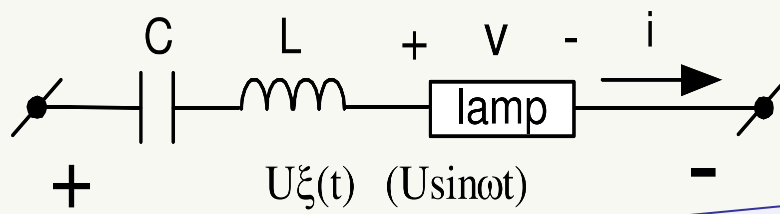
where ξ is a *T*-periodic (usually, sine) given function or "wave-form", normed in some way.

Equation (3) plays an important role in a **nonlinear theory of fluorescent lamp circuits** – very important power consumers [see, e.g., **[6] E. Gluskin, "On the theory of an integral equation",** ***Advances in Applied Mathematics*****, 15(3), 1994 (305-335), and also:** ***IEEE Trans. CAS, Pt.1*****, May 1999**].

As well, a mechanical version of (3) is known in the theory of systems with "**Coulomb friction**". (Then, energy consumption is not the main topic.)

The 50-60 Hz L-C-fluorescent lamp circuit:





"U" is changed using laboratory "varjak".

The KVL gives

$$L\frac{di}{dt} + A\,\mathrm{sign}[i(t)] + \frac{1}{C}\int i(t)\,dt \;=\; U\sin\omega t \qquad (3a)$$

where *L*, *A*, *C* and *U* are positive constants.

Eq.(3a) and the circuit are nonlinear, obviously, and

$$x = A/U$$

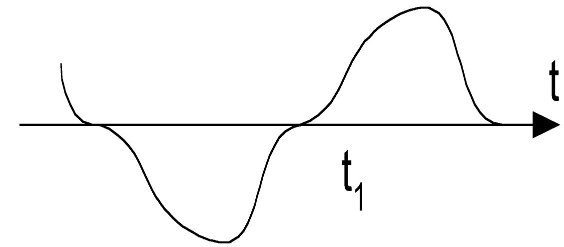
is the parameter of the nonlinearity.

The resistive (light-emitting) term is physically most important; LC is the lamp's "ballast".

!

For $x = A/U$ properly limited, *i*(*t*) is ([6]) a zerocrossing function:

and then, $Asign[i(t)]$ is a rectangular wave.



Using also that (similarly to the input function):

$$i(t+T/2) = -i(t)\ ,$$

we have *A*sign[*i*(*t*)] as the simple *square wave*

$$Asign[i(t)] = \frac{4A}{\pi}\sum_{1,3,5,...}\frac{\sin n\omega(t-t_1(x))}{n}\ .$$

This equality has the form of

$$F(t,\{t_k\}) = A\zeta(\{t-t_k(x)\});\quad t_k = t_k(\mathrm{mod}\,T),$$

with the zerocrossings of *i*(*t*) as parameters, and ζ known. Thus, (3) becomes

$$L\frac{di}{dt} + F(t,\{t_k\}) + \frac{1}{C}\int i(t)\,dt \ = \ U\xi(t)\ . \qquad \text{(3b)}$$

If

$$\omega_o \equiv \frac{1}{\sqrt{LC}} = 2\omega$$

where ω is the basic frequency of the periodic input, then (see [6]) t_k are **constant**, i.e. ***unmoved*** with the permitted change of *U*.

That is,

$$t_k(x) \equiv t_k(0) \quad [x = A/U],$$

This means that when U is changed, t_k are not shifted

thus

$$\zeta(\{t - t_k(x)\}) \equiv \zeta(\{t - t_k(0)\}),$$

and

$$F = A\zeta = Asign[i(t)] \sim A$$

is known before *i*(*t*) is determined.

In this case, after rewriting (3) as

The whole right-hand side is completely known

$$L\frac{di}{dt} + \frac{1}{C}\int i(t)\,dt = U\xi(t) - A\zeta(\{t - t_k(0)\})$$

one can mistakenly conclude that this is a linear equation (system).

This is the *ANN*

However, since the lamp (or the mechanical Coulomb-friction unit) remains in the actual circuit, the nonlinearity *must remain*, and in fact

$$L\frac{di}{dt}+\frac{1}{C}\int i(t)\,dt \;=\; U\xi(t)-A\zeta(\{t-t_k(0)\})$$

is an *ANN* equation:

$$(\hat{T}i)(t)+F(t,\{t_k(0)\}) \;=\; U\xi(t),$$

$\hat{T}$ is the linear operator of the *L-C* sub-circuit

or

$$\boxed{\hat{L}_1 y(t)=\hat{L}_2 x(t)+\psi(t)\,.}$$

Of course, the nonlinearity also has to be well seen via *power features* of the circuit, and indeed for ω_o/ω, as for any other ω_o/ω, we do ***not*** have (see the references) for the average power $P \sim U^2$, which would be necessary for any linear circuit in the periodic steady state.

You see that *ANN* can take the duties of a singular nonlinearity!

Let us return, however, to the algebraic characteristic, introducing now a *quantitative measure* for *ANN.*

Then, we shall a bit complete the classical theory of 1-ports to which the concept of *ANN* belong.

Observe that we deal with the algebraic 1-ports, -- not necessarily resistive, possibly also magnetic and ferroelectric.
(Consider, e.g., the "magnetic circuits" with "reluctances").

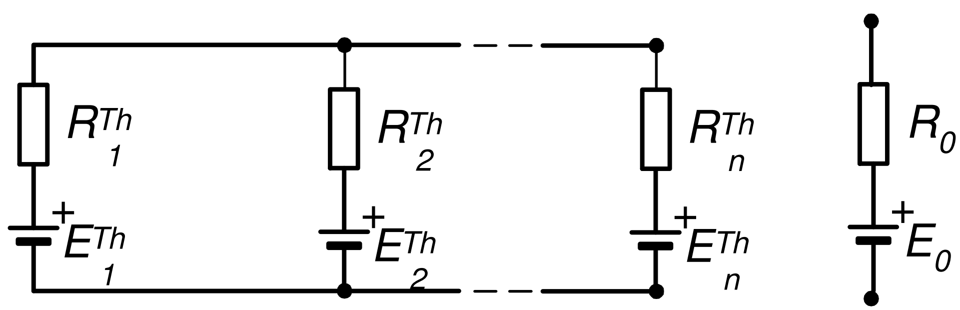
The quantitative ANN

Returning to the simple case of $v = \mp Ri + E$, let us define, for some quantitative estimations, the *measure of affine nonlinearity as*

$$ANN^e \equiv \frac{E}{Ri_o}$$

using some standard i_o .

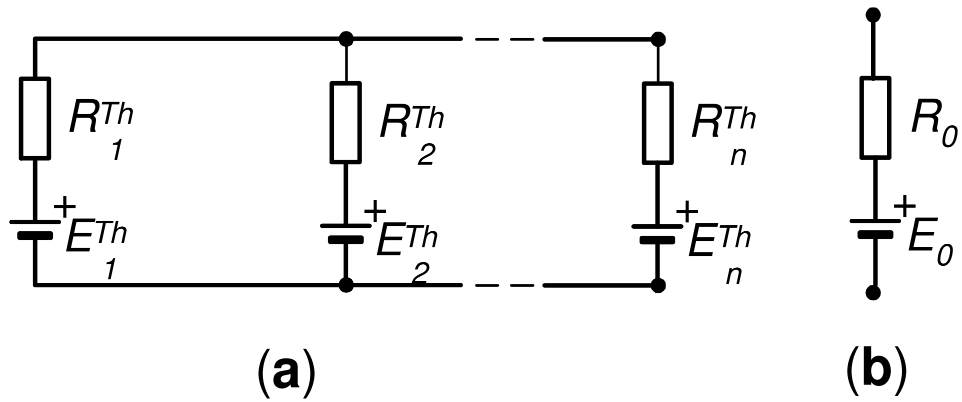
Thus defined, ANN^e can be changed, for instance, by means of parallel connections of Thevenin's 1-ports:



(**a**) (**b**)

Parallel connection of the 1-ports

For Fig. (**b**) here:



we have

$$ANN^e = \frac{E_o}{R_o i_o}$$

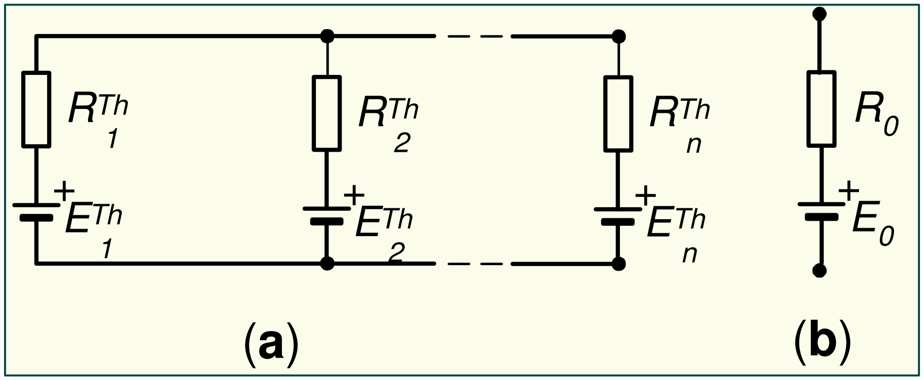
with

$$E_o = \frac{\sum_{k=1}^{n} (E_{Th\,k} / R_{Th\,k})}{\sum_{k=1}^{n} 1 / R_{Th\,k}}$$

and

$$R_o = \left(\sum_{k=1}^{n} 1/R_{Th\,k} \right)^{-1}$$

The parallel connection (cont)



Taking, for simplicity, all '*E*' and all '*R*' in Fig.(**a**) similar, we have in (**b**):

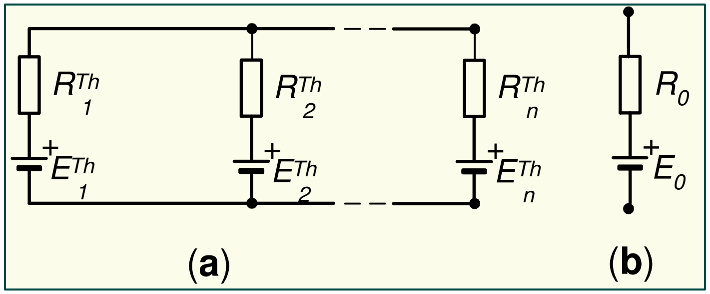
$$E_o = E, \quad and \quad R_o = R/n$$

which gives

$$ANN^e = \frac{E_o}{R_o i_o} = n\frac{E}{R i_o} \sim n\,.$$

While on the analytical side we obtain, as $n \to \infty$,

$$(ANN^e \sim n) \to \infty ,$$



on the structural side we have

$$R_o = R/n \to 0 ,$$

i.e. circuit (**b**) becomes pure **voltage hardlimiter** of

$$E_o = E.$$

A new far-reaching point starts here!

To this simple transfer to the hardlimiter we find a nontrivial analogy in elements with (generally very important, see below) power-law *i*(*v*) characteristic

$$i \sim v^{\alpha} , \qquad (4)$$

Note, here, this is a single element.

and just as it is/was with the solar-cell characteristic in Slide 13, *ANN* can be connected with (4), i.e. an affine kernel like *E is* observed in (4).

Indeed, rewriting the power-law conductivity characteristic

$$i \sim v^{\alpha}$$

as the dimensionally more reasonable:

$$i / i_o = (v / v_o)^{\alpha},$$

with some given i_o and v_o, we have for the respective $v(i)$, the limit

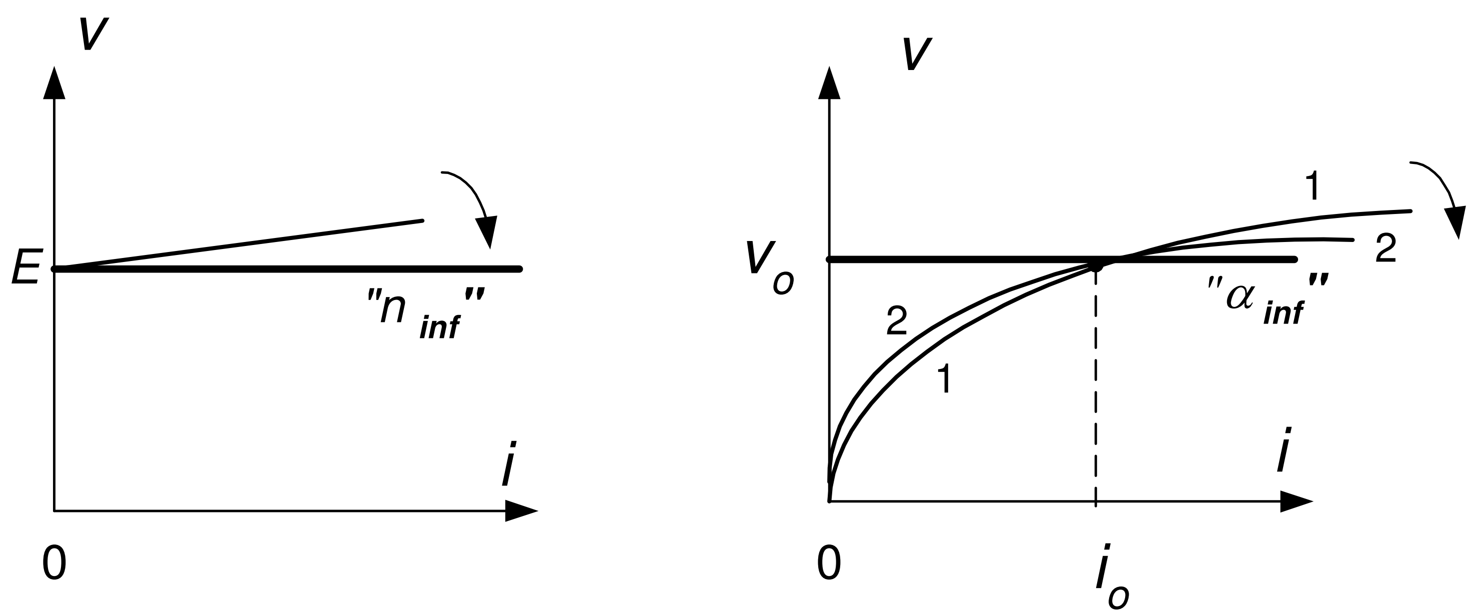
$$v / v_o = (i / i_o)^{1/\alpha} \underset{\alpha \to \infty}{\to} 1,$$

that is,

$$v \to v_o$$

as it is for *ANN* when $R_o \sim n^{-1} \to 0$.

Compare the two transfers to hard-limiters:



$$v = (R/n)i + E$$

$$(n \to \infty)$$

$$v/v_o = (i/i_o)^{1/\alpha}$$

$$(\alpha \to \infty)$$

<u>Remark</u>: For $0 < i < \infty$, both transfers are non-uniform.

The affine kernel

Using that for $\varepsilon \ln a << 1$ $a^{\varepsilon} \approx 1 + \varepsilon \ln a,$

we obtain for

$$v / v_o = (i / i_o)^{1/\alpha}$$,

$$v = v_o + \frac{v_o}{\alpha} \ln(i / i_o) , \quad \alpha >> \ln(i / i_o) ,$$

i.e. the **affine kernel** v_o is separated in the power-law characteristic.

Thus, for the mutual limitation on α and *i,* a circuit model of the power-law element (consuming energy, i.e. with *i* directed *inside*) can involve voltage hardlimiter or battery.

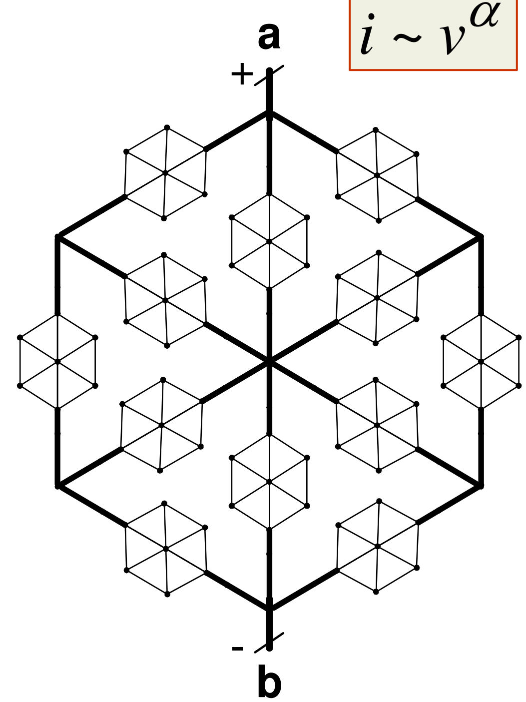
Thus, it appears that the kernel feature of *ANN* can be instructive also for this nonlinearity.

E. G. **A step towards circuit complexity (pardon!): The power-law characteristic and fractal 1-ports** "NDES 2012

Dealing with *ANN* circuits, we deal with the very basic concept of *1-port*, and now we are in position to observe two remarkable features of 1-ports (not mentioned in the classical theory of algebraic circuits) the second of which is associated with a new circuit connection.

1. Observe that since each circuit branch is a 1-port, *each 1-port is a specific potential fractal*. There is the possibility of repeating the whole structure in each branch, or in some of the branches. This recursive repetition works very well with the power-law elements, $i \sim v^{\alpha}$, because then the input conductivity, is of the same type ($\sim v_{inp}^{\alpha}$), just as it is for $\alpha = 1$ (linear resistors) or only linear capacitors or indictors.

Advice: make computer simulations of such recursive procedure, and study $i_{inp} = F(v_{inp})$.

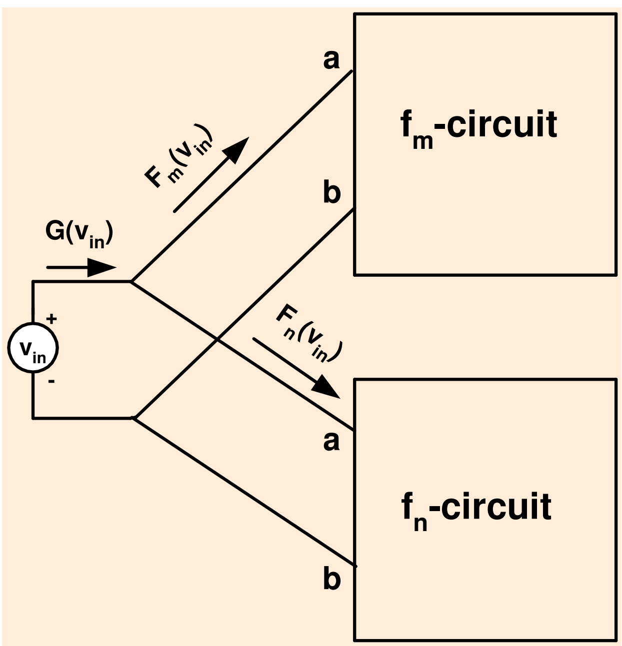


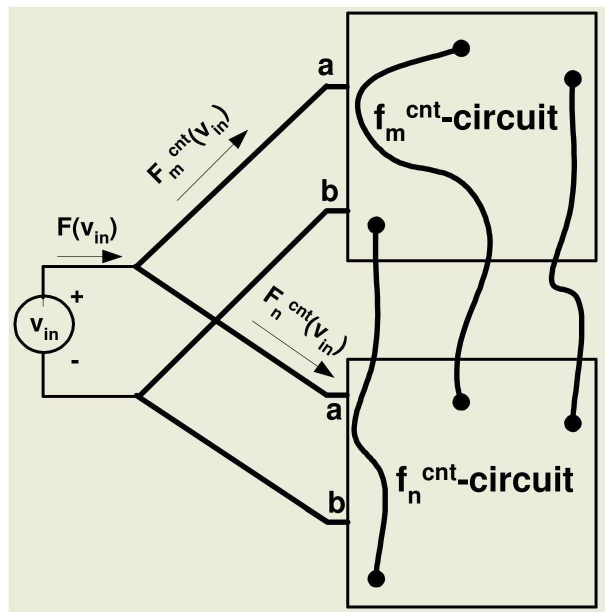
$i \sim v^{\alpha}$

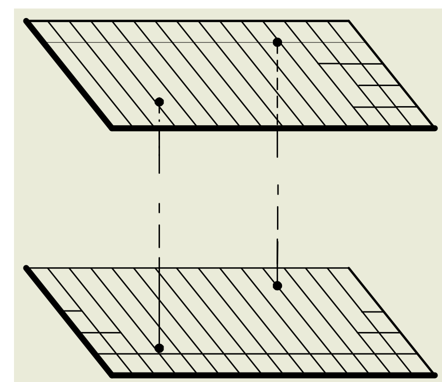
The specific features of the 1-ports with $i \sim v^{\alpha}$ suggest working with the more flexible for applications

$$i(v) = D_1 v^{\alpha_1} + D_2 v^{\alpha_2}$$

To make some conclusion of this model, let us introduce a new circuit connection, named "*f*-connection", which relates to circuits **of the same topology** and is a generalization of the usual parallel connection (in general, of not necessarily 1-ports).







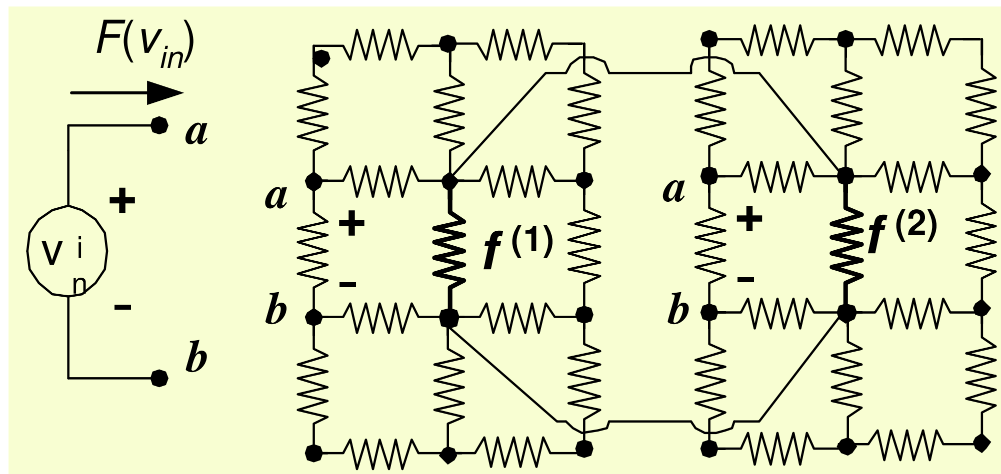
 Usual parallel connection

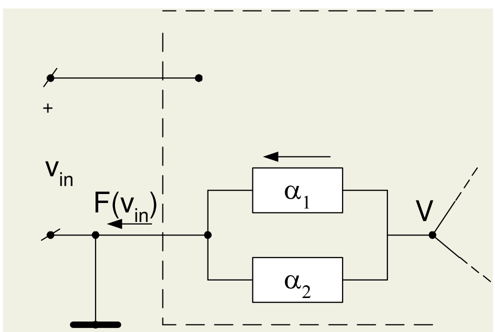
***f*-connection**

"*f*-connection" and a use of the power-law characteristic

The obtaining of a circuit with the prescribed topology, composed of elements $i(v) = D_1 v^{\alpha_1} + D_2 v^{\alpha_2}$:

Here, each f-connected circuit has similar elements in its branches.





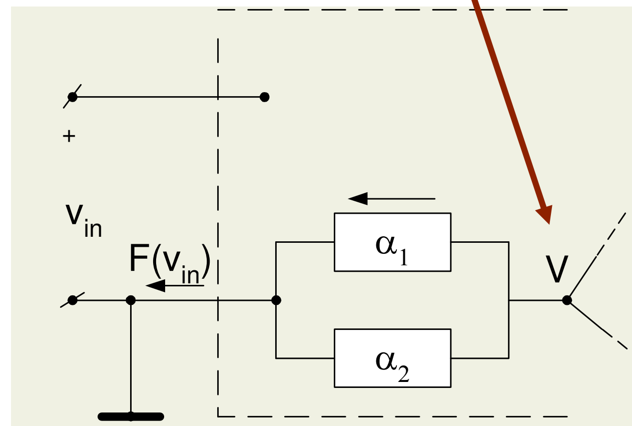
$$i = D_1 v^{\alpha_1}$$

$$i = D_2 v^{\alpha_2}$$

 ## *f*-connection and the input current of the "α-circuits"

For the two basic (individual) initial states we have at the node in focus:

{v_1, or v_2, or v}



$$v_1: \quad i = D_1 v^{\alpha_1};$$

$$v_2: \quad i = D_2 v^{\alpha_2};$$

Remark: Values of D_1 and D_2 do *not* influence the initial voltage distributions, just scale the input currents, i.e. $F \sim D$.

After *f*-connection, we have

hypothesis

$$v_1 < v < v_2;$$

$$D_1 v^{\alpha_1} + D_2 v^{\alpha_2} \approx D_1 v_1^{\alpha_1} + D_2 v_2^{\alpha_2}$$

The right-hand side relates to the usual parallel connection

That is,

$$F_{f-conct}(v_{inp}) \approx F_1(v_{inp}) + F_2(v_{inp})$$

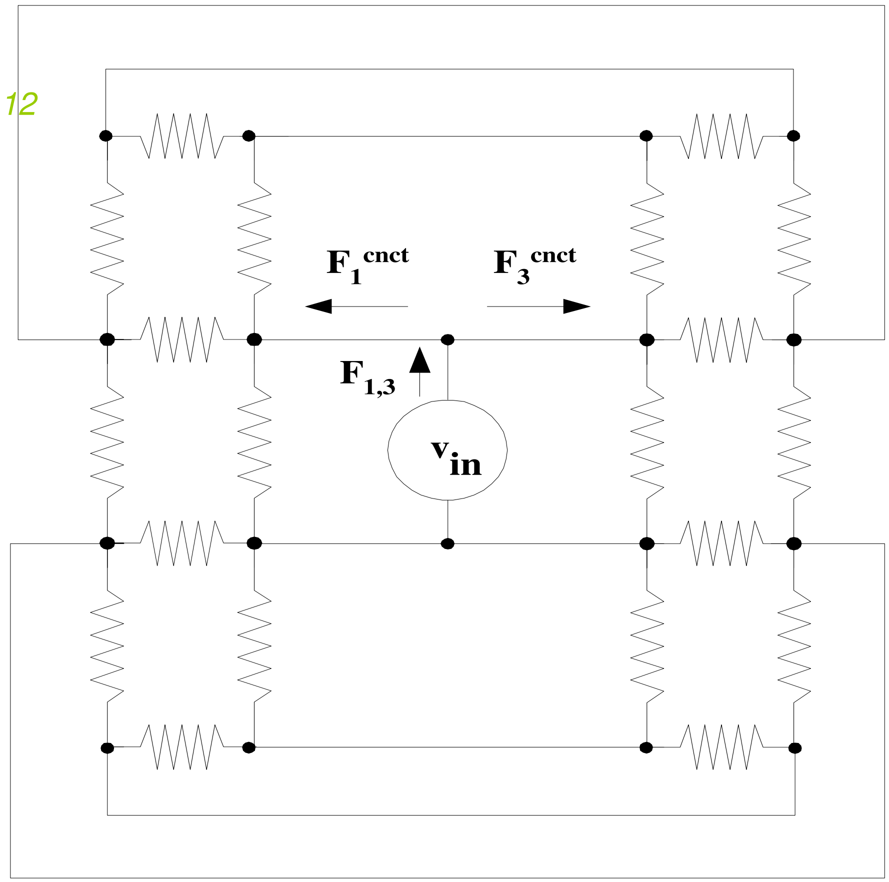
"***approximate analytical superposition***".

Possible applications:
1. Percolation theory (power degrees).
2. Spatial filtering (homogeneous structures).

?

Error in the analytical superposition

(observe the two *vertical* circuits)



f-connection

Left side: α =1

Right side: α =3

($v_{in} = 1$, $D_1 = D_3 = 1$).

Results:

The circuit	*F*	*F* *cnct*	Percent change
α =1:	**1.4**	**1.466**	**+4.7%**
α = 3:	**1.14**	**1.044**	**-8.4%**
f-connection	**2.511**		

MatLab simulation:

Usual parallel connection gave **2.54** ⟶ **1.15%** error versus **2.511**, i.e.

$$F_{1,3} = F_{f-connected} \lesssim F_1 + F_2$$

Some works on the circuits composed of the elements with the power-law characteristic $i \sim v^{\alpha}$:

"*One-ports composed of power-law resistors*", IEEE Trans. on Circuits and Systems II: Express Briefs 51(9), 2004 (464-467).

"*On the symmetry features of some electrical circuits*", Int'l. J. of Circuit Theory and Applications – 34, 2006 (637-644).

"*f - connection: a new circuit concept*", IEEE 25th Convention of Electrical and Electronics Engineers in Israel ("IEEEI 2008"), 2008, 3-5 Dec., pp: 056 – 060.

"An estimation of the input conductivity characteristic of some resistive (percolation) structures composed of elements having a two-term polynomial characteristic", Physica A, 381, 2007 (431-443).

"An approximation for the input conductivity function of the nonlinear resistive grid", Int'l. J. of Circuit Theory and Applications, 29, 2001 (517-526).

See also my ArXiv works devoted to "α-circuits" and "approximate analytical superposition".

END of the lecture