Two new constants μ, θ and a new formula $\pi=\frac{1}{2}e^{\theta}$

Chen Wenwei

（Room 301,No.48, ShiCen road, BaiYun district, GuangZhou ，China Postcode 510430）

Abstract: This paper brings to light a new constant μ hidden by Euler's constant γ. Euler's constant γ is the limit of the difference between harmonic series and ln*n*. The constant μ is the sum of the series of the remainder terms of the difference between harmonic series, ln*n*, Euler's constant γ and $\frac{1}{2n}$. Since both constant μ and the Euler's constant γ are relevant to the difference between harmonic series and ln*n*, we define a new constant θ =1+ γ + 2μ. This is a singular constant, together with π and e, we found a new perfect formula $\pi=\frac{1}{2}e^{\theta}$

Key words: γ, Euler's constant; μ, new constant; θ, new constant; π, the ratio of the circumference of a circle to its diameter; e, the natural base of logarithm; Formula $\pi=\frac{1}{2}e^{\theta}$.

1. The definition of the new constant μ and its formula

The sum of a harmonic series is,[1]

$$\sum_{k=1}^{n}\frac{1}{k}=\gamma+\ln n+\frac{1}{2n}-\sum_{k=2}^{\infty}\frac{A_k}{n(n+1)\cdots(n+k-1)} \qquad (1)$$

where $A_k=\frac{1}{k}\int_0^1 x(1-x)(2-x)\cdots(k-1-x)\,dx \qquad (2)$

We denote the remainder term as

$$\varepsilon_n=\sum_{k=2}^{\infty}\frac{A_k}{n(n+1)\cdots(n+k-1)} \qquad (3)$$

1.1 The lemma about remainder term ε_n

Lemma 1: The limit of $n\varepsilon_n$ is exist as n tends infinity , that is $\lim_{n\to\infty}n\varepsilon_n=0$

Proof：

The series expression of $n\varepsilon_n$ is：

$$n\varepsilon_n=\frac{A_2}{n+1}+\sum_{k=3}^{\infty}\frac{A_k}{(n+1)\cdots(n+k-1)} \qquad (4)$$

where, the general term can be expressed as

$$\frac{A_k}{(n+1)\cdots(n+k-1)}=\frac{\int_0^1 x(1-x)(2-x)\cdots(k-1-x)\,dx}{(n+1)(n+2)\cdots(n+k-1)k}$$

$$=\frac{1}{(n+k-2)(n+k-1)}\int_0^1\frac{x(1-x)(2-x)\cdots(k-1-x)}{(n+1)\cdots(n+k-3)k}dx$$

As $n\geq 2$, we have：

$$\int_0^1 \frac{x(1-x)(2-x)\cdots(k-1-x)}{(n+1)\cdots(n+k-3)k}dx \le \int_0^1 \frac{x(1-x)(2-x)\cdots(k-1-x)}{3\cdot 4\cdots(k-1)k}dx$$
$$= 2\int_0^1 \frac{x(1-x)(2-x)\cdots(k-1-x)}{1\cdot 2\cdots(k-1)k}dx < 2$$

The general term meets: $\frac{A_k}{(n+1)\cdots(n+k-1)} < \frac{2}{(n+k-2)(n+k-1)}$

The series meet: $\sum_{k=3}^{\infty}\frac{A_k}{(n+1)\cdots(n+k-1)} < \sum_{k=3}^{\infty}\frac{2}{(n+k-2)(n+k-1)} = \frac{2}{n+1}$

Therefore, $\lim_{n\to\infty} n\varepsilon_n = \lim_{n\to\infty}\frac{A_2}{n+1} + \lim_{n\to\infty}\sum_{k=3}^{\infty}\frac{A_k}{(n+1)\cdots(n+k-1)} \le \lim_{n\to\infty}\frac{A_2}{n+1} + \lim_{n\to\infty}\frac{2}{n+1} = 0$

Lemma 2: A new expression of the series of the remainder term

$$\sum_{n=1}^{\infty}\varepsilon_n = \sum_{k=1}^{\infty}\frac{A_{k+1}}{k\cdot k!} \qquad (5)$$

Proof:

Because both A_k and the general terms in above series are nonnegative, and the order of summing with respect to k or n are exchangeable, using the theorem about sum of the double series[2], we have,

$$\sum_{n=1}^{\infty}\varepsilon_n = \sum_{n=1}^{\infty}\sum_{k=2}^{\infty}\frac{A_k}{n(n+1)\cdots(n+k-1)}$$
$$=A_2\left(\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots\right)$$
$$+A_3\left(\frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\frac{1}{3\cdot 4\cdot 5}+\cdots\right)$$
$$+A_4\left(\frac{1}{1\cdot 2\cdot 3\cdot 4}+\frac{1}{2\cdot 3\cdot 4\cdot 5}+\frac{1}{3\cdot 4\cdot 5\cdot 6}+\cdots\right)+\cdots$$
$$=A_2+\frac{A_3}{2\cdot 2!}+\frac{A_4}{3\cdot 3!}++\frac{A_5}{4\cdot 4!}+\cdots+\frac{A_{k+1}}{k\cdot k!}+\cdots$$
$$=\sum_{k=1}^{\infty}\frac{A_{k+1}}{k\cdot k!}$$

Lemma 3: The series of the remainder terms is convergence

Proof:

In lemma 2, the general term of the series of the remainder terms ε_n is

$$\frac{A_{k+1}}{k\cdot k!} = \frac{1}{k(k+1)k!}\int_0^1 x(1-x)(2-x)\cdots(k-x)dx$$
$$= \frac{1}{k(k+1)}\int_0^1 x\frac{(1-x)}{1}\cdot\frac{(2-x)}{2}\cdots\frac{(k-x)}{k}dx$$

Since $0\le x\le 1$, we have $0\le\frac{k-x}{k}\le 1$, and,

$$\frac{1}{k!}\int_0^1 x(1-x)(2-x)\cdots(k-x)dx \le 1$$

Therefore, the general term should meet

$$\frac{A_{k+1}}{k\cdot k!}\le\frac{1}{k(k+1)}<\frac{1}{k^2}$$

Since A_k is nonnegative, the convergence theorem tell us the series of the tail terms is convergence

1.2 The definition of a new constant, μ.

By Lemma 3, the series of the remainder term ε_n, which is the remainder term of harmonic series, must be convergence to a constant, say μ, we have the following definition.

Definition 1. A new constant μ is the sum of the series of the remainder term ε_n, that is

$$\mu=\sum_{n=1}^{\infty}\varepsilon_n \qquad (6)$$

（1）The first formula to calculate constant μ

The formula (5) of lemma 2 directly leads to： $\mu=\sum_{k=1}^{\infty}\frac{A_{k+1}}{k\cdot k!}$ (7)

Where, A_k follows from formula (2)

（2）The second formula to calculate constant μ

The constant μ can also be calculated by the sum and remainder of the harmonic series, see formula (1) and (3).

$$\mu=\sum_{n=1}^{\infty}\sum_{k=2}^{\infty}\frac{A_k}{n(n+1)\cdots(n+k-1)} \qquad (8)$$

Where, A_k follows from formula (2)

（3）The third formula for constant μ

Revise the formula for the sum of the harmonic series, the constant μ can also be calculated by

$$\mu=\sum_{n=1}^{\infty}\left(\gamma+\ln n+\frac{1}{2n}-\sum_{k=1}^{n}\frac{1}{k}\right) \qquad (9)$$

This formula clue us on how the new constant μ and the Euler's constant γ are interrelated.

2. The definition and formula for new constant θ

（1）The formula for Euler's constant γ and its value[3]

Euler's constant γ is the limit of the difference between harmonic series and ln*n*. The formula and its value are

$$\gamma=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\ln n\right)=0.57721566490153286060651\cdots \qquad (10)$$

（2）The value of the new constant μ

A computer program, using formula (9), has found the value of the new constant μ is

$$\mu=0.13033070075390631147707\cdots \qquad (11)$$

（3）The definition of the new constant θ

The formula (9) tells us that both Euler's constant γ and the new constant μ are related to the difference between harmonic series and ln*n*. Combine the formula (9), (10) and (1), we have found the remainder term, ε_n of the harmonic series is a very small number which has proved by the lemma 1. We have also found that the sum of the remainder series is a constant and we define is as a new constant μ. We realize that constant μ is a correlative constant of Euler's constant γ, and so there must be a new constant, say θ, also hides behind γ. Or we can define θ using γ and μ.

Definition 2: New constant θ is defined as

$$\theta=1+\gamma+2\mu=1.83787706640934548356065\cdots \qquad (12)$$

This is a singular constant and will be explained in detail later.

3. A new formula combined by θ, π and e

3.1 A new approximation to the factorial n!

Theorem 1: A new formula to approximate n!

$$n! = e^{\frac{1}{2}\theta} n^{n+\frac{1}{2}} e^{-n} e^{\eta_n} \qquad (13)$$

Proof: The formula (14) is Abel' formula for the sum

$$\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n-1} S_k \Delta b_k + S_n b_n \qquad (14)$$

Let

$$a_k = \frac{1}{k}, \quad b_k = k$$

We have

$$S_k = \sum_{m=1}^{k} a_m = \sum_{m=1}^{k} \frac{1}{m}, \quad \Delta b_k = b_k - b_{k+1} = -1$$

Substitute them back to (14), we have

$$n = (n+1)\sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{n}\sum_{m=1}^{k} \frac{1}{m} \qquad (15)$$

Summing the sum in formula (1) again (where the remainder is replaced by ε_k), we have

$$\begin{aligned}\sum_{k=1}^{n}\sum_{m=1}^{k}\frac{1}{m} &= \sum_{k=1}^{n}\left(\gamma + \ln k + \frac{1}{2k} - \varepsilon_k\right)\\ &= n\gamma + \sum_{k=1}^{n}\ln k + \frac{1}{2}\sum_{k=1}^{n}\frac{1}{k} - \sum_{k=1}^{n}\varepsilon_k \qquad (16)\\ &= n\gamma + \ln(n!) + \frac{1}{2}\left(\gamma + \ln n + \frac{1}{2n} - \varepsilon_n\right) - \sum_{k=1}^{n}\varepsilon_k\end{aligned}$$

Substitute expression (16) to (15), and sort it out, we get

$$\begin{aligned} n &= (n+1)\left(\gamma + \ln n + \frac{1}{2n} - \varepsilon_n\right) - n\gamma - \ln(n!) - \frac{1}{2}\left(\gamma + \ln n + \frac{1}{2n} - \varepsilon_n\right) + \sum_{k=1}^{n}\varepsilon_k\\ &= (n+1)\gamma + (n+1)\ln n + \frac{(n+1)}{2n} - (n+1)\varepsilon_n - n\gamma - \ln(n!) - \frac{1}{2}\gamma - \frac{1}{2}\ln n - \frac{1}{4n} + \frac{1}{2}\varepsilon_n + \sum_{k=1}^{n}\varepsilon_k \qquad (17)\\ &= \frac{1+\gamma}{2} + \frac{1}{4n} + \ln n^{n+1} - \ln(n!) - \ln n^{\frac{1}{2}} - n\varepsilon_n - \frac{1}{2}\varepsilon_n + \sum_{k=1}^{n}\varepsilon_k\\ &= \frac{1+\gamma}{2} + \mu + \ln n^{n+1} - \ln(n!) - \ln n^{\frac{1}{2}} + \eta_n\end{aligned}$$

Where, $\eta_n = \dfrac{1}{4n} - n\varepsilon_n - \dfrac{\varepsilon_n}{2} - u_n$ (18)

$$\sum_{k=1}^{n}\varepsilon_k = \mu - u_n \qquad (19)$$

Lemma 3 leads to

$$\lim_{n\to\infty} u_n = 0$$

Lemma 1 leads to

$$\lim_{n\to\infty} \eta_n = 0$$

Take both sides of (17) as exponents of e, and reform it, we have the expression (13)
Expression (13) is a new formula for n!, which is different with Stirling's formula.
Corollary: The following limit is hold

$$\lim_{n\to\infty}\frac{n!e^n}{n^n\sqrt{n}}=e^{\frac{1}{2}\theta} \qquad (20)$$

3. 2 A new formula combining π, e, and θ

Theorem 2: There is a very meaningful relationship between π, e, and the new constant θ

$$\pi=\frac{1}{2}e^{\theta} \qquad (21)$$

Proof: Stirling's formula holds for all n[3], or:

$$\lim_{n\to\infty}\frac{n!e^n}{n^n\sqrt{n}}=\sqrt{2\pi} \qquad (22)$$

Comparing with formula (20), the uniqueness theorem of a limit tell us

$$\sqrt{2\pi}=e^{\frac{1}{2}\theta} \qquad (23)$$

That is same to formula (21)

4. Discussion about the new constant and the new formula

(1) The inherent relationship between π, e and θ, even they come from different sources

We know, π is the ratio of the circumference of a circle to its diameter. The e is the natural base of logarithm, and e is invariant in the process of differentiating and integrating. Even e and θ, come from different sources, it is found that they are connected together by imaginary number i in the wonderful formula $e^{\pi i}=-1$.

Here, the research in harmonic series and the remainder series has lead to a constant μ, which is a new constant behind the Euler's constant γ. All π, e and θ are related to ln*n*, and are combined to a new constant θ. A further research leads to a new approximation for n!, and another wonderful formula $\pi=\frac{1}{2}e^{\theta}$.

This new formula widens our field of view, and reveals the natural relationship between π and e.

(2) The transform between π and e.

In probability and statistics, many formulas include natural exponential function e^x and $\sqrt{2\pi}$

If we use formula (23) to replace $\sqrt{2\pi}$, the natural exponential function e^x and constant e^{θ} will be combined tightly. For example, The Euler-Poisson integration formula

$$\int_{-\infty}^{\infty}e^{-\frac{x^2}{2}}dx=\sqrt{2\pi} \quad \text{will transformed as} \quad \int_{-\infty}^{\infty}e^{-\frac{1}{2}\left(x^2+\theta\right)}dx=1$$

The Fourier transformation

$$F(\lambda)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(t)e^{-i\lambda t}dt \quad \text{will transformed as} \quad F(\lambda)=\int_{-\infty}^{\infty}f(t)e^{-(i\lambda t+\frac{\theta}{2})}dt$$

Normal distributing function

$$\phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-\frac{t^2}{2}}\,d \quad \text{will transformed as} \quad \phi(x)=\int_{-\infty}^{x}e^{-(\frac{t^2}{2}+\theta)}dt$$

(3) The transform between γ and μ

Professor Xi, Zixing , Fudan University, Shanghai China, reads my paper and thinks that the Euler's constant can be used for approximating Gamma function. However, there are no enough exquisite way to calculate γ. It is easier to calculate μ instead of γ. Therefore, he suggest starting from μ to calculate constant γ. To this end, he has programmed this approximation process and verified his idea, himself. Both his and my results are very close to the true value of the μ. Here I am grateful to professor Xi. I am especially appreciative his suggestion and verifications.

References:

[1] И.M. Leirike etc, Table for functions and Integrations，Higher education publishing company, 1959

[2] Hua Luogeng ,Introduction to the Advanced Mathematics, Science publishing company,1979

[3] The hand book for Mathematics, Dumb millions education publishing company,1979