A conjecture about real density

As mentioned previously, I'm using the mathematical term density rather loosely; here, what I mean by density is positive Lebesgue measure. It's clear the density of the reals is based on the density of its proper subset – the irrationals. But what of those? As with the previous article, let's return to binary representation since that's the simplest form:

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1. sqrt(10)      = 1.011010100000100...
2. sqrt(10)/10 =  .1 011010100000100...
3. sqrt(p) =     w.uis
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Before we discuss above, I need to cover another related topic: irrationality measure. Please visit the link and 'try to make some sense of it'. According to that article, there are basically three kinds of irrational numbers: algebraic, transcendental, and functional. An example of an *algebraic irrational* number is sqrt(2) (which is above on line one in binary format). An example of a *transcendental* irrational number is π (pi, the ratio of circumference to diameter of any circle). Please consult the link for examples of functional irrational numbers. Ready? Okay.. Now let's revisit what Jodie Foster talked about in Contact: the prime numbers: 2, 3, 5, 7, 11,... In binary: 10, 11, 101, 111, 1011,... Why? We'll see.. Prime numbers are unique whole numbers that are only divisible (with whole number result) by themselves and one. So now we're ready to review the statements above. Line one basically states in *binary form* that the square-root of two is equal to the value on the right-hand-side. Line two says what the square-root of two is *divided by two* – again, in binary representation. Notice *the value on line two is simply the value on line one with the 'binary point' moved one* – ***to the left***. Also notice that *both numbers are irrational*. Finally notice that *we can perform this operation forever* – dividing an irrational by two – *to get **another unique** irrational number*.

What about transcendental numbers? Off the top of my head, I can think of only two: π and *e*. That's pretty bad for a math major! One more thing before we state the conjecture. Line three above states the square-root of a prime number is equal to a ***whole*** number and ***u**nique-**i**nfinite-**s**equence* fractional part. We're ready to state the conjecture:

The density of the real numbers is based on the density of the algebraic irrationals which in turn is based on the (countably) infinite primes.

The point of this article is not to impress you with my mathematical fluency .. Allow me to digress.. Many years ago, I realized there was *absolutely nothing* special about me. The ***only*** things that would allow me to 'stand out' in a crowd was my mind and perspectives. I firmly believe I have an average IQ: 100, but I've tested far above.. Why? Because I've *trained my mind* .. More than this, I've come to understand *we all have gifts*; we simply need to discover them. There are mathematicians in the past who seemed connected to an 'infinite fountain of inspiration'. This is a historical fact. They give evidence, as does above, that we are merely *conduits* of inspiration and creativity. And finally, conduits of love. What more evidence of God do you need?

I challenge you to *prove* the conjecture above; I challenge you to find *your* unique gifts and inspirations – *whatever* your interests may be..

in true-love, sam

Open questions about irrational numbers

Is $\pi+e$ irrational? Is sqrt(2)+sqrt(3)? One of the open questions (they don't know either way) dealing with irrational numbers is: *is the sum of of two irrational numbers irrational?* To address this, let's think about the crucial property of irrationals: the unique-infinite-sequence (or uis) following the decimal/binary point representing the unique fractional part of an irrational number (as explained in the previous article). So in decimal format, $3-e$ and $4-\pi$ are what I call the uis-complements of e and π respectively. The reason I choose this term is because under addition, they produce whole numbers. It's not hard to visualize this critical feature in binary format; please try it.

When we fully understand $3-e$, $4-\pi$, 2-sqrt(2),.. are *unique* irrational numbers themselves, this so-called open question becomes almost trivial. From a probability standpoint anyways.. If we randomly choose a number from the real line, chances are it will be irrational is 100%! This is the astounding result from real analysis. What are the chances we choose a *particular* irrational number? 0%. So.. The chances $\pi+e$ or $\pi-e$ is *rational* is zero. When we think about the uniqueness of irrational complements, we're Confident about this claim.

Now we're ready to state another conjecture. Here, we introduce another term: distinct. For purposes of this discussion, I define distinct analogously to linear-independence. Two irrational numbers are distinct if you cannot make a liner combination of one out of the other using rational numbers. More explicitly, two irrational numbers are *distinct*, x and y, if you cannot find *any* non-zero rational numbers, a and b, such that $y = ax + b$ is true. Conjecture 1:

The sum of two distinct irrational numbers is irrational.

Conjecture 2:

The product or quotient of two distinct irrational numbers is irrational.

Conjecture 3:

The exponentiation of two distinct irrational numbers is irrational.

One and two together equate with conjecture 4:

The set of distinct irrational numbers is a field.

Why is this important? In a sense, we don't 'need' rational numbers in math! (Remember, the rationals include the integers – the 'counting numbers'.) Any rational number can be 'approximated' with a nearby irrational. In a very real sense, we can dispense with *all* other numbers! ..Of course, these statements are facetious computationally but illustrate the overwhelming density of the irrationals.

Convince yourself: if you doubt above, try the operations above on your calculator. For instance, try using sqrt(2) and sqrt(3) as components and look at the results you get. Use any two distinct irrational numbers.. If you find two that violate a statement above, *let me know!* We all can make mistakes!

Counting infinities

This may not be the way they 'do it' in number theory nor may it be 'standard notation' but.. It seems the most intuitive approach toward measuring the actual density of irrationals and rationals. The existing notation in number theory seems to be deliberately obfuscating and I will deliberately avoid it here .. From the previously outlined view, the simplest algebraic irrational described in the simplest way is the square-root of 2 in binary:
sqrt(10) = 1.011010100000100...
This is a *single instance* of an irrational number. The whole portion is immaterial at this moment. The irrationality of the number is *completely defined* by the fractional portion. If we flip the nth bit, this implies with this *one* irrational number, we can create a countably infinite set. Let's notate this with the symbol inf^1-sub-c or simply inf^1-c. Now if we flip the first *and* nth bit, this creates a *whole new* countably infinite set of irrationals with size inf^1-c. Carry this operation on forever implies the size of the set of irrationals we can create from the *single* irrational number sqrt(10) is inf^2-c! This is already quite revealing! From a *single instance*, we've created an order 2 infinite set of irrationals! (If not obvious, inf = infinity – the symbol for infinity, the 'lazy eight'. ^ = super-script or exponentiation. sub-c or -c indicate a sub-script – here, it means 'c' for countable.)

This directly implies the O(sqrt(p)) = 3 or S(sqrt(p)) = inf^3-c where O and S mean order and size respectively. We've made tremendous progress simply by using convenient and intuitive notation! We've begun to actually *count* the irrational numbers! Now, let's use the field properties of the irrationals to count some more! From those, we know O(sqrt(p1)+sqrt(p2)) = 2 and O(sqrt(p1)sqrt(p2)) = 2 which implies S(*,+ o sqrt(p)) ≥ 2inf^2-c. In English, the size of the set with multiplication and addition operating on the set of the root-primes is greater than or equal to two times order-two countably infinite. Before we stop this tack, let's consider one final implication. If we consider sequences of operations on distinct root-primes, they produce order 4 sets:
∑(2,inf)S(∑(i,j)sqrt(p-i)) = inf^4-c and ∑(2,inf)S(∏(i,j)sqrt(p-i)) = inf^4-c
which implies S(**I**) ≥ 2inf^4-c. The irrational numbers are at least order 4.

Let's try another way of looking at this.. Let's explicitly look at addition, subtraction, multiplication, and division as operations on root-primes: +, -, *, 1/ . Let's denote distinct root-primes as a1, a2, a3, and a4. There's only one way to add: +a1+a2+a3+a4. But there are four ways to add: -a1+a2+a3+a4 because we can move the minus sign to four distinct positions. Similarly, -a1-a2+a3+a4 can be added six ways based on the positions of the two minus signs.. And, -a1-a2-a3+a4, 4. Finally, -a1-a2-a3-a4 can only be added one way. That totals 16 ways to add/subtract four distinct numbers. Similarly, we can explore the 16 ways to combine a1a2a3a4. The interesting question is: how many ways can we arithmetically combine four distinct numbers based on the *four* basic operations? The answer is 4^4. Examples are: a1-a2a3/a4 and (1/a1)a2-a3+a4. This implies S(+,-,*,1/ o sqrt(p)) = inf^4-c. So we've shown another way that the irrationals are order 4.

What about the rationals? What's their order? Let's consider the unit-interval. If we look at the whole number 1 then divide by two progressively until that limit is zero, we get the size of that subset as inf^1-c. But we don't have to start with 1; we can start with *any* rational number and move toward another limit rationally. That implies the size of the rationals is inf^2-c. Convention claims they're countably infinite which they are but.. They're clearly not order 1 .. I spent some time trying to convince myself the order of the rationals is greater than 2. I was trying to enumerate all unique complementary sequences. The challenging problem is to find complementary sequences so that their union is the rationals. Try this yourself. Here, what we mean by complementary is disjoint – no common elements. And, the notion relates to completeness so.. We're ready for a new conjecture:

The union of all complementary rational sequences is the set of rational numbers with order 2.

What's the point of all this? To simply befuddle? [wink] No. The purpose of this approach is to get a 'handle' (intuitive idea) of the respective densities of the rationals and irrationals. To say one is countably infinite and sparsely dense while the other is uncountably infinite and dense is *not* adequate. We need new measures that differentiate *exactly why* they're different. The notions introduced above allow us to create an analogy so we can understand the relative densities *completely* .. Here is the analogy we've been looking for.. The density of the irrationals compared to the density of the rationals is *perfectly analogous* to the following scenario. Imagine you pick a random point in **R^3**, what are the chances your number is on the x-axis? Clearly zero. What are the chances it's elsewhere? 100%. **R^3** is two orders of magnitude 'more infinite' than **R^1** (order 12 vs order 4). So we're ready for another conjecture:

The order of the reals is 4; the size is inf^4-c + inf^2-c based on the respective sizes of irrationals and rationals.

This approach gives us a much more intuitive understanding of relative densities. No longer are sets simply 'countably infinite' or 'uncountably dense'.. We have a very precise language now to describe different 'levels of infinities' .. Let's practice our new skills and identify sizes and orders of common sets.. Ready?

What's the size and order of **I^2**? It should be clear the size is *not* 2inf^4-c but inf^8-c. The order is therefore 8.

What's the size and order of **R^2**? By now, the 'technique' should be clear:

S-**R^2** = (inf^4-c + inf^2-c)^2 = (S-**I**)^2 + (S-**Q**)^2 + 2(S-**I**)(S-**Q**)

inf^8-c + inf^4-c + 2inf^6-c

The corresponding order is 8 clearly from the dominating subset **I^2**.

Calculating the size and order of **R^3** is left to the reader.

What about **C**, the set of complex numbers? This may appear confusing at first because a complex number is typically written as a+bi where a is the real part and b is the coefficient of the imaginary part. Don't worry, the complex-plane corresponds to **R^2** so we can 'cheat' by simply inspecting those associated values. S-**C** = S-**R^2** and O-**C** = O-**R^2**.

Size is an appropriate term because it essentially equates with the number of elements in a set. But the old notation typically found in texts on number theory simply makes the knowledge *inaccessible* to many, if not most, students. If the arguments in these last few articles are sound, we have made number theory and real analysis accessible to high-school students. It's a tremendous leap forward for the mathematical sciences and education .. With this perspective, there's not many open questions left, but thank God there are:

Is π an element of {+,-,*,1/ o sqrt(p)}?

Is *e* an element of {+,-,*,1/ o sqrt(p)}?

What's S-**T** where **T** are the transcendentals?

From one perspective, they're inconsequential. From another, 'just' as important as sqrt(2) (possibly more-so). For instance, if they're not elements of the set based on root-primes, they may represent a *subfield* of **I**. As the primes may represent a 'basis' for **I**, π and *e* may be a basis for **T**. It's an interesting problem I pose for the reader.