New Expression of the Factorial of n ($n!$, $n \in N$)

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Abstract

New Expression of the factorial of n ($n!$, $n \in N$) is given in this article. The general expression of it has been proved with help of the Principle of Mathematical Induction. It is found in the form

$$1 + \sum_{i=1}^{n} a_i + \sum_{\substack{i,j=1 \\ (i<j)}}^{n} a_i a_j + \sum_{\substack{i,j,k=1 \\ (i<j<k)}}^{n} a_i a_j a_k + \cdots + a_1 a_2 \cdots a_n, \tag{1}$$

where $a_i = i - 1$ for $i = 1,\ 2,\ \cdots,\ n$. More convenient expression of this form is provided in Appendix.
Keywords: Factorial, new expression of factorial

1 Introduction

In mathematics, the factorial of a non-negative integer n is denoted by $n!$. It is defined by the product of all positive integers less than or equal to n. Thus $n! = 1 \times 2 \times 3 \times \cdots \times n$. For example, $1! = 1$, $2! = 2$, $3! = 6$, $4! = 24$ etc; while the value of $0!$ is 1 according to the convention for an empty product [1]. The most basic occurrence of factorial function is the fact that there are $n!$ ways to arrange n distinct objects into a sequence (i.e., number of permutations of the objects). To Indian scholars this fact was well known at least as early as the 12th century [2]. Although the factorial function has its roots in combinatorics, the factorial operation is encountered in many different areas of mathematics such as permutations, algebra, calculus, probability theory and number theory.

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2 Theorem

The following expression holds for factorial of n ($n!$, $n \in N$):

$$n! = 1 + \sum_{i=1}^{n} a_i + \sum_{\substack{i,j=1 \\ (i<j)}}^{n} a_i a_j + \sum_{\substack{i,j,k=1 \\ (i<j<k)}}^{n} a_i a_j a_k + \cdots + a_1 a_2 \cdots a_n, \tag{2}$$

where $a_i = i - 1$ for $i = 1,\ 2,\ \cdots,\ n$.

3 Proof

Case: $n = 1$

$a_1 = 0$, $\sum_{i=1}^{n} a_i = a_1 = a_1 a_2 \cdots a_n = 0$ for this case. The value of right hand side (RHS) of (2) can then be obtained as 1, while we know $n! = 1$ for $n = 1$. These show that the formula is valid for $n = 1$.

Case: $n = 2$

To prove the formula for this case, the values $a_1 = 0$, $a_2 = 1$, $\sum_{i=1}^{n} a_i = a_1 + a_2 = 1$, $\sum_{\substack{i,j=1 \\ (i<j)}}^{n} a_i a_j = a_1 a_2 = a_1 a_2 \cdots a_n = 0$ have been used at RHS of (2). The computed value is then given as 2 that is the exact value of $n!$ for $n = 2$. Thus the formula is valid for $n = 2$.

Case: $n = 3$

We have $a_1 = 0$, $a_2 = 1$, $a_3 = 2$, $\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 = 3$, $\sum_{\substack{i,j=1 \\ (i \neq j)}}^{n} a_i a_j = a_1 a_2 + a_2 a_3 + a_3 a_1 = 2$, $\sum_{\substack{i,j,k=1 \\ (i \neq j \neq k)}}^{n} a_i a_j a_k = a_1 a_2 a_3 = a_1 a_2 \cdots a_n = 0$. The value of RHS of (2) can be evaluated as 6 and we have $n! = 6$ for $n = 3$. So the formula has been verified for this case.

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Case: inductive step $n = m$

Keeping the formula general, the help of the Principle of Mathematical Induction will be considered to prove the theorem for all natural values of n. It has been assumed that the formula is true for an arbitrary natural number $n = m$. Then

$$m! = 1 + \sum_{i=1}^{m} a_i + \sum_{\substack{i,j=1 \\ (i<j)}}^{m} a_i a_j + \sum_{\substack{i,j,k=1 \\ (i<j<k)}}^{m} a_i a_j a_k + \cdots + a_1 a_2 \cdots a_m, \tag{3}$$

Case: $n = m + 1$

To prove the formula for arbitrary natural number n, we have to prove the formula for $n = m+1$ when it is true for $n = 1$ and is assumed true for an arbitrary $n = m$. Now from RHS of (2), we have

$$1 + \sum_{i=1}^{m+1} a_i + \sum_{\substack{i,j=1 \\ (i<j)}}^{m+1} a_i a_j + \sum_{\substack{i,j,k=1 \\ (i<j<k)}}^{m+1} a_i a_j a_k + \cdots + \sum_{\substack{i,j,\cdots\ s_t=1 \\ (i<j<\cdots<s_r)}}^{m+1} a_i a_j \cdots a_{s_t} + a_1 a_2 \cdots a_{m+1},$$

where s_t stands for m;

$$\begin{aligned} = \;& 1 + \left(\sum_{i=1}^{m} a_i + a_{m+1}\right) + \left(\sum_{\substack{i,j=1 \\ (i<j)}}^{m} a_i a_j + a_{m+1} \sum_{i=1}^{m} a_i\right) + \left(\sum_{\substack{i,j,k=1 \\ (i<j<k)}}^{m} a_i a_j a_k + a_{m+1} \sum_{\substack{i,j=1 \\ (i<j)}}^{m} a_i a_j\right) \\ + \;& \cdots + \left(a_1 a_2 \cdots a_m + a_{m+1} \sum_{\substack{i,j,\cdots,s_r=1 \\ (i<j<\cdots<s_r)}}^{m} a_i a_j \cdots a_{s_r}\right) + a_1 a_2 \cdots a_{m+1}, \end{aligned}$$

where $s_r = m - 1$;

$$\begin{aligned} = \;& \left(1 + \sum_{i=1}^{m} a_i + \sum_{\substack{i,j=1 \\ (i<j)}}^{m} a_i a_j + \sum_{\substack{i,j,k=1 \\ (i<j<k)}}^{m} a_i a_j a_k + \cdots + a_1 a_2 \cdots a_m\right) \\ + \;& a_{m+1} \left(1 + \sum_{i=1}^{m} a_i + \sum_{\substack{i,j=1 \\ (i<j)}}^{m} a_i a_j + \sum_{\substack{i,j,k=1 \\ (i<j<k)}}^{m} a_i a_j a_k + \cdots + a_1 a_2 \cdots a_m\right) \end{aligned}$$

$$= m! + m \times m! = (m+1)!, \tag{4}$$

which is the desired value of $n!$ for $n = m+1$. Hence the new expression of $n!$ ($n \in N$) has been proved by the Principle of Mathematical Induction.

Appendix:

Since $a_1 = 0$ in the new expression (2) of $n!$ for all $n \in N$, the formula can be represented as

$$n! = 1 + \sum_{i=1}^{n-1} a_i + \sum_{\substack{i,j=1 \\ (i<j)}}^{n-1} a_i a_j + \sum_{\substack{i,j,k=1 \\ (i<j<k)}}^{n-1} a_i a_j a_k + \cdots + a_1 a_2 \cdots a_{n-1}, \tag{5}$$

where $a_i = i$ for $i = 1,\ 2,\ \cdots,\ n-1$.

References

[1] Graham, R. L., Knuth, D. E., Patashnik and O. (1988) Concrete Mathematics, Addison-Wesley, Reading MA. ISBN 0-201-14236-8, pp. 111

[2] Biggs, N. L., The roots of combinatorics, Historia Math. 6 (1979) 109136