Zorn's Lemma

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Abstract. We give a short proof of Zorn's Lemma.

Let P be a poset. Assume that all well-ordered subsets of P have an upper bound, and that P has no maximal element. We'll get a contradiction.

For any pair $I \subset S$ of subsets of P, say that I is an **initial segment** of S if $S \ni s < i \in I$ implies $s \in I$. For any well-ordered subset W of P choose an element $p(W)$ in $P_{>W}$, *i.e.* $p(W) \in P$ and $p(W) > w$ for all w in W. Let $\mathcal{W}$ be the set of those well-ordered subsets W of P such that $p(W_{<w}) = w$ [self-explanatory notation] for all w in W, and let $U \subset P$ be the union of $\mathcal{W}$.

We claim that U is in $\mathcal{W}$. This will give the contradiction $U \cup \{p(U)\} \in \mathcal{W}$.

We have:

(a) if W, X are in $\mathcal{W}$, then W is an initial segment of X, or X is an initial segment of W; in particular U is totally ordered;

(b) any $W \in \mathcal{W}$ is an initial segment of U;

(c) U is in $\mathcal{W}$.

To prove (a) let I be the set of those p in P which belong to some initial segment common to W and X. Then I is the largest such initial segment. Moreover I is equal to W or to X because otherwise $I \cup \{p(I)\}$ would contradict the maximality of I, for, W and X being well-ordered, we would have $W_{<w} = I = X_{<x}$ for some w in W and some x in X, yielding $w = p(I) = x$.

Now (b) follows from (a).

We prove (c). To check that U is well-ordered, let A be a nonempty subset of U, choose a W in $\mathcal{W}$ which meets A, let m be the minimum of $W \cap A$, and let a be in A. We must show $m \leq a$. If such was not the case, (a) would imply $a < m$, in contradiction with (b). It remains to prove $p(U_{<u}) = u$ for u in U, that is, $U_{<u} \subset W$ for $u \in W \in \mathcal{W}$. This follows from (b).

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