Universes

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Abstract. The book **Categories and Sheaves** by Kashiwara and Schapira starts with a few statements which are not proved, a reference being given instead. We spell out the proofs in a short and self-contained way.

Feeling the proofs are easier to read if they are printed on a single page, I left this page almost blank. Please, go to the next page.

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A **universe** is a set $\mathcal{U}$ satisfying

(i) $\varnothing \in \mathcal{U}$,

(ii) $u \in U \in \mathcal{U} \implies u \in \mathcal{U}$,

(iii) $U \in \mathcal{U} \implies \{U\} \in \mathcal{U}$,

(iv) $U \in \mathcal{U} \implies \mathcal{P}(U) \in \mathcal{U}$,

(v) $I \in \mathcal{U}$ and $U_i \in \mathcal{U}$ for all $i \implies \bigcup_{i \in I} U_i \in \mathcal{U}$,

(vi) $\mathbb{N} \in \mathcal{U}$.

We want to prove:

(vii) $U \in \mathcal{U} \implies \bigcup_{u \in U} u \in \mathcal{U}$,

(viii) $U, V \in \mathcal{U} \implies U \times V \in \mathcal{U}$,

(ix) $U \subset V \in \mathcal{U} \implies U \in \mathcal{U}$,

(x) $I \in \mathcal{U}$ and $U_i \in \mathcal{U}$ for all $i \implies \prod_{i \in I} U_i \in \mathcal{U}$.

[We have kept Kashiwara and Schapira's numbering of Conditions (i) to (x).]

Obviously, (ii) and (v) imply (vii), whereas (iv) and (ii) imply (ix). Axioms (iii), (vi), and (v) imply

(a) $U, V \in \mathcal{U} \implies \{U, V\} \in \mathcal{U}$,

and thus

(b) $U, V \in \mathcal{U} \implies (U, V) := \{\{U\}, \{U, V\}\} \in \mathcal{U}$.

Proof of (viii). If $u \in U$ and $v \in V$, then $\{(u, v)\} \in \mathcal{U}$ by (ii), (b), and (iii). Now (v) yields

$$U \times V = \bigcup_{u \in U} \bigcup_{v \in V} \{(u, v)\} \in \mathcal{U}. \ \square$$

Assume $U, V \in \mathcal{U}$, and let V^U be the set of all maps from U to V. As $V^U \in \mathcal{P}(U \times V)$, Statements (viii), (iv), and (ii) give

(c) $U, V \in \mathcal{U} \implies V^U \in \mathcal{U}$.

Proof of (x). As

$$\prod_{i \in I} U_i \in \mathcal{P}\left(\left(\bigcup_{i \in I} U_i\right)^I\right),$$

(x) follows from (v), (c), and (iv). $\square$