

THE TWIN PRIMES CONJECTURE - SOME SOLUTIONS

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ABSTRACT

The author had published a paper on the solutions for the twin primes conjecture in an international mathematics journal in 2003. This paper, which consists of 2 parts that are each self-contained, presents some approaches to the twin primes problem.

MSC: 11-XX (Number Theory)

PART 1

Keywords: matched; building-blocks; recurrent, identical mathematical structures

Theorem: - There is an infinitude of twin primes pairs.

Proof 1:-

Lemma: By Georg Cantor's definition of infinity something that is undeniably smaller than an infinite quantity could also be infinite. Cantor, who had worked alongside Hilbert, defined infinity as the size of the never-ending list of counting numbers (1, 2, 3, 4, 5, 6, 7, 8, ...). Anything that is comparable in size is equally infinite.

By this definition, the number of odd counting numbers, that would intuitively appear to be smaller, is also infinite. It could be shown that the quantity of counting numbers and the quantity of odd numbers are comparable as we could pair off each counting number with a corresponding odd number, as follows:-

1	2	3	4	5	6	7	8	9	...
↓	↓	↓	↓	↓	↓	↓	↓	↓	...
1	3	5	7	9	11	13	15	17	...

If every member of the counting numbers list could be matched with a member of the odd numbers list then the two lists must be the same size. As the list of counting numbers is infinite the list of odd numbers must also be infinite.

The primes (with the exception of the number "2" which is prime by definition only) and twin primes are also odd numbers and belong to the list of the odd numbers. As the list of the odd numbers is infinite the list of the primes and twin primes must also be infinite. (In fact, every odd number (in the infinite list of odd numbers) is either a prime number or a composite of prime numbers (i.e., it consists of prime factors), the primes being the "atoms" or building-blocks of the odd numbers (the primes could also be regarded as the building-blocks of all the whole numbers or integers, including the even numbers - vide the Goldbach conjecture, which, it appears, virtually all mathematicians believe to be true, all the even numbers with the exception of 2 are each the sum of two prime numbers - all the even numbers with the exception of 2 are also each the product of prime numbers (composite).) The twin primes could be found scattered all over the infinite list of odd numbers. All this means that just by the expediency of enumerating the infinite odd numbers, we could "generate" an infinite number of primes and twin primes. With regards to the twin primes, it is as if the innumerable 'twin primes' fruits are waiting to be plucked from the infinitely big 'odd numbers' tree.

Proof 2:-

Lemma: The development of the self-similarity concept has brought fame to Mitchell Feigenbaum, who has worked in the Los Alamos Laboratory in the early 1970s. The "distinguished presence" of the twin primes amongst the prime numbers, as described in the paper, is analogous to the "self-similar mathematical pattern or structure" (which is the shape of the fig-tree itself) of the various parts of the fig-tree, i.e., its trunk to bough section, bough to branch section, branch to twig section and twig to twiglet section, in Feigenbaum's famous fig-tree example, and, such self-similar mathematical pattern or structure, or, fractal characteristic, could also be found in other aspects of nature, e.g., waves, turbulence or chaos, the structures of viruses and bacteria, polymers and ceramic materials, the universe and many others, even the movements of prices in financial markets, the growths of populations, the sound of music, the flow of blood through our circulatory system, the behaviour of people en masse, etc., which have all spawned a relatively new and important branch of mathematics with wide practical applications known as fractal geometry, which has been pioneered by Benoit Mandelbrot. As a matter of fact, self-similarity or fractal characteristic could be regarded as the fundamental mathematical aspect found in practically everything in nature including the numbers such as the prime numbers and twin primes which are the subjects

of our investigation here, and, this new branch of mathematics, fractal geometry, besides having a great practical impact on us also gives us a deeper vision of the universe in which we live and our place in it.

By Dirichlet's Theorem, the arithmetic progressions (arithmetic sequences) of terms $a + im$ and $(a + 2) + im$, $i = 0, 1, 2, 3, 4, \dots$, contain infinitely many primes if a and $a + 2$ have no factors or divisors in common with m other than $+1$ and -1 (i.e., the integers, a and m , are relatively prime or coprime, as well as the integers, $a + 2$ and m). We now, e.g., let $a = 5$ and $m = 12$ (we could always assign values to a and m such that both a and $a + 2$ have no factors or divisors in common with m other than $+1$ and -1). Then we have the following sequences ($a + im$, $i = 0, 1, 2, 3, 4, \dots$, and, $(a + 2) + im$, $i = 0, 1, 2, 3, 4, \dots$):-

5, 17, 29, 41, 53, 65, 77, 89, 101, 113, 125, 137, 149, 161, 173,

7, 19, 31, 43, 55, 67, 79, 91, 103, 115, 127, 139, 151, 163, 175,

The integers in the above sequences which are underlined are prime numbers. The first sequence contains infinitely many primes. The second sequence contains infinitely many primes too. The corresponding integers in both the sequences, e.g., 5 and 7, 17 and 19, 29 and 31, and so on, differ from one another by 2, which is a characteristic of the twin primes, and, a number of twin primes is evident in both the sequences.

Since the twin primes (5 & 7, 17 & 19, 29 & 31, 41 & 43, 101 & 103, 137 & 139, and, 149 & 151) are evident in both the sequences, by the infinitude of the primes in both the sequences as postulated by Dirichlet's Theorem, and, the above lemma, which postulates that the elements of an infinite subset of an infinite set contain all the recursive significant properties of that set unless the process that selects the elements of the subset directly excludes a property, infinitely many of them must appear thereafter in both the sequences; so, there must be infinitely many twin primes, the twin primes representing a characteristic or feature of the prime numbers, the recurrent, identical mathematical structures postulated by the well-established self-similarity concept.

PART 2

Keywords: induction; contradiction; recurrent, identical mathematical structures

Theorem:- There is an infinitude of twin primes.

Proof 1:-

Lemma: The development of the self-similarity concept has brought fame to Mitchell Feigenbaum, who has worked in the Los Alamos Laboratory in the early 1970s. The "distinguished presence" of the twin primes amongst the prime numbers, as described in the paper, is analogous to the "self-similar mathematical pattern or structure" (which is the shape of the fig-tree itself) of the various parts of the fig-tree, i.e., its trunk to bough section, bough to branch section, branch to twig section and twig to twiglet section, in Feigenbaum's famous fig-tree example, and, such self-similar mathematical pattern or structure, or, fractal characteristic, could also be found in other aspects of nature, e.g., waves, turbulence or chaos, the structures of viruses and bacteria, polymers and ceramic materials, the universe and many others, even the movements of prices in financial markets, the growths of populations, the sound of music, the flow of blood through our circulatory system, the behaviour of people en masse, etc., which have all spawned a relatively new and important branch of mathematics with wide practical applications known as fractal geometry, which has been pioneered by Benoit Mandelbrot. As a matter of fact, self-similarity or fractal characteristic could be regarded as the fundamental mathematical aspect found in practically everything in nature including the numbers such as the prime numbers and twin primes which are the subjects of our investigation here, and, this new branch of mathematics, fractal geometry, besides having a great practical impact on us also gives us a deeper vision of the universe in which we live and our place in it.

We could generate an infinite list of primes, Q , by utilising Euclid's indirect (reductio ad absurdum) method of proving the infinity of the primes. In his proof, he first assumed that there is a finite list of prime numbers and then showed that there must exist an infinite number of additions to this list. The proof is as follows:-

Assume that n is the largest existing prime number in the following list of primes:

$$x = 2, 3, 5, 7, 11, 13, 17, 19, \dots, n$$

There could be a prime, q , which is larger than n , computed as follows:

$$q = (2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times \dots \times n) + 1$$

This number, q , could not be divided exactly by any of our primes (from 2 to n) and must be either a prime number itself which is larger than n or must be divisible by prime factors each of which is larger than n , both of which cases contradicting our original assumption that n is the largest existing prime, thus implying the infinity of the primes. In this manner we have added to our original

list of primes, which is x . We could now repeat the process by including our new prime, q , or its prime factors if q is not prime in our list and generate a new number that is either a prime number that is larger than q or a composite number with prime factors that are each larger than q , i.e., we could generate some other new prime or primes that are not on our list of known primes. No matter how long our list of prime numbers is it is possible to find some new prime or primes. Thus, the list of primes is never-ending and infinite. In other words, Q is an infinite list of prime numbers.

We could in fact modify Euclid's indirect method slightly, but use the same reasoning, to generate another infinite list of prime numbers, P . (This method would have served Euclid equally well in proving the infinity of the primes.) This proof takes the following form:-

Assume that n is the largest existing prime in the following list of prime numbers:

$$x = 2, 3, 5, 7, 11, 13, 17, 19, \dots, n$$

There could be a prime number, p , which is larger than n , computed as follows:

$$p = (2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times \dots \times n) - 1$$

(p is either a prime that is larger than n , or, it has prime factors each of which is larger than n ; therefore, n could not be the largest existing prime, and, P is an infinite list of prime numbers.)

It is evident that $Q - P = 2$. In other words, $q_1 - p_1 = 2$, $q_2 - p_2 = 2$, $q_3 - p_3 = 2$, $q_4 - p_4 = 2$, and so on. This is the characteristic of twin primes.

We would here match P and Q for $n = 2$ to 19 , which is as follows:-

$n = 2$ to 19	P	Q
1] $(2 \times 3) \pm 1$	5 = Prime	7 = Prime (Direct twin primes)
2] $(2 \times 3 \times 5) \pm 1$	29 = Prime	31 = Prime (Direct twin primes)
3] $(2 \times 3 \times 5 \times 7) \pm 1$	209 = 11 x 19 (Indirect twin primes: 11 & 13; 17 & 19)	211 = Prime
4] $(2 \times 3 \times 5 \times 7 \times 11) \pm 1$	2,309 = Prime	2,311 = Prime (Direct twin primes)
5] $(2 \times 3 \times 5 \times 7 \times 11 \times 13) \pm 1$	30,029 = Prime	30,031 = 59 x 509 (Indirect twin primes: 59 & 61)
6] $(2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17) \pm 1$	510,509 = 61 x 8,369 (Indirect twin primes: 59 & 61)	510,511 = 19 x 97 x 277 (Indirect twin primes: 17 & 19)
7] $(2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19) \pm 1$	9,699,689 = 53 x 197 x 929 (Indirect twin primes: 197 & 199)	9,699,691 = 347 x 27,953 (Indirect twin primes: 347 & 349)
	.	.
	.	.
	.	.

It is evident that P and Q , for $n = 2$ to 19 , yield three pairs of direct twin primes and five pairs of different indirect twin primes which are the result of pairing the prime factors of the p 's and the q 's with the prime numbers which differ from each of them respectively by 2. Moreover, all these direct and indirect twin primes, a total of eight pairs of them, are found between n and Q and are of course larger than n , the assumed largest existing prime number; they are scattered amongst the twenty-five newly found prime numbers (which are all larger than n). (Note: The above computations are performed with an ordinary hand-held calculator which is capable of yielding results up to $n = 19$ only. For results for $n > 19$, a more powerful calculator or computer would be needed.)

Beside these direct and indirect twin primes between n and Q , more twin primes larger than n , and relatively plenty of them too, could be generated between n and Q by resorting to the Sieve of Eratosthenes - we shall call these twin primes the extraneous twin primes. For example, in (3) above, where $n = 7$ and $Q = 211$, 41 & 43, 71 & 73, 101 & 103, etc., are extraneous twin primes which

are all larger than n ($n = 7$) and $n - 2 = 5$, 5 & 7 here being twin primes, and, in (5) above, where $n = 13$ and $Q = 30,031$, 107 & 109 , 137 & 139 , 227 & 229 , 311 & 313 , etc., are extraneous twin primes which are all larger than n ($n = 13$) and $n - 2 = 11$, 11 & 13 being twin primes. Hence, if we assume that any of these $n - 2$'s and n 's are the largest existing twin primes in the spirit of Euclid we would encounter contradictions from the extraneous twin primes (as well as the direct twin primes and/or the indirect twin primes) which are all larger (these being indirect proofs of the infinitude of the twin primes, which would be described below).

By induction:

P contains an infinite listing of direct primes, the p 's, and, prime factors of the p 's wherever the p 's are non-prime, as proven above.

Q also contains an infinite listing of direct primes, the q 's, and, prime factors of the q 's wherever the q 's are non-prime, as proven above.

$Q - P = 2$, which is a characteristic of the twin primes.

Both P and Q contain a number of direct twin primes (the result of the pairing of the direct primes in P and Q) and indirect twin primes (the result of the pairing of the prime factors of the p 's and the q 's in both P and Q with the prime numbers which differ from each of them respectively by 2) for $n = 2$ to 19, between n and Q , as is shown in the above example - in this case, a total of eight pairs of them scattered amongst the twenty-five newly found prime numbers, and everyone of these, twin primes and newly found prime numbers, is larger than n , the assumed largest existing prime, a contradiction. Beside these direct and indirect twin primes, more twin primes, the evidently more abundant extraneous twin primes between n and Q , are generated by the Sieve of Eratosthenes - these extraneous twin primes are all larger than n as well. All these are in line with the concept behind our proofs by contradiction, the indirect proofs - in these indirect proofs, contradiction implies an infinite number of primes, not excluding the twin primes, which are two primes that differ from one another by 2. (The indirect proof of the twin primes' infinitude is described below.) From the above-mentioned example of twin primes, and, many, many other examples of twin primes beside this, it is evident that twin primes represent a characteristic or feature of the prime numbers. This is in accordance with the above lemma, upon which the method of renormalization in perturbation theory is based, developed by Mitchell Feigenbaum in the 1970s, which postulates that there is a tendency of identical mathematical structures to recur on many levels. Within a given structure, there would be smaller copies of the same structure, their sizes being determined by the scaling factor. Feigenbaum found that at the utmost tips of the fig-tree, there is some mathematical structure which remains the same when its size is changed (enlarged) by a scaling factor of 4.669, which is found to be a constant like π (3.142); this structure is the shape of the fig-tree itself; in other words, little whorls could be found within big whorls. Renormalization has been a well-established technique in chaos theory/fractal geometry and is a mathematical trick which functions rather like a microscope, zooming in on the self-similar structure, removing any approximations, and filtering out everything else. All this shows the universality of some features of chaos. That is, some kind of order or pattern could be found in or is inherent in disorder or chaos. Likewise, the "distinguished" appearances of the twin primes could be found amongst the chaotic (disorderly, random, patternless) array of infinite prime numbers, the twin primes being comparable to the little whorls found within the big whorls - they represent the recurrent, identical mathematical structures postulated by the above lemma.

Since by Euclid's proof the list of the primes is infinite, and, since the twin primes represent a characteristic or feature of the prime numbers as is described above, it could thus be concluded that there is an infinitude of twin primes. This is the result of the "reflection" principle. (For example, the characteristic of a mountain or infinite volume of sand is reflected in the characteristic of some grains of sand found there so that studying the characteristic of some grains of sand there is sufficient for deducing the characteristic of the mountain or infinite volume of sand. The twin primes found in P and Q above could be taken to represent the "grains of sand" found in the "infinite volume of sand" which is represented by the infinite list of the primes. The "grains of sand" (which are in effect a subset) and the "infinite volume of sand" (which are in effect a set), here, share this same characteristic - the presence of twin primes within them.)

Thus, in general, some of the direct primes in P and Q would pair to form direct twin primes and some of the prime factors of the p 's and the q 's in both P and Q would pair with other prime numbers (which differ from each of them respectively by 2) to form indirect twin primes, not to mention the presence of the relatively abundant extraneous twin primes generated by the Sieve of Eratosthenes. All these results are representative of the afore-mentioned characteristic or "pattern" of the prime numbers, and, they are in accordance with the above lemma - they represent the recurrent, identical mathematical structures postulated by the above lemma.

By the above lemma and the "reflection" principle, the infinity of the primes in P (direct primes and prime factors) and the infinity of the primes in Q (direct primes and prime factors) imply that there is an infinitude of twin primes (direct twin primes, indirect twin primes and extraneous twin primes) generated by both P and Q together, and, the Sieve of Eratosthenes.

Hence, it is clear that the twin primes are infinite.

Proof 2:-

The following is indirect proof (proof by contradiction) of the infinity of the twin primes. As P and Q contain an infinitude of twin primes (direct and indirect), including the extraneous twin primes between n and Q generated by the Sieve of Eratosthenes, which are all larger than n, the assumed largest existing prime number, as proven above, this implies that if this n, and, n - 2, are both prime numbers (twin primes), they could not be the largest existing pair of twin primes. If we assume that these n and n - 2 are the largest existing twin primes, we would be contradicted by the above-mentioned proof. This is the indirect proof of the infinity of the twin primes. In the above-mentioned example, for n = 2 to 19, our respective assumed largest existing pairs of twin primes (n - 2 & n), 3 & 5, 5 & 7, 11 & 13, and, 17 & 19, are superceded (and "contradicted") by 29 & 31 (direct twin primes); 11 & 13, &, 17 & 19 (indirect twin primes); 59 & 61 (indirect twin primes); and, 197 & 199, &, 347 & 349 (indirect twin primes); respectively; not to mention the respective larger extraneous twin primes generated by the Sieve of Eratosthenes. In other words, it could be expected that all the direct twin primes and the indirect twin primes so produced (or generated), and the extraneous twin primes generated by the Sieve of Eratosthenes, as described above, would be larger than n (the assumed largest existing prime number), or, n - 2 and n (the assumed largest existing pair of twin primes) - this is incontrovertible indirect proof of the infinitude of the primes and twin primes.

With the two formulae, $(2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times \dots \times n) - 1$ and $(2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times \dots \times n) + 1$ (and, the help of the Sieve of Eratosthenes), we could indeed generate two infinite lists of prime numbers, with an infinite number of twin primes (direct, indirect and extraneous) scattered here and there all over the place.

We state the following theorem:-

If we let n represent any successive prime number in a list of consecutive prime numbers which commences from the prime number 2 and assume it to be the largest existing prime number, then by the following statements:

$$p = (2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times \dots \times n) - 1$$

$$q = (2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times \dots \times n) + 1$$

since $q - p = 2$, if both p and q are prime numbers, in which case they are twin primes, they would be larger than n, and, if n - 2 is also a prime number, in which case these n and n - 2 are twin primes, and if we then assume that these n and n - 2 are the largest existing twin primes pair, we would find that these twin primes, p and q, would be larger than these n and n - 2, the assumed largest existing twin primes pair, hence contradicting the assumption that these n and n - 2 are the largest existing twin primes pair and proving the infinitude of the twin primes according to the principle of "reductio ad absurdum"; if, on the other hand, either of these p and q or both these p and q are not prime (i.e., they are composite numbers), then the prime factors of either of these p and q, or, both these p and q would be larger than these n and n - 2, and, if any of these prime factors pair with other prime numbers that differ from them by 2 then we would have a twin primes pair or pairs consisting of these prime factor or factors of p and/or q and another prime number or numbers that differ from these prime factor or factors by 2 which would be larger than these n and n - 2, the assumed largest existing twin primes pair, thereby contradicting this assumption and proving the infinity of the twin primes through the principle of "reductio ad absurdum"; this rule would always apply whenever p and q are both primes, i.e., they are twin primes, or, whenever p and/or q are non-prime, i.e., they are composite numbers, and their prime factor or factors pair with another prime number or numbers that differ from them by 2 to form a twin primes pair or pairs, when, at the same time, n and n - 2 are also prime numbers, i.e., twin primes. In other words: If p and q are both primes, i.e., twin primes, or, if a prime factor or factors of p and/or q pair with another prime number or numbers that differ from them by 2 to form a twin primes pair or pairs (when p and/or q are composite, i.e., non-prime), then n and n - 2 could not be the largest existing twin primes pair if it happened that n and n - 2 are both primes, i.e., twin primes.

Proof 3:-

The following describes two other ways of proving the infinity of the primes:-

- (1) Whatever number one picks, there is a prime larger than it.
- (2) Pick a number - N.
- (3) Multiply all the positive integers starting with 1 and ending with N.
- (4) I.e., form the factorial of N (written N!).
- (5) What one gets is divisible by every number up to N.
- (6) When one adds 1 to N!, the result:

- (a) can't be a multiple of 2 (because it leaves 1 over, when you divide by 2).
- (b) can't be a multiple of 3 (because it leaves 1 over, when you divide by 3).
- (c) can't be a multiple of 4 (because it leaves 1 over, when you divide by 4).

(n) can't be a multiple of N (because it leaves 1 over, when you divide by N).

- (7) In other words, $N! + 1$, if it is divisible at all (other than by 1 and itself) only is divisible by numbers greater than N.
- (8) So, either $N! + 1$ is itself prime, or its prime divisors are greater than N.
- (9) In either case, there must exist a prime bigger than N.
- (10) The process holds no matter what number N is.
- (11) Whatever N is, there is a prime greater than N.
- (12) Thus, the infinitude of the primes. (Generalisation)

However, if we modify (6) above as follows the conclusion would still be the same:-

- (6) When one minuses 1 from $N!$, the result:
- (a) can't be a multiple of 2 (because it leaves 1 over, when you divide by 2).
- (b) can't be a multiple of 3 (because it leaves 2 over, when you divide by 3).
- (c) can't be a multiple of 4 (because it leaves 3 over, when you divide by 4).
- .
- .
- .
- (n) can't be a multiple of N (because it leaves $N - 1$ over, when you divide by N).
- (7) In other words, $N! - 1$, if it is divisible at all (other than by 1 and itself) only is divisible by numbers greater than N.
- (8) So, either $N! - 1$ is itself prime, or its prime divisors are greater than N.
- (9) In either case, there must exist a prime bigger than N.
- (10) The process holds no matter what number N is.
- (11) Whatever N is, there is a prime greater than N.
- (12) Thus, the infinitude of the primes. (Generalisation)

Therefore: When $N! - 1$ and $N! + 1$ are both primes, or, when any of their prime divisors are partnered by primes which differ from them by 2 the result would be new twin primes pairs. By the infinitude of the primes generated by both $N! + 1$ and $N! - 1$ (which differ from one another by 2, a characteristic of the twin primes), as described above, and, the well-established self-similarity concept stated above, there would be an infinitude of such twin primes pairs, the twin primes being a characteristic or feature of the prime numbers, the recurrent, identical mathematical structures postulated by the self-similarity concept.

Proof 4:-

All odd numbers are either prime numbers or composite numbers (i.e., they contain prime factors), the prime numbers being the "atoms" or building-blocks of the odd numbers. Based on this premise, we carry on with the following steps:-

- (1) It is possible to generate new prime numbers. All we have to do is to first create a composite number by multiplying the consecutive odd numbers together, starting from the lowest odd number, 3 (which is a prime number), in consecutive, ascending order, and ending at any odd number (N), for example (a hand-held calculator yields results up to $N = 17$ only):
- (i) $3 \times 5 = 15$ ($N = 5$)
- (ii) $3 \times 5 \times 7 = 105$ ($N = 7$)
- (iii) $3 \times 5 \times 7 \times 9 = 945$ ($N = 9$)
- (iv) $3 \times 5 \times 7 \times 9 \times 11 = 10,395$ ($N = 11$)
- (v) $3 \times 5 \times 7 \times 9 \times 11 \times 13 = 135,135$ ($N = 13$)
- (vi) $3 \times 5 \times 7 \times 9 \times 11 \times 13 \times 15 = 2,027,025$ ($N = 15$)
- (vii) $3 \times 5 \times 7 \times 9 \times 11 \times 13 \times 15 \times 17 = 34,459,425$ ($N = 17$)
- .
- .
- .
- (n) $3 \times 5 \times 7 \times 9 \times 11 \times 13 \times 15 \times 17 \times \dots \times N$

- (2) The results in (1) above would be composite numbers which are divisible by any of their odd factors (primes and multiples of primes), for example, 3 or 5 in (i), and, 3, 5, 7, 9, 11, 13, 15 or 17 in (vii).
- (3) Next, we attempt to make each of these results non-composite, i.e., turn each of these results into a prime number, by taking away 2 from each of them to get result X (which could turn out to be a prime number).
- (4) Then, we take away 2 from X, and, we could get another prime number, Y (which would of course be a twin prime if it is a prime number and if X is also a prime number).
- (5) For example:

- (a) In (1)(i) above, we would get the following twin primes:

$$\begin{aligned}(3 \times 5) - 2 &= 13 && (X) \\ (3 \times 5) - 2 - 2 &= 11 && (Y)\end{aligned}$$

- (b) In (1)(ii) above, we would get the following twin primes:

$$\begin{aligned}(3 \times 5 \times 7) - 2 &= 103 && (X) \\ (3 \times 5 \times 7) - 2 - 2 &= 101 && (Y)\end{aligned}$$

- (c) In (1)(iii) above, we would get the following composite number and prime:

$$\begin{aligned}(3 \times 5 \times 7 \times 9) - 2 &= 943 && (X) - \text{Composite Number} \\ (3 \times 5 \times 7 \times 9) - 2 - 2 &= 941 && (Y) - \text{Prime Number}\end{aligned}$$

- (d) In (1)(iv) above, we would get the following composite number and prime:

$$\begin{aligned}(3 \times 5 \times 7 \times 9 \times 11) - 2 &= 10,393 && (X) - \text{Composite Number} \\ (3 \times 5 \times 7 \times 9 \times 11) - 2 - 2 &= 10,391 && (Y) - \text{Prime Number}\end{aligned}$$

- (e) In (1)(v) above, we would get the following composite number and prime:

$$\begin{aligned}(3 \times 5 \times 7 \times 9 \times 11 \times 13) - 2 &= 135,133 && (X) - \text{Composite Number} \\ (3 \times 5 \times 7 \times 9 \times 11 \times 13) - 2 - 2 &= 135,131 && (Y) - \text{Prime Number}\end{aligned}$$

- (f) In (1)(vi) above, we would get the following twin primes:

$$\begin{aligned}(3 \times 5 \times 7 \times 9 \times 11 \times 13 \times 15) - 2 &= 2,027,023 && (X) \\ (3 \times 5 \times 7 \times 9 \times 11 \times 13 \times 15) - 2 - 2 &= 2,027,021 && (Y)\end{aligned}$$

- (g) In (1)(vii) above, we would get the following composite number and prime:

$$\begin{aligned}(3 \times 5 \times 7 \times 9 \times 11 \times 13 \times 15 \times 17) - 2 &= 34,459,423 && (X) - \text{Prime Number} \\ (3 \times 5 \times 7 \times 9 \times 11 \times 13 \times 15 \times 17) - 2 - 2 &= 34,459,421 && (Y) - \text{Composite Number}\end{aligned}$$

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- (6) X and Y would be larger than N and would never be divisible by any of the consecutive odd numbers, 3 to N. They are either prime numbers (which may pair to become twin primes) or composite numbers. The fact that X and Y are not divisible by any of the consecutive odd numbers, 3 to N, means that if they are each composite numbers their prime factors would be larger than N and if each of these prime factors pair with another prime to form a twin primes pair this twin pair would of course also be larger than N.
- (7) Prime numbers, and, twin primes are likely to appear in X and Y for the following reasons:

- (a) The lists of numbers to be multiplied together to create composite numbers in (1) above do not include even numbers and all the numbers are consecutive odd numbers (whereas Euclid's list of prime numbers in his indirect proof also includes the prime 2 which is an even number), leaving relatively little room or opportunity for composite numbers with prime factors larger than N to form after executing steps (3) and (4) above. The inclusion of even number(s) would cause an increase in the numerical value of the composite number consisting of odd factors, this composite number being only divisible by odd numbers, for example:

In the following list of numbers which includes even numbers: $2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 \times 11$, the composite

of odd numbers ($3 \times 5 \times 7 \times 9 \times 11 = 10,395$), 10,395, would be increased in numerical value by ($2 \times 4 \times 6 \times 8 \times 10 = 3,840$) 3,840 times to become ($10,395 \times 3,840 = 39,916,800$) 39,916,800, due to the inclusion of the even numbers, 2, 4, 6, 8 and 10.

(If we, for example, attempt to convert the result of multiplying the composite of odd numbers by the even numbers, in this case 39,916,800, into twin primes by subtracting 1 from it ($39,916,800 - 1 = 39,916,799$ which is not a prime but a composite number) and then subtracting 2 from the last result ($39,916,799 - 2 = 39,916,797$ which is also a composite number), we are more likely to end up with composite numbers rather than primes or twin primes.)

- (b) X and Y are each smaller than the respective composites of the consecutive odd numbers, 3 to N (shown in (1) above), by 2 and 4 respectively, evidently leaving even less room or opportunity for composite numbers with prime factors larger than N to form after executing steps (3) and (4).
- (c) (5) above evidently shows that X and Y are likely to contain primes and twin primes.
- (8) Since X and Y if they are primes or if they pair to become twin primes, the prime factors of X and/or Y if X and/or Y are non-prime (composite) and any twin pairs formed by the prime factors of X and Y with other primes would be larger than N, this means that N would never be the largest existing prime number if it is prime and N - 2 and N would never be the largest existing twin primes pair if they are both primes. This is indirect proof of the infinity of the primes and twin primes. In the indirect proof, the proof by contradiction, we assume that N, and, N - 2 and N are the largest existing prime and twin primes pair respectively. Then we find a larger prime and twin primes pair to contradict this assumption and prove the infinity of the primes and twin primes.
- (9) In the above-mentioned manner, we could go on creating new primes and twin primes ad infinitum.
- (10) Thus, by (8) above and the well-established self-similarity concept stated above, we conclude the infinity of the primes and twin primes.

APPENDIX

The Sieve Of Eratosthenes

The Sieve of Eratosthenes is an algorithm for making tables of primes. The process of determining all the primes not greater than a number N is carried out by writing down all the numbers from 2 to N, removing those after 2 which are multiples of 2, those after 3 which are multiples of 3, and continuing until all multiples of primes not greater than the square root of N, except the primes themselves, have been removed. Only prime numbers would remain.

The following is an example of the generation of primes (and twin primes) by the Sieve of Eratosthenes:-

The following numbers from 2 to 50 are sieved by removing those which are multiples of 2, 3, 5 and 7 (square root of 50):

1 2 3 **4** 5 6 7 8 9 **10** 11 **12** 13 **14** **15** **16**
 17 **18** 19 **20** **21** **22** 23 **24** **25** **26** **27** **28** 29 **30**
 31 **32** **33** **34** **35** **36** 37 **38** **39** **40** 41 **42** 43 **44**
45 **46** 47 **48** **49** 50

The bold, italicised numbers have been sieved and removed.

The following prime numbers remain after sieving:

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43 and 47

The following are twin primes:

3 & 5, 5 & 7, 11 & 13, 17 & 19, 29 & 31, and, 41 & 43

CONCLUSION

A number of methods have been adopted in this paper in solving the twin primes problem, which here lead us to conclude that the twin primes are infinite. Importantly, quite a number of ways of finding twin primes have been presented.

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