DISPROOF OF THE RIEMANN HYPOTHESIS

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Abstract

Bernhard Riemann has written down a very mysterious work "Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse" since 1859. This paper of Riemann tried to show some functional equations related to prime numbers without proof. Let us investigate those functional equations together about how and where they came from. And at the same time let us find out whether or not the Riemann Zeta Function $\zeta(s) = 2^s(\pi)^{(s-1)}\sin(\pi\frac{s}{2})\Gamma(1-s)\zeta(1-s)$ really has zeroes at negative even integers $(-2, -4, -6 \ldots)$, which are called the trivial zeroes, and the nontrivial zeroes of Riemann Zeta Function which are in the critical strip $(0 < \mathfrak{R}(\mathbf{s}) < 1)$ all lie on the critical line $(\mathfrak{R}(\mathbf{s}) = \frac{1}{2})$ (or the nontrivial zeroes of Riemann Zeta Function are complex numbers of the form $(\frac{1}{2} + \propto i)$). Step by step, you will not believe your eyes to see that Riemann has made such unbelievable mistakes in his work. Finally, you can easily find out that there are no trivial and nontrivial zeroes of Riemann zeta function at all.

1.Introduction

Prime numbers are the most interesting and useful numbers. Many great mathematicians try to work with them in several ways. One of them, Bernhard Riemann, has written down a very famous work "Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse" since 1859 showing a functional equation ζ(s) or Riemann Zeta Function without proof. He believed that with the assistance of his functional equation and all of the methods shown in his paper, the number of prime numbers that are

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smaller than x can be determined.

Someone believes that by using analytic continuation technique, he or she can extend a domain of a powerful analytic function, derived from two or more ordinary expressions or equations, which can help him or her reach the shore he or she tries to. One of them, Riemann, might has thought for about 150 years ago that he could extend the domain of his new analytic function, which was the composition of Riemann Zeta Function and Pi or Gamma function, to the entire complex plane by using this technique. But this technique, just like others, needs to be checked or proved for the essential conditions of the former equations and of the new functional equation itself. Until now usages of Riemann Hypothesis in mathematics and physics are still found more and more, despite the truth that it is just a "hard to solve" problem, not a proven one!

2. What are $\zeta(s)$, $\Gamma(s)$ and $\prod(s)$? What is the relation between $\Gamma(s)$ and $\prod(s)$? How can we derive $2\sin \pi s\, \zeta(s)\prod(s-1) = i\int_{+\infty}^{+\infty} \frac{(-x)^{(s-1)}}{(e^{x}-1)}dx$ and $\zeta(s) = 2^{s}\pi^{(s-1)}\sin\frac{\pi s}{2}\Gamma(1-s)\zeta(1-s)$? Can we really find out trivial zeroes of $\zeta(s)$?

2.1 Let's begin from the great observation **"The Euler Product"**

$$\prod\left(1-\frac{1}{P^{s}}\right)^{-1} = \sum_{n=1}^{+\infty}\left(\frac{1}{n^{s}}\right)$$

For p = all prime numbers = 2, 3, 5,...

n = all whole numbers = 1, 2, 3,...,+∞

Leonard Euler proved this "Euler Product Formula" in 1737.

Let us start following his proof from the series

$$\sum_{n=1}^{+\infty}\left(\frac{1}{n^{s}}\right) = 1+\frac{1}{2^{s}}+\frac{1}{3^{s}}+\frac{1}{4^{s}}+\frac{1}{5^{s}}+\ldots \qquad \textbf{...(A)}$$

Multiply ...(A) by $\frac{1}{2^{s}}$ both sides

$$\frac{1}{2^{s}}\sum_{n=1}^{+\infty}\left(\frac{1}{n^{s}}\right) = \frac{1}{2^{s}}+\frac{1}{4^{s}}+\frac{1}{6^{s}}+\frac{1}{8^{s}}+\frac{1}{10^{s}}+\ldots \qquad \textbf{...(B)}$$

Subtract… (A) by …(B) to remove all elements that have factors of 2

$$(1-\frac{1}{2^s})\sum_{n=1}^{+\infty} (\frac{1}{n^s}) = 1+\frac{1}{3^s}+\frac{1}{5^s}+\frac{1}{7^s}+\frac{1}{9^s}+\frac{1}{11^s}+\ldots \qquad \text{...(C)}$$

Multiply…(C) by $\frac{1}{3^s}$ both sides

$$\frac{1}{3^s}(1-\frac{1}{2^s})\sum_{n=1}^{+\infty} (\frac{1}{n^s}) = \frac{1}{3^s}+\frac{1}{9^s}+\frac{1}{15^s}+\frac{1}{21^s}+\frac{1}{27^s}+\ldots \qquad \text{...(D)}$$

Subtract …(C) by… (D) to remove all elements that have factors of 3 or 2 or both

$$(1-\frac{1}{3^s})(1-\frac{1}{2^s})\sum_{n=1}^{+\infty} (\frac{1}{n^s}) = 1+\frac{1}{5^s}+\frac{1}{7^s}+\frac{1}{11^s}+\frac{1}{13^s}+\frac{1}{17^s}+\ldots \qquad \text{...(E)}$$

Repeating the process infinitely, we will get

$$\ldots(1-\frac{1}{11^s})(1-\frac{1}{7^s})(1-\frac{1}{5^s})(1-\frac{1}{3^s})(1-\frac{1}{2^s})\sum_{n=1}^{+\infty} (\frac{1}{n^s}) = 1$$

Or $$\sum_{n=1}^{+\infty} \left(\frac{1}{n^s}\right) = \frac{1}{\left(1-\frac{1}{2^s}\right)\left(1-\frac{1}{3^s}\right)\left(1-\frac{1}{5^s}\right)\left(1-\frac{1}{7^s}\right)\left(1-\frac{1}{11^s}\right)\ldots}$$

$$\sum_{n=1}^{+\infty} (\frac{1}{n^s}) = \prod_{p\ prime} [\frac{1}{(1-\frac{1}{p^s})}]$$

$$\sum_{n=1}^{+\infty} (\frac{1}{n^s}) = \prod_{p\ prime} (1-\frac{1}{p^s})^{-1}$$

Riemann denoted this relation $\zeta(s) = \sum_{n=1}^{+\infty} (\frac{1}{n^s}) = \prod_{p\ prime} (1-\frac{1}{p^s})^{-1}$ **would converge only when real part of s was greater than 1($\Re(s) > 1$)** in his paper **"**Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse**"** in 1859.

ζ(s)(Riemann Zeta Function) will diverge for all $s \leq 1$, for example

If $\Re(s) = 1$, $\zeta(s) = \sum_{n=1}^{+\infty} (\frac{1}{n^s}) = \frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\ldots$

$$\zeta(1) = \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$$

By comparison test $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\cdots$

$$> \frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{16} + \cdots$$

But $\frac{1}{1} + \frac{1}{2} + (\frac{1}{4} + \frac{1}{4}) + (\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}) + (\frac{1}{16} + \cdots)$

$$= \frac{1}{1} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$$

$$= +\infty$$

so $\zeta(1) = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$

$$= +\infty$$

(finally diverges to $+\infty$)

If $\Re(s) = 0$, $\zeta(s) = \sum_{n=1}^{+\infty} (\frac{1}{n^s}) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \ldots$

$$\zeta(0) = \frac{1}{1^0} + \frac{1}{2^0} + \frac{1}{3^0} + \frac{1}{4^0} + \ldots$$

$$= 1 + 1 + 1 + 1 + \ldots$$

$$= +\infty$$

(finally diverges to $+\infty$)

If $\Re(s) = -1$, $\zeta(s) = \sum_{n=1}^{+\infty} (\frac{1}{n^s}) = \frac{1}{1^{-1}} + \frac{1}{2^{-1}} + \frac{1}{3^{-1}} + \frac{1}{4^{-1}} + \ldots$

$$\zeta(-1) = 1 + 2 + 3 + 4 + \ldots$$

$$= +\infty$$

(finally diverges to $+\infty$)

2.2 Next, let us consider $\Gamma(s)$ (Gamma function)

2.2.1 $\Gamma(s)$ when $s > 0$

Gamma function was first introduced by Leonhard Euler (1707-1783) in his goal to generalize the factorial to non integer values, and was studied more by Adrien-Marie Legendre (1752-1833).

$$\Gamma(s) = \int_0^{+\infty} (e)^{(-u)} (u)^{(s-1)} \, du$$

Which will converge if real part of s is greater than 0 ($\Re$(s) >0,

and can be rewritten as

$$\Gamma(s) = \frac{\Gamma(s+1)}{s}$$

or $\Gamma(s+1) = (s)\Gamma(s)$; converges if $\boldsymbol{\Re(s) > 0}$

Let us prove using integration by parts.

$$\Gamma(s+1) = \int_0^{+\infty} (e)^{(-u)} (u)^{(s)} \, du \quad \text{for } \boldsymbol{\Re(s) > 0}$$

$$= -[(u)^{(s)}(e)^{(-u)}]_0^{+\infty} + \int_0^{+\infty} (e)^{(-u)} (s)(u)^{(s-1)} \, du$$

$$= \lim_{u\to+\infty} -[(u)^{(s)(e)^{(-u)}}] - \lim_{u\to 0} -[(u)^{(s)}(e)^{(-u)}]$$

$$+ \int_0^{+\infty} (e)^{(-u)} (s)(u)^{(s-1)} \, du$$

$$= [\frac{-\infty}{\infty} + \frac{0}{1}] + s\int_0^{+\infty} (e)^{(-u)} (u)^{(s-1)} \, du$$

Use **L'Hospital's Rule** to find the value of $\frac{-\infty}{\infty}$ (indeterminate form).

$$\lim_{u\to+\infty}[\frac{-(u)^{(s)}}{(e)^{(u)}}] = \lim_{u\to+\infty}[\frac{(-1)(s)(u)^{(s-1)}}{(e)^{(u)}}]$$

Repeat differentiation $(l + 1)$times until$(u)^{(s-(l+1))} \to (u)^{(0)}$

$$\text{Then } \lim_{u\to+\infty}[\frac{-(u)^{(s)}}{(e)^{(u)}}] = \lim_{u\to+\infty}[\frac{(-1)(s)(s-1)\ldots(s-l)(u)^{(0)}}{(e)^{(u)}}]$$

$$= \frac{(-1)(s)(s-1)\ldots(s-l)(1)}{(e)^{(+\infty)}}$$

$$= 0$$

Thus $\Gamma(s+1) = 0 + s\int_0^{+\infty} (e)^{(-u)} (u)^{(s-1)} \, du$

So $\Gamma(s+1) = s\,\Gamma(s)$ $\boldsymbol{\Re(s) > 0}$

Find $\Gamma(\frac{1}{2})$

From $\Gamma(s) = \int_0^{+\infty} (e)^{(-u)} (u)^{(s-1)} \, du$

$$\Gamma(\tfrac{1}{2}) = \int_0^{+\infty} (e)^{(-u)}\,(u)^{(-\frac{1}{2})}\,du$$

Let $r = (u)^{(\frac{1}{2})}$

Then $du = 2rdr$, $(u)^{(-\frac{1}{2})} = \frac{1}{(u)^{(\frac{1}{2})}} = \frac{1}{r}$

$$\int_0^{+\infty} (e)^{(-u)}\,(u)^{(-\frac{1}{2})}\,du = \int_0^{+\infty} (e)^{-(r^2)} \left(\frac{1}{r}\right) 2r\,dr$$

$$= 2\int_0^{+\infty} (e)^{-(r^2)}\,dr$$

Let $I = \int_0^{+\infty} (e)^{-(r^2)}\,dr$

$= \int_0^{+\infty} (e)^{-(x^2)}\,dx$, on x- axis

$= \int_0^{+\infty} (e)^{-(y^2)}\,dy$, on y- axis

Then $(I)^2 = (\int_0^{+\infty} (e)^{-(x^2)}\,dx)(\int_0^{+\infty} (e)^{-(y^2)}\,dy)$

$$= \int_0^{+\infty}\int_0^{+\infty} (e)^{-(x^2+y^2)}dxdy$$

Change from rectangular to polar coordinate

$x = r\cos\theta$, $y = r\sin\theta$

So $x^2 + y^2 = (r\cos\theta)^2 + (r\sin\theta)^2$

$= r^2$

For a very small θ (in radian), dθ , and dr

$\cos\theta \approx 1$, $\sin\theta \approx \theta$; $(\sin\theta)^2$, $(d\theta)^2$,and $(dr)^2$ are negligible

$dx = dr\cos\theta = \cos\theta dr + r d\cos\theta = \cos\theta dr - r\sin\theta d\theta$

$dy = dr\sin\theta = \sin\theta dr + r d\sin\theta = \sin\theta dr + r\cos\theta d\theta$

$dxdy = (\cos\theta dr - r\sin\theta d\theta)(\sin\theta dr + r\cos\theta d\theta)$

$= \cos\theta\sin\theta(dr)^2 + r(\cos\theta)^2 drd\theta - r(\sin\theta)^2 drd\theta - (r)^2\sin\theta\cos\theta(d\theta)^2$

$= 0 + rdrd\theta - 0 - 0$

$= \mathrm{rdrd\theta}$

Then $(\mathrm{I})^2 = \int_0^{+\infty}\int_0^{+\infty}(\mathrm{e})^{-(\mathrm{x}^2+\mathrm{y}^2)}\mathrm{dxdy}$

$= \frac{2}{2}\int_0^{\frac{\pi}{2}}\int_0^{+\infty}(\mathrm{e})^{-(\mathrm{r}^2)}\mathrm{rdrd\theta}$

$= \frac{1}{2}\int_0^{\frac{\pi}{2}}(1)\mathrm{d\theta}$

$= \frac{\pi}{4}$

$\mathrm{I} = \frac{\sqrt{\pi}}{2}$

So $\Gamma(\frac{1}{2}) = \int_0^{+\infty}(\mathrm{e})^{(-\mathrm{u})}\,(\mathrm{u})^{(-\frac{1}{2})}\,\mathrm{du}$

$= 2\int_0^{+\infty}(\mathrm{e})^{-(\mathrm{r}^2)}\,\mathrm{dr}$

$= 2\mathrm{I}$

$= \sqrt{\pi}$

Or from **Euler's Reflection Formula**

$$\Gamma(\mathrm{s})\,\Gamma(1-\mathrm{s}) = \frac{\pi}{\sin\pi s} \quad , \quad 0 < \mathrm{s} < 1$$

$$\Gamma(\tfrac{1}{2})\,\Gamma(1-\tfrac{1}{2}) = \frac{\pi}{\sin\frac{\pi}{2}} = \pi$$

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

$$= 1.772$$

Find $\Gamma(1)$

$$\Gamma(\mathrm{s}+1) = \int_0^{+\infty}(\mathrm{e})^{(-\mathrm{u})}\,(\mathrm{u})^{(\mathrm{s})}\,\mathrm{du}$$

$$\Gamma(0+1) = \int_0^{+\infty}(\mathrm{e})^{(-\mathrm{u})}\,(\mathrm{u})^{(0)}\,\mathrm{du}$$

$$= -[(\mathrm{e})^{(-\mathrm{u})}]\,{}_{0}^{+\infty}$$

$$= \lim_{u\to+\infty} -[(\mathrm{e})^{(-\mathrm{u})}] - \lim_{u\to 0} -[(\mathrm{e})^{(-\mathrm{u})}]$$

$$= -0 + 1$$

$$\Gamma(1) \;=\; 1$$

Find $\Gamma(2)$

From $\quad \Gamma(s+1) \;=\; s\,\Gamma(s)$

$$\Gamma(1+1) \;=\; 1\,\Gamma(1)$$

$$\Gamma(2) \;=\; 1$$

And for $s \;=\;$ positive integers 1, 2, 3…, the relation between gamma function and factorial can be found from

$$\Gamma(s+1) \;=\; s\,\Gamma(s) \qquad \text{for} \quad s = 1, 2, 3, \ldots$$

$$= s(s-1)\Gamma(s-1)$$

$$= s(s-1)(s-2)\ldots(1)\,\Gamma(1)$$

$$= s! \qquad \text{for } s = \text{positive integers } 1, 2, 3, \ldots$$

From $\Gamma(s+1) \;=\; s\,\Gamma(s)$ for $s =$ positive integers 1, 2, 3,…

, for example $\Gamma(2) = 1\Gamma(1) \;=\; 1!\,\Gamma(1)$

$$\Gamma(3) = 2\Gamma(2) \;=\; (2)(1)\Gamma(1) \;=\; 2!\,\Gamma(1)$$

$$\Gamma(4) = 3\Gamma(3) \;=\; (3)(2)(1)\Gamma(1) \;=\; 3!\,\Gamma(1)$$

We can rewrite

$$\Gamma(1) = \frac{\Gamma(2)}{1!} = \frac{\Gamma(1+1)}{1!} = \frac{\Gamma(1+1)}{1}$$

$$= \frac{\Gamma(3)}{2!} = \frac{\Gamma(1+2)}{2!} = \frac{\Gamma(1+2)}{1(2)}$$

$$= \frac{\Gamma(4)}{3!} = \frac{\Gamma(1+3)}{3!} = \frac{\Gamma(1+3)}{1(2)(3)}$$

Or $\quad \Gamma(s) = \frac{\Gamma(s+1)}{s}$

$$= \frac{\Gamma(s+2)}{s(s+1)}$$

$$= \frac{\Gamma(s+3)}{s(s+1)(s+2)}$$

The identity $\Gamma(\mathrm{s}) = \frac{\Gamma(\mathrm{s+1})}{s}$ can be used (by analytic continuation technique) to extended the integral formulation for Γ(s) to a meromorphic function defined for all real (and complex) numbers except s = zero and negative integers $(0, -1, -2 \ldots)$ which are poles of the function.

Thus we can evaluate Γ(s+1) for $(\mathrm{s+1}) > 0$ or $\mathrm{s} > -1$ from

$$\Gamma(\mathrm{s}) = \frac{\Gamma(\mathrm{s+1})}{s} \quad , \ \mathrm{s} > -1, \mathrm{s} \neq 0$$

And we can evaluate Γ(s+2) for $(\mathrm{s+2}) > 0$ or $\mathrm{s} > -2$ from

$$\Gamma(\mathrm{s}) = \frac{\Gamma(\mathrm{s+2})}{s(s+1)} \quad , \ \mathrm{s} > -2, \mathrm{s} \neq 0, -1$$

In general, we can evaluate Γ(s) for $(\mathrm{s+k}) > 0$ or $\mathrm{s} > -\mathrm{k}$, k = 1, 2, 3… (all positive integers) from

$$\Gamma(\mathrm{s}) = \frac{\Gamma(\mathrm{s+k})}{s(s+1)(s+2)\ldots(s+k-1)} \quad , \mathrm{s} > -\mathrm{k}, \mathrm{s} \neq 0, -1, -2 \ldots, -(k-1)$$

Hence we can evaluate Γ(s) from all s = real numbers (negative non integer or positive integers and non integers) except s = zero and negative integers $(0, -1, -2, \ldots)$ which are poles of the function.

2.2.2 Γ(s) when s = 0

Find Γ(0) (pole of the function)

From **Euler's Reflection Formula**

$$\Gamma(\mathrm{s})\,\Gamma(1-\mathrm{s}) = \frac{\pi}{\sin \pi s}$$

$$\lim_{s\to(0)^-} \Gamma(\mathrm{s})\,\Gamma(1-\mathrm{s}) = \lim_{s\to(0)^-} \frac{\pi}{\sin \pi s}$$

$$\Gamma[(0)^-]\Gamma[1-(0)^-] = \frac{\pi}{\sin \pi (0)^-}$$

$$\Gamma[(0)^-]\,\Gamma(1) = \frac{\pi}{-\sin(0)}$$

$$= \frac{\pi}{-0} \quad \text{, undefined}$$

$$\lim_{s\to(0)^+} \Gamma(\mathrm{s})\,\Gamma(1-\mathrm{s}) = \lim_{s\to(0)^+} \frac{\pi}{\sin \pi s}$$

$$\Gamma[(0)^{+}]\Gamma[1-(0)^{+}] = \frac{\pi}{\sin\pi(0)^{+}}$$

$$\Gamma[(0)^{+}]\,\Gamma(1) = \frac{\pi}{\sin(0)}$$

$$= \frac{\pi}{0} \quad \text{, undefined}$$

And $\Gamma(1) = 1$

So $\Gamma(0) = \mp\frac{\pi}{0}$, undefined

($\mp\frac{\pi}{0} = \mp\infty$ in the Riemann sphere)

2.2.3 $\Gamma(s)$ when $s < 0$

Find $\Gamma(-\frac{1}{2})$

From $\Gamma(s) = \frac{\Gamma(s+k)}{s(s+1)(s+2)\ldots(s+k-1)}$, $s > -k, s \neq 0, -1, -2, \ldots, -(k-1)$

$= \frac{\Gamma(s+1)}{s}$, for $s > -1, s \neq 0$

$$\Gamma(s) = \frac{\Gamma(s+1)}{s}$$

$$\Gamma(-\tfrac{1}{2}) = \frac{\Gamma(-\frac{1}{2}+1)}{(\frac{-1}{2})}$$

$$= \frac{\Gamma(\frac{1}{2})}{(\frac{-1}{2})}$$

$$= (-2)\sqrt{\pi}$$

$$= -3.545$$

Find $\Gamma(-1)$ (pole of the function)

From $\Gamma(s) = \frac{\Gamma(s+k)}{s(s+1)(s+2)\ldots(s+k-1)}$, $s > -k, s \neq 0, -1, -2, \ldots, -(k-1)$

$= \frac{\Gamma(s+2)}{s(s+1)}$, for $s > -2, s \neq 0, -1, -2$

$$\lim_{s\to(-1)^{+}} \Gamma(s) = \lim_{s\to(-1)^{+}} \frac{\Gamma(s+2)}{s(s+1)}$$

$$= \frac{\Gamma[(-1)^{+}+2]}{(-1)^{+}[(-1)^{+}+1]}$$

$$= \frac{[1]}{-[0]}$$

$$\lim_{s\to(-1)^-} \Gamma(s) = \lim_{s\to(-1)^-} \frac{\Gamma(s+2)}{s(s+1)}$$

$$= \frac{\Gamma[(-1)^{-}+2]}{(-1)^{-}[(-1)^{-}+1]}$$

$$= \frac{[1]}{[0]}$$

So $\Gamma(-1) = \mp\frac{1}{0}$, undefined

($\mp\frac{1}{0} = \mp\infty$ in the Riemann sphere)

Find $\Gamma(-\frac{3}{2})$

From $\Gamma(s) = \frac{\Gamma(s+k)}{s(s+1)(s+2)\ldots(s+k-1)}$, $> -k, s \neq 0, -1, -2 \ldots, -(k-1)$

$= \frac{\Gamma(s+2)}{s(s+1)}$, for $s > -2, s \neq 0, -1, -2$

$$\Gamma(-\frac{3}{2}) = \frac{\Gamma(-\frac{3}{2}+2)}{(-\frac{3}{2})(-\frac{3}{2}+1)}$$

$$= \frac{\Gamma(\frac{1}{2})}{(-\frac{3}{2})(-\frac{1}{2})}$$

$$= \left(-\frac{2}{3}\right)(-2)\sqrt{\pi}$$

$$= 2.363$$

Find $\Gamma(-2)$ (pole of the function)

From $\Gamma(s) = \frac{\Gamma(s+k)}{s(s+1)(s+2)\ldots(s+k-1)}$, $s > -k, s \neq 0, -1, -2, \ldots, -(k-1)$

$= \frac{\Gamma(s+3)}{s(s+1)(s+2)}$, for $s > -3, s \neq 0, -1, -2, -3$

$$\lim_{s\to(-2)^+} \Gamma(s) = \lim_{s\to(-2)^+} \frac{\Gamma(s+3)}{s(s+1)(s+2)}$$

$$= \frac{\Gamma[(-2)^{+}+3]}{(-2)^{+}[(-2)^{+}+1][(-2)^{+}+2]}$$

$$= \frac{[1]}{[0]}$$

$$\lim_{s\to(-2)^{-}} \Gamma(s) = \lim_{s\to(-2)^{-}} \frac{\Gamma(s+3)}{s(s+1)(s+2)}$$

$$= \frac{\Gamma[(-2)^{-}+3]}{(-2)^{-}[(-2)^{-}+1][(-2)^{-}+2]}$$

$$= \frac{[1]}{-[0]}$$

So $\Gamma(-2) = \pm\frac{1}{0}$, undefined

($\pm\frac{1}{0} = \pm\infty$ in the Riemann sphere)

2.3 Consider $\prod(s)$ = Pi function

Pi function has been denoted by Carl Friedrich Gauss since 1813

$\prod(s) = \int_0^{+\infty} (e)^{(-u)} (u)^{(s)} du$, converses if **$\Re(s) > 0$**

The relation between Pi and Gamma functions is

$$\prod(s-1) = \int_0^{+\infty} (e)^{(-u)} (u)^{(s-1)} du \qquad \text{... (1)}$$

$$= \Gamma(s)$$

The two functions will converge if real part of s is greater than 0, ($\Re(s) > 0$)

2.4 Finding the product of $\zeta(s)\prod(s-1)$ and corresponding value of $\Re(s)$

From equation ... (1) $\prod(s-1) = \int_0^{+\infty} (e)^{(-u)} (u)^{(s-1)} du$, **$\Re(s) > 0$,**

For $+\infty \geq u \geq 0$

Let $u = nx$, $n = 1, 2, 3, \ldots$

Then $+\infty \geq x \geq 0$

Multiply equation ... (1) by $\frac{1}{n^s}$ both sides

$$(\frac{1}{n^s})\prod(s-1) = (\frac{1}{n^s}) \int_0^{+\infty}(e)^{(-u)} (u)^{(s-1)} du$$

$$= \int_0^{+\infty}(e)^{(-nx)} (nx)^{(s-1)} (n)^{(-s)} dnx$$

$$= \int_0^{+\infty}(e)^{(-nx)} (nx)^{(s-1)} (n)^{(-s)} ndx$$

$$= \int_0^{+\infty}(e)^{(-nx)} (nx)^{(s-1)} (n)^{-(s-1)}dx$$

$$= \int_0^{+\infty}(e)^{(-nx)} (x)^{(s-1)} dx \qquad ...(1.1)$$

Important: 1. To make sure that the result of $(\mathbf{nx})^{(s-1)}$ multiplies by$(\mathbf{n})^{-(s-1)}$ of equation ...(1.1) will exactly be $(\mathbf{x})^{(s-1)}$ without $(\mathbf{n})^{(s-1)}$left, the value of s from $(\frac{1}{n^s})$ of $\zeta(s)$ which $1 < \Re(s) \leq +\infty$, and from $(u)^{(s-1)}$ of $\prod(s-1)$ which $0 < \Re(s) \leq +\infty$ must be the same number or 1< $\Re$(s $\leq +\infty$.

2. Next we have to prove that the values of all the real parts of s of the product $(\frac{1}{n^s})\prod(s-1)$ or new analytic function that will make the new function converge have to be only those numbers which are larger than 1 or 1< $\Re$(s $\leq +\infty$ as it's original function $(\frac{1}{n^s})$ and $\prod(s-1)$ or not.

Then try to make infinite summation of $(\frac{1}{n^s})\prod(s-1)$

$$\sum_{n=1}^{+\infty} (\frac{1}{n^s})\prod(s-1) = \sum_{n=1}^{+\infty} \int_0^{+\infty}(e)^{(-nx)} (x)^{(s-1)} dx \qquad ... (1.2)$$

Or $$\zeta(s)\prod(s-1) = \sum_{n=1}^{+\infty} \int_0^{+\infty}(e)^{(-nx)} (x)^{(s-1)} dx \qquad ... (1.3)$$

And from $$(e)^{(-nx)} = (e^{(-x)})^{(n)}$$

$$= (e^{(-x)})^{(n)} [\frac{(e)^{(-x)}}{(e)^{(-x)}}]$$

$$= (e^{(-x)})^{(n-1)}(e)^{(-x)}$$

Then $$\zeta(s)\prod(s-1) = \sum_{n=1}^{+\infty} \int_0^{+\infty} (e^{(-x)})^{(n-1)}(e)^{(-x)} (x)^{(s-1)}dx$$

$$= \int_0^{+\infty} \sum_{n=1}^{+\infty} (e^{(-x)})^{(n-1)})(e)^{(-x)}(x)^{(s-1)}dx$$ **... (1.4)**

Let $\sum_{n=1}^{+\infty}(e^{(-x)})^{(n-1)} \quad = \quad \sum_{n=1}^{+\infty} ar^{(n-1)}$

From **Geometric Series**

$$\sum_{n=1}^{+\infty} ar^{(n-1)} \quad = \quad \lim_{n\to+\infty} Sn$$

$$= \lim_{n\to+\infty} \frac{(a-ar^n)}{(1-r)}$$

$$= \lim_{n\to+\infty} [\frac{a}{(1-r)} - \frac{ar^n}{(1-r)}] \qquad , a = 1 \; , r = (e)^{(-x)}, (r < 1)$$

$$, [0 \leq x \leq +\infty)$$

But $\lim_{n\to+\infty} \frac{ar^n}{(1-r)} \quad = \quad \lim_{n\to+\infty} \frac{(e^{(-x)})^n}{(1-e^{(-x)})} \quad = \quad 0 \qquad , [0 \leq x \leq +\infty)$

So $\sum_{n=1}^{+\infty} ar^{(n-1)} \; = \; \frac{a}{(1-r)}$, $a = 1, r = (e)^{(-x)}, (r < 1), [0 \leq x \leq +\infty)$

And then $\sum_{n=1}^{+\infty} \; (e^{(-x)})^{(n-1)} \; = \; \frac{1}{(1-e^{(-x)})}$

Thus $\qquad \zeta(s)\prod(s-1) \quad = \int_0^{+\infty} \frac{(e)^{(-x)}(x)^{(s-1)}}{(1-e^{(-x)})}dx$

$$= \int_0^{+\infty} \frac{(x)^{(s-1)}}{(1-e^{(-x)})(e^x)}dx$$

or $\qquad \zeta(s)\prod(s-1) \quad = \int_0^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)}dx$ **... (2)**

2.5 Riemann's attempt to extend the analytic equation $\zeta(s)\prod(s-1)$ to the negative side of real axis, the formation of the equation

$$2\sin \pi s \; \zeta(s)\prod(s-1) \quad = \quad i\int_{+\infty}^{+\infty} \frac{(-x)^{(s-1)}}{(e^x-1)}dx$$

Riemann substituted $(-x)$ into $(x)^{(s-1)}$ of equation ... (2) and took consideration in positive sense around a domain $(+\infty,+\infty)$. Next, by **Cauchy's theorem** " if two different paths connect the same two points, and a function is holomorphic everywhere "in between" the two paths, then the two path integrals of the function will be the same" and briefly "the path integral along a Jordan curve of a function, holomorphic in the interior of the curve, is zero" , we have

$$\oint_c f(u)du \;=\; 0$$

If a and b are two points on Jordan curve (simple closed curve) C, then

$$\oint_c f(u)du \;=\; \int_a^b f(u)du \;+\; \int_b^a f(u)du$$

$$=\; 0$$

And let us consider the improper integral when b→ +∞ , a = 0, then

$$\oint_c f(u)du \;=\; \lim_{b\to+\infty}\int_0^b f(u)du + \lim_{b\to+\infty}\int_b^0 f(u)du$$

Or $$\oint_c f(u)du \;=\; \int_0^{+\infty} f(u)du \;+\; \int_{+\infty}^0 f(u)du$$

$$=\; 0$$

And for $f(u)du \;=\; \frac{(-x)^{(s-1)}}{(e^x-1)}dx$ (from Riemann),

then $$\int_{+\infty}^{+\infty}\frac{(-x)^{(s-1)}}{(e^x-1)}dx \;=\; \int_0^{+\infty}\frac{(-x)^{(s-1)}}{(e^x-1)}dx \;+\; \int_{+\infty}^0\frac{(-x)^{(s-1)}}{(e^x-1)}dx$$

$$=\; 0$$

Or $$\int_{+\infty}^{+\infty}\frac{(-x)^{(s-1)}}{(e^x-1)}dx \;=\; \int_0^{+\infty}\frac{(-x)^{(s-1)}}{(e^x-1)}dx \;-\; \int_0^{+\infty}\frac{(-x)^{(s-1)}}{(e^x-1)}dx$$

$$=\; 0$$

That means the value of the equation $\zeta(s)\prod(s-1) = \int_0^{+\infty}\frac{(x)^{(s-1)}}{(e^x-1)}dx$ after extending to $\int_{+\infty}^{+\infty}\frac{(-x)^{(s-1)}}{(e^x-1)}dx$ is always equal to zero.

Now, let us go further from the above equation

$$\int_{+\infty}^{+\infty}\frac{(-x)^{(s-1)}}{(e^x-1)}dx \;=\; \int_0^{+\infty}\frac{(-1)^{(s-1)}\,(x)^{(s-1)}}{(e^x-1)}dx - \int_0^{+\infty}\frac{(-1)^{(s-1)}\,(x)^{(s-1)}}{(e^x-1)}dx$$

$$=\; \frac{(-1)^{(s)}}{(-1)}\int_0^{+\infty}\frac{(x)^{(s-1)}}{(e^x-1)}dx \;-\; \frac{(-1)^{(s)}}{(-1)}\int_0^{+\infty}\frac{(x)^{(s-1)}}{(e^x-1)}dx$$

From **Euler's Formula**

$$(e)^{\pm i\pi} \quad = \; -1$$

$$(\cos\pi \;\pm i\sin\pi) \;=\; -1$$

$$\cos\pi \;=\; -1\;, \quad \sin\pi \;=\; 0$$

Hence $\int_{+\infty}^{+\infty}\frac{(-\mathrm{x})^{(\mathrm{s}-1)}}{(\mathrm{e}^{\mathrm{x}}-1)}\,\mathrm{dx} \;=\; 0$

$$= \frac{(-1)^{(s)}}{(-1)}\int_{0}^{+\infty}\frac{(\mathrm{x})^{(\mathrm{s}-1)}}{(\mathrm{e}^{\mathrm{x}}-1)}\,\mathrm{dx} \;-\; \frac{(-1)^{(s)}}{(-1)}\int_{0}^{+\infty}\frac{(\mathrm{x})^{(\mathrm{s}-1)}}{(\mathrm{e}^{\mathrm{x}}-1)}\,\mathrm{dx}$$

$$= \frac{(e^{i\pi})^{(s)}}{(-1)}\int_{0}^{+\infty}\frac{(\mathrm{x})^{(\mathrm{s}-1)}}{(\mathrm{e}^{\mathrm{x}}-1)}\,\mathrm{dx} \;-\; \frac{(e^{-i\pi})^{(s)}}{(-1)}\int_{0}^{+\infty}\frac{(\mathrm{x})^{(\mathrm{s}-1)}}{(\mathrm{e}^{\mathrm{x}}-1)}\,\mathrm{dx}$$

$$= [\frac{(e^{i\pi})^{(s)}}{(-1)} - \frac{(e^{-i\pi})^{(s)}}{(-1)}]\int_{0}^{+\infty}\frac{(\mathrm{x})^{(\mathrm{s}-1)}}{(\mathrm{e}^{\mathrm{x}}-1)}\,\mathrm{dx}$$

$$\mathbf{[\,0 \;=(0)\int_{0}^{+\infty}\frac{(x)^{(s-1)}}{(e^{x}-1)}\,dx\;]}$$

$$= [-\left(e^{i\pi}\right)^{(s)} + \left(e^{-i\pi}\right)^{(s)}]\int_{0}^{+\infty}\frac{(\mathrm{x})^{(\mathrm{s}-1)}}{(\mathrm{e}^{\mathrm{x}}-1)}\,\mathrm{dx}$$

$$= [\left(e^{-i\pi}\right)^{(s)} - \left(e^{i\pi}\right)^{(s)}]\int_{0}^{+\infty}\frac{(\mathrm{x})^{(\mathrm{s}-1)}}{(\mathrm{e}^{\mathrm{x}}-1)}\,\mathrm{dx} \qquad \textbf{... (3)}$$

$$= [(\cos\pi s - i\sin\pi s) - (\cos\pi s + i\sin\pi s)]\int_{0}^{+\infty}\frac{(\mathrm{x})^{(\mathrm{s}-1)}}{(\mathrm{e}^{\mathrm{x}}-1)}\,\mathrm{dx}$$

$$= \;-2i\sin\pi s\int_{0}^{+\infty}\frac{(\mathrm{x})^{(\mathrm{s}-1)}}{(\mathrm{e}^{\mathrm{x}}-1)}\,\mathrm{dx}$$

$$= \;-2i\sin\pi s\;\zeta(\mathrm{s})\prod(\mathrm{s}-1) \qquad \textbf{... (4)}$$

Or $\int_{+\infty}^{+\infty}\frac{(-\mathrm{x})^{(\mathrm{s}-1)}}{(\mathrm{e}^{\mathrm{x}}-1)}\,\mathrm{dx} \;=\; -2i\sin\pi s\;\zeta(\mathrm{s})\prod(\mathrm{s}-1)$

$$\mathbf{[\,0 \;=\; (0)\,\zeta(s)\prod(s-1)\,]}$$

Multiply by i both sides

$$i\int_{+\infty}^{+\infty}\frac{(-\mathrm{x})^{(\mathrm{s}-1)}}{(\mathrm{e}^{\mathrm{x}}-1)}\,\mathrm{dx} \;=\; -2(i)^{2}\sin\pi s\;\zeta(\mathrm{s})\prod(\mathrm{s}-1)$$

$$= \;-2(-1)\sin\pi s\;\zeta(\mathrm{s})\prod(\mathrm{s}-1)$$

Or $2\sin\pi s\,\zeta(s)\prod(s-1) = i\int_{+\infty}^{+\infty}\frac{(-x)^{(s-1)}}{(e^{x}-1)}dx = 0$ **... (5)**

$$[\,2(0)\zeta(s)\prod(s-1) = 0\,]$$

That means the value of the equation $2\sin\pi s\,\zeta(s)\prod(s-1) = i\int_{+\infty}^{+\infty}\frac{(-x)^{(s-1)}}{(e^{x}-1)}dx$ must always equal zero by the way that we have used to derive the equation, and $\sin\pi s$ always equal zero in this case.

Riemann observed the many valued function from the above equation

$$(-x)^{(s-1)} = (e)^{(s-1)\mathrm{Log}(-x)}$$

and said that the logarithm of $(-x)$ was determined to be real only when x was negative. Therefore Riemann tried to show that the integral $\int_{+\infty}^{+\infty}\frac{(-x)^{(s-1)}}{(e^{x}-1)}dx$ would be valuable if $x < 0$ in contrary with the domain $(+\infty, +\infty)$ of the integral. This looked strange and confused.

Big confusion was that Riemann did not change (x) of the denominator $(e^{x}-1)$ of his equation $\int\frac{(-x)^{(s-1)}}{(e^{x}-1)}dx$ to $(-x)$ at the same time when he changed (x) of the numerator $(x)^{(s-1)}$ to $(-x)$. Actually (x) of both denominator and numerator came from the same function $\prod(s-1)$ or both are the same (x), so they had to be changed to $(-x)$ simultaneously.

I do not know what was in his mind at that time, but if one looks carefully at the first page of his original paper **"Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse",** you can see the traces of confusion and hesitation which urged him to change the boundary of the integral $\int\frac{(-x)^{(s-1)}}{(e^{x}-1)}dx$ from $(+\infty, +\infty)$ to $(-\infty, +\infty)$ and back to $+\infty, +\infty)$ again.

2.5.1 Firstly, he might try to extend the functional equation $\zeta(s)\prod(s-1)$ to the negative values along the x-axis which meant that he had tried to consider the integral on the domain $(-\infty, +\infty)$, but failed.

Let us prove together start from equation ... (1.4)

$$\zeta(s)\prod(s-1) = \int_0^{+\infty} \sum_{n=1}^{+\infty} (e^{(-x)})^{(n-1)} (e)^{(-x)} (x)^{(s-1)} dx$$

Riemann extended it to negative values along x-axis

$$\zeta(s)\prod(s-1) = \int_0^{+\infty} \sum_{n=1}^{+\infty} (e^{(-x)})^{(n-1)} (e)^{(-x)} (x)^{(s-1)} dx$$
$$+ \int_{-\infty}^{0} \sum_{n=1}^{+\infty} (e^{(-x)})^{(n-1)} (e)^{(-x)} (x)^{(s-1)} dx$$

Consider $\int_{-\infty}^{0} \sum_{n=1}^{+\infty} (e^{(-x)})^{(n-1)} (e)^{(-x)} (x)^{(s-1)} dx$

Let $\sum_{n=1}^{+\infty} (e^{(-x)})^{(n-1)} = \sum_{n=1}^{+\infty} ar^{(n-1)}$, $a = 1$, $r = (e)^{(-x)}$, $(r > 1)$,

$(-\infty \le x \le 0]$

From **Geometric Series**

$$\sum_{n=1}^{+\infty} ar^{(n-1)} = a + ar + ar^2 + \ldots + ar^{(n-1)}$$, $a = 1$, $r = (e)^{(-x)}$, $(r > 1)$,

$(-\infty \le x \le 0]$

$$= 1 + (e)^{(-x)} + (e)^{(-x)2} + \ldots + (e)^{(-x)(n-1)}$$

$$\sum_{n=1}^{+\infty} ar^{(n-1)} = 1 + (e)^{(0)} + (e)^{(0)2} + \ldots (e)^{(0)(\infty-1)}$$, for $x = 0$

$$= (+\infty)$$

$$\sum_{n=1}^{+\infty} ar^{(n-1)} = 1 + (e)^{(1)} + (e)^{(1)2} + \ldots (e)^{(1)(\infty-1)}$$, for $x = -1$

$$= (+\infty)$$

...

So $\sum_{n=1}^{+\infty} ar^{(n-1)} = \sum_{n=1}^{+\infty} (e^{(-x)})^{(n-1)}$

$= (+\infty)$ for $a = 1$, $r = (e)^{(-x)}$ $(r > 1)$,$(-\infty \le x \le 0]$

Thus $\int_{-\infty}^{0} \sum_{n=1}^{+\infty} (\mathrm{e}^{(-\mathrm{x})})^{(\mathrm{n}-1)} (\mathrm{e})^{(-\mathrm{x})} (\mathrm{x})^{(\mathrm{s}-1)} \mathrm{dx}$

$$= \int_{-\infty}^{0} (+\infty)(\mathrm{e})^{(-\mathrm{x})} (\mathrm{x})^{(\mathrm{s}-1)} \mathrm{dx}$$

$$= (+\infty) \int_{-\infty}^{0} (\mathrm{e})^{(-\mathrm{x})} (\mathrm{x})^{(\mathrm{s}-1)} \mathrm{dx}$$

$$= (+\infty)$$

diverges to$(+\infty)$ for $(-\infty \leq \mathrm{x} \leq 0]$

Then $\zeta(\mathrm{s})\prod(\mathrm{s}-1) = \int_{0}^{+\infty} \sum_{n=1}^{+\infty} (\mathrm{e}^{(-\mathrm{x})})^{(\mathrm{n}-1)})(\mathrm{e})^{(-\mathrm{x})} (\mathrm{x})^{(\mathrm{s}-1)} \mathrm{dx}$

$$+ \int_{-\infty}^{0} (+\infty)(\mathrm{e})^{(-\mathrm{x})} (\mathrm{x})^{(\mathrm{s}-1)} \mathrm{dx}$$

$$= \int_{0}^{+\infty} \sum_{n=1}^{+\infty} (\mathrm{e}^{(-\mathrm{x})})^{(\mathrm{n}-1)})(\mathrm{e})^{(-\mathrm{x})} (\mathrm{x})^{(\mathrm{s}-1)} \mathrm{dx}$$

$$+ (+\infty)$$

$$= (+\infty)$$

diverges to$(+\infty)$ for $(-\infty \leq \mathrm{x} \leq 0]$

So extending $\zeta(\mathrm{s})\prod(\mathrm{s}-1) = \int_{0^+}^{+\infty} \sum_{n=1}^{+\infty} (\mathrm{e}^{(-\mathrm{x})})^{(\mathrm{n}-1)} (\mathrm{e})^{(-\mathrm{x})} (\mathrm{x})^{(\mathrm{s}-1)} \mathrm{dx}$ to $\int_{0}^{+\infty} \sum_{n=1}^{+\infty} (\mathrm{e}^{(-\mathrm{x})})^{(\mathrm{n}-1)} (\mathrm{e})^{(-\mathrm{x})} (\mathrm{x})^{(\mathrm{s}-1)} \mathrm{dx} + \int_{-\infty}^{0} (+\infty)(\mathrm{e})^{(-\mathrm{x})} (\mathrm{x})^{(\mathrm{s}-1)} \mathrm{dx}$ will cause it to diverge to $(+\infty)$.

2.5.2 Secondly, he might try to take integration along a closed curve C covered the domain $(+\infty , +\infty)$. By famous **Cauchy's theorem** " if two different paths connect the same two points, and a function is holomorphic everywhere "in between" the two paths, then the two path integrals of the function will be the same" and briefly "the path integral along a Jordan curve of a function, holomorphic in the interior of the curve, is zero", we get

$$\oint_c f(u)du = 0$$

If a and b are two points on Jordan curve (simple closed curve) C,

then $\oint_c f(u)du = \int_a^b f(u)du + \int_b^a f(u)du$

$= 0$

And let us consider the improper integral when b→ +∞ , a = 0

Then $\oint_c f(u)du = \lim_{b\to+\infty} \int_0^b f(u)du + \lim_{b\to+\infty} \int_b^0 f(u)du$

Or $\oint_c f(u)du = \int_0^{+\infty} f(u)du + \int_{+\infty}^0 f(u)du$

$= 0$

For $f(u)du = \frac{(x)^{(s-1)}}{(e^x-1)} dx$

Then $\int_{+\infty}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)} dx = \int_0^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)} dx + \int_{+\infty}^0 \frac{(x)^{(s-1)}}{(e^x-1)} dx$

$= 0$

$= \int_0^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)} dx - \int_0^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)} dx$

$= \int_0^{+\infty} \frac{(1)^{(s)}}{(1)} \frac{(x)^{(s-1)}}{(e^x-1)} dx - \int_0^{+\infty} \frac{(1)^{(s)}}{(1)} \frac{(x)^{(s-1)}}{(e^x-1)} dx$

$= (1)^{(s)} \int_0^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)} dx - (1)^{(s)} \int_0^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)} dx$

From **Euler's Formula** again

$(e)^{\pm i\pi} = -1$

$(-e)^{\pm i\pi} = 1$

$(\cos\pi \pm i\sin\pi) = -1$

$\cos\pi = -1$, $\sin\pi = 0$

Hence $\int_{+\infty}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)} dx = 0$

$= (1)^{(s)} \int_0^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)} dx - (1)^{(s)} \int_0^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)} dx$

$$= \left(-e^{i\pi}\right)^{(s)} \int_0^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)} dx - \left(-e^{-i\pi}\right)^{(s)} \int_0^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)} dx$$

$$= [\left(-e^{i\pi}\right)^{(s)} - \left(-e^{-i\pi}\right)^{(s)}] \int_0^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)} dx \qquad \textbf{...(6)}$$

$$\mathbf{[\,0 \;=\; [0] \int_0^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)} dx\,]}$$

$$= [(-\cos \pi s - i\sin \pi s) - (-\cos \pi s + i\sin \pi s)] \int_0^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)} dx$$

$$= -2i\sin \pi s \int_0^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)} dx \qquad \textbf{... (7)}$$

Or $$\int_{+\infty}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)} dx = -2i\sin \pi s \int_0^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)} dx$$

$$= -2i\sin \pi s\ \zeta(s) \prod(s-1)$$

$$\mathbf{[\,0 \;=\; (0)\zeta(s)\prod(s-1)]}$$

Multiply by i both sides

$$i \int_{+\infty}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)} dx = -2(i)^2 \sin \pi s\ \zeta(s) \prod(s-1)$$

$$= 2\sin \pi s\ \zeta(s) \prod(s-1)$$

Or $$2\sin \pi s\ \zeta(s) \prod(s-1) = i \int_{+\infty}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)} dx \qquad \textbf{... (8)}$$

$$\mathbf{[\,(0)\ \zeta(s) \prod(s-1) \;=\; 0\,]}$$

That means the value of the equation $2\sin \pi s\ \zeta(s)\prod(s-1) = i \int_{+\infty}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)} dx$ must always equal zero, and $\sin \pi s$ always equal zero.

Now look at the many valued function again

$$(x)^{(s-1)} = (e)^{(s-1)\mathrm{Log}(x)}$$

The logarithm of x is determined to be real when x is positive number within the domain $(+\infty\,, +\infty)$.

2.6 Can we really get trivial zeroes ($-2, -4, -6, \ldots$) from

Riemann Zeta Function $\zeta(\mathrm{s}) = 2^{s}(\pi)^{(s-1)}\sin(\frac{\pi s}{2})\Gamma(1-\mathrm{s})\zeta(1-\mathrm{s})$?

To answer this question, we have to study four functional equations and their relationships.

1. $(\pi)^{(-\frac{s}{2})}\Gamma(\frac{s}{2})\zeta(\mathrm{s}) = \int_0^{+\infty}(\mathrm{e})^{(-\mathrm{nn}\pi x)}(\mathrm{x})^{(\frac{s}{2}-1)}\,\mathrm{dx}$

2. $(\pi)^{-(\frac{1-s}{2})}\Gamma(\frac{1-s}{2})\zeta(1-\mathrm{s}) = \int_0^{+\infty}(\mathrm{e})^{(-\mathrm{nn}\pi\mathrm{x})}\,(\mathrm{x})^{(\frac{1-s}{2}-1)}\,\mathrm{dx}$

3. $(\pi)^{-(s)}\Gamma(s)\zeta(\mathrm{s}) = \int_0^{+\infty}(\mathrm{e})^{(-\mathrm{n}\pi x)}(\mathrm{x})^{(s-1)}\mathrm{dx}$

4. $(\pi)^{-(1-s)}\Gamma(1-s)\zeta(1-\mathrm{s}) = \int_0^{+\infty}(\mathrm{e})^{(-\mathrm{n}\pi x)}(\mathrm{x})^{(1-s-1)}\mathrm{dx}$

Firstly, you have to pay attention to interesting facts which are hidden in those equations.

2.6.1 Let us start from trying to change $\prod(\mathrm{s}-1)$ of equation ...(1) to $\prod(\frac{s}{2}-1)$.

From $\prod(\mathrm{s}-1) = \int_0^{+\infty}(\mathrm{e})^{(-\mathrm{u})}\,(\mathrm{u})^{(\mathrm{s}-1)}\,\mathrm{du}$ **...(1)**

Or $\Gamma(\mathrm{s}) = \int_0^{+\infty}(\mathrm{e})^{(-\mathrm{u})}\,(\mathrm{u})^{(\mathrm{s}-1)}\,\mathrm{du}$

Which converges when $\mathfrak{R}(\mathbf{s}) > 0$, $+\infty \geq \mathrm{u} \geq 0$

Thus $\prod(\frac{s}{2}-1) = \int_0^{+\infty}(\mathrm{e})^{(-\mathrm{u})}(\mathrm{u})^{(\frac{s}{2}-1)}\,\mathrm{du}$ $\quad \mathfrak{R}(\mathbf{s}) > 0$

Multiply by $\left(\frac{1}{n^{s}}\right)(\pi)^{(-\frac{s}{2})}$ both sides and let $\mathrm{u} = \mathrm{nn}\pi\mathrm{x}$ (as Riemann tried to)

$$\left(\frac{1}{n^{s}}\right)(\pi)^{(-\frac{s}{2})}\prod\left(\frac{s}{2}-1\right) = \int_0^{+\infty}\frac{1}{(\pi)^{\left(\frac{s}{2}\right)}}\left(\frac{1}{n^{s}}\right)(\mathrm{e})^{(-\mathrm{u})}\,(\mathrm{u})^{(\frac{s}{2}-1)}\,\mathrm{du}$$

$$= \int_0^{+\infty}\frac{1}{(\pi)^{\left(\frac{s}{2}\right)}}\left(\frac{1}{(\mathrm{nn})^{\left(\frac{s}{2}\right)}}\right)(\mathrm{e})^{(-\mathrm{nn}\pi\mathrm{x})}\,(\mathrm{nn}\pi\mathrm{x})^{(\frac{s}{2}-1)}\,\mathrm{d}(\mathrm{nn}\pi\mathrm{x})$$

$$= \int_0^{+\infty}\frac{(\mathrm{nn}\pi\mathrm{x})^{\left(\frac{s}{2}-1\right)}}{(\mathrm{nn}\pi)^{\left(\frac{s}{2}\right)}}(\mathrm{e})^{(-\mathrm{nn}\pi\mathrm{x})}\mathrm{nn}\pi\,\mathrm{dx}$$

$$= \int_0^{+\infty} \frac{(\mathrm{nn}\pi \mathrm{x})^{\left(\frac{s}{2}-1\right)}}{(\mathrm{nn}\pi)^{\left(\frac{s}{2}-1\right)}} (\mathrm{e})^{(-\mathrm{nn}\pi \mathrm{x})}\, \mathrm{dx}$$

$$= \int_0^{+\infty} (\mathrm{e})^{(-\mathrm{nn}\pi x)} (\mathrm{x})^{\left(\frac{s}{2}-1\right)}$$

<u>Important</u>: **1. To make sure that the result of $(\mathbf{nn}\boldsymbol{\pi x})^{\left(\frac{s}{2}-1\right)}$ devides by $(\mathbf{nn}\boldsymbol{\pi})^{\left(\frac{s}{2}-1\right)}$ of equation above will exactly be $(\boldsymbol{x})^{\left(\frac{s}{2}-1\right)}$ without $(\mathbf{nn}\boldsymbol{\pi})^{\left(\frac{s}{2}-1\right)}$left, the value of s from $(\frac{1}{n^s})$ of $\zeta(s)$ which $1 < \Re(s) \leq +\infty$, and from $(\mathbf{u})^{\left(\frac{s}{2}-1\right)}$ of $\prod\left(\frac{s}{2}-1\right)$ which $0 < \Re(s) \leq +\infty$ must be the same number or 1< $\Re$(s $\leq +\infty$.**

2. Next we have to prove that the values of all the real parts of s of the product $(\frac{1}{n^s})\prod\left(\frac{s}{2}-1\right)$ or new analytic function that will make the new function converge have to be only those numbers which are larger than 1 or 1< $\Re$(s $\leq +\infty$ as it's original function $(\frac{1}{n^s})$ and $\prod\left(\frac{s}{2}-1\right)$ or not.

Then take infinite summation both sides

$$\sum_{n=1}^{+\infty} \left(\frac{1}{n^s}\right)(\pi)^{\left(-\frac{s}{2}\right)} \prod\left(\frac{s}{2}-1\right) = \sum_{n=1}^{+\infty} \int_0^{+\infty} (\mathrm{e})^{(-\mathrm{nn}\pi \mathrm{x})} (\mathrm{x})^{\left(\frac{s}{2}-1\right)}\, \mathrm{dx}$$

$$= \int_0^{+\infty} \sum_{n=1}^{+\infty} (\mathrm{e})^{(-\mathrm{nn}\pi \mathrm{x})} (\mathrm{x})^{\left(\frac{s}{2}-1\right)}\, \mathrm{dx}$$

But **Riemann** denoted $\sum_{n=1}^{+\infty} (\mathrm{e})^{(-\mathrm{nn}\pi \mathrm{x})} = \psi(\mathrm{x})$

Then $\sum_{n=1}^{+\infty} \left(\frac{1}{n^s}\right)(\pi)^{\left(-\frac{s}{2}\right)} \prod\left(\frac{s}{2}-1\right) = \int_0^{+\infty} \psi(\mathrm{x})(\mathrm{x})^{\left(\frac{s}{2}-1\right)}\, \mathrm{dx}$

Or $(\pi)^{\left(-\frac{s}{2}\right)}\prod\left(\frac{s}{2}-1\right)\zeta(s) = \int_0^{+\infty} \psi(\mathrm{x})(\mathrm{x})^{\left(\frac{s}{2}-1\right)}\, \mathrm{dx}$ **...(9)**

Let's consider the value of $\psi(\mathrm{x})$.

Evaluate $\psi(\mathrm{x})$ by **Euler's Formula** for $[0 \leq \mathrm{x} \leq +\infty)$

$$\psi(\mathrm{x}) = \sum_{n=1}^{+\infty} (\mathrm{e})^{(-\mathrm{nn}\pi \mathrm{x})}$$

$$= \sum_{n=1}^{+\infty} [(\mathrm{e})^{(-\mathrm{nn}\pi)}]^{(\mathrm{x})}$$

$= \sum_{n=1}^{+\infty} \ [(\mathrm{e})^{(-1)(\mathrm{nn}\pi)}]^{(\mathrm{x})}$

$= \sum_{n=1}^{+\infty} \ [(\mathrm{e})^{(i)^2(\pi)(\mathrm{nn})}]^{(\mathrm{x})}$

$= \sum_{n=1}^{+\infty} \ [(\mathrm{e})^{(i\pi)}]^{(i)(\mathrm{nn})(\mathrm{x})}$

$= \sum_{n=1}^{+\infty} \ [\cos(\pi) + \mathrm{i}\sin(\pi)]^{(nn\mathrm{x}i)}$

$= \sum_{n=1}^{+\infty} \ (-1)^{(nn\mathrm{x}i)}$

From $(a)^{(ni)} = \cos(\mathrm{Ln}(a)^n) + i\sin(\mathrm{Ln}(a)^n)$

And $\mathrm{Ln}(\mathrm{z}) = \mathrm{Ln}(|\mathrm{z}|) + i\,\mathrm{Arg}(\mathrm{z})$, for all complex z

$\mathrm{Z} = -1 + (0)i$

So $\mathrm{Ln}(-1) = \mathrm{Ln}(|-1|) + i\,\mathrm{Arg}(-1)$

$= i\pi$

And $\mathrm{Ln}(1) = 0$

Then for $[0 \leq \mathrm{x} \leq +\infty)$ if n = odd , x = odd then nnx = odd

$(-1)^{(nn\mathrm{x}i)} = \cos(\mathrm{Ln}(-1)^{nn\mathrm{x}}) + i\sin(\mathrm{Ln}(-1)^{nn\mathrm{x}})$

And $\sum_{n=1}^{+\infty} \ (-1)^{(nn\mathrm{x}i)} = \sum_{n=1}^{+\infty} \ [\cos(\mathrm{Ln}(-1)^{nn\mathrm{x}}) + i\sin(\mathrm{Ln}(-1)^{nn\mathrm{x}})]$

$= [\cos(\mathrm{Ln}(-1)^{\mathrm{x}}) + i\sin(\mathrm{Ln}(-1)^{\mathrm{x}})]$

$+ [\cos(\mathrm{Ln}(-1)^{9\mathrm{x}}) + i\sin(\mathrm{Ln}(-1)^{9\mathrm{x}})]$

$+ [\cos(\mathrm{Ln}(-1)^{25\mathrm{x}}) + i\sin(\mathrm{Ln}(-1)^{25\mathrm{x}})]$

$+ \ldots$

$= [\cos(\mathrm{Ln}(-1)) + i\sin(\mathrm{Ln}(-1))]$

$+ [\cos(\mathrm{Ln}(-1)) + i\sin(\mathrm{Ln}(-1))]$

$+ [\cos(\mathrm{Ln}(-1)) + i\sin(\mathrm{Ln}(-1))]$

$+ \ldots$

$= [\cos(i\pi) + i\sin(i\pi)]$

$+ [\cos(i\pi) + i\sin(i\pi)]$

$$+ [\cos(i\pi) + \ i \sin(i\pi)]$$

$$+ \ldots$$

$$= \ (\mathrm{e})^{(ii\pi)} + (\mathrm{e})^{(ii\pi)} + (\mathrm{e})^{(ii\pi)} + \ldots$$

$$= \ (\mathrm{e})^{(-\pi)} + (\mathrm{e})^{(-\pi)} + (\mathrm{e})^{(-\pi)} + \ldots$$

$$= \ [\frac{1}{(\mathrm{e})^{(\pi)}} + \frac{1}{(\mathrm{e})^{(\pi)}} + \frac{1}{(\mathrm{e})^{(\pi)}} + \ldots]$$

$$+[\frac{1}{(\mathrm{e})^{(\pi)}} + \frac{1}{(\mathrm{e})^{(\pi)}} + \frac{1}{(\mathrm{e})^{(\pi)}} + \ldots]$$

$$+[\frac{1}{(\mathrm{e})^{(\pi)}} + \frac{1}{(\mathrm{e})^{(\pi)}} + \frac{1}{(\mathrm{e})^{(\pi)}} + \ldots]$$

$$+ \ldots$$

$$= \ [\approx 1] + [\approx 1] + [\approx 1] + \ldots$$

$$= \ (+\infty)$$

for $[0 \leq \mathrm{x} \leq +\infty)$ if n $=$ even , x = odd then nnx $=$ even

$$(-1)^{(nn\mathrm{x}i)} = \ \cos(\mathrm{Ln}(-1)^{nn\mathrm{x}}) + i \sin(\mathrm{Ln}(-1)^{nn\mathrm{x}})$$

And $\sum_{n=1}^{+\infty} (-1)^{(nn\mathrm{x}i)} = \sum_{n=1}^{+\infty}[\cos((-1)^{nn\mathrm{x}}) \ + \ i \sin((-1)^{nn\mathrm{x}})]$

$$= \ [\cos((-1)^{4\mathrm{x}}) + \ i \sin((-1)^{4\mathrm{x}})]$$

$$+ [\cos((-1)^{16\mathrm{x}}) + \ i \sin((-1)^{16\mathrm{x}})]$$

$$+ [\cos((-1)^{36\mathrm{x}}) + \ i \sin((-1)^{36\mathrm{x}})]$$

$$+ \ \ldots$$

$$= \ [\cos(\mathrm{Ln}(1)) \ + \ i \sin(\mathrm{Ln}(1))]$$

$$+ [\cos(\mathrm{Ln}(1)) \ + \ i \sin(\mathrm{Ln}(1))]$$

$$+ [\cos(\mathrm{Ln}(1)) \ + \ i \sin(\mathrm{Ln}(1))]$$

$$+ \ldots$$

$$= \ [\cos(0) \ + \ i \sin(0)]$$

$$+ [\cos(0) \ + \ i \sin(0)]$$

$$+ [\cos(0) + i\sin(0)]$$

$$+ \ldots$$

$$= \frac{(+\infty)}{2}[1 + i(0)]$$

$$= (+\infty)$$

For $[0 \leq x \leq +\infty)$ if n = odd , x = zero or even then nnx = zero or even

$$(-1)^{(nnxi)} = \cos(\mathrm{Ln}(-1)^{nnx}) + i\sin(\mathrm{Ln}(-1)^{nnx})$$

And $\sum_{n=1}^{+\infty} (-1)^{(nnxi)} = \sum_{n=1}^{+\infty} [\cos(\mathrm{Ln}(-1)^{nnx}) + i\sin(\mathrm{Ln}(-1)^{nnx})]$

$$= [\cos((-1)^{x}) + i\sin((-1)^{x})]$$

$$+ [\cos((-1)^{9x}) + i\sin((-1)^{9x})]$$

$$+ [\cos((-1)^{25x}) + i\sin((-1)^{25x})]$$

$$+ \ldots$$

$$= [\cos(\mathrm{Ln}(1)) + i\sin(\mathrm{Ln}(1))]$$

$$+ [\cos(\mathrm{Ln}(1)) + i\sin(\mathrm{Ln}(1))]$$

$$+ [\cos(\mathrm{Ln}(1)) + i\sin(\mathrm{Ln}(1))]$$

$$+ \ldots$$

$$= [\cos(0) + i\sin(0)]$$

$$+ [\cos(0) + i\sin(0)]$$

$$+ [\cos(0) + i\sin(0)]$$

$$+ \ldots$$

$$= \frac{(+\infty)}{2}[1 + i(0)]$$

$$= (+\infty)$$

For $[0 \leq x \leq +\infty)$ if n = even , x = zero or even then nnx = zero or even

$$(-1)^{(nnxi)} = \cos(\mathrm{Ln}(-1)^{nnx}) + i\sin(\mathrm{Ln}(-1)^{nnx})$$

And $\sum_{n=1}^{+\infty} (-1)^{(nnxi)} = \sum_{n=1}^{+\infty}[\cos((-1)^{nnx}) + i\sin((-1)^{nnx})]$

$$= [\cos((-1)^{4x}) + i\sin((-1)^{4x})]$$
$$+ [\cos((-1)^{16x}) + i\sin((-1)^{16x})]$$
$$+ [\cos((-1)^{36x}) + i\sin((-1)^{36x})]$$
$$+ \ldots$$
$$= [\cos(\mathrm{Ln}(1)) + i\sin(\mathrm{Ln}(1))]$$
$$+ [\cos(\mathrm{Ln}(1)) + i\sin(\mathrm{Ln}(1))]$$
$$+ [\cos(\mathrm{Ln}(1)) + i\sin(\mathrm{Ln}(1))]$$
$$+ \ldots$$
$$= [\cos(0) + i\sin(0)]$$
$$+ [\cos(0) + i\sin(0)]$$
$$+ [\cos(0) + i\sin(0)]$$
$$+ \ldots$$
$$= \frac{(+\infty)}{2}[1 + i(0)]$$
$$= (+\infty)$$

So $\psi(x) = \sum_{n=1}^{+\infty} (-1)^{(nnxi)}$ for $[0 \le x \le +\infty)$ (zero, odd and even)

$= [(+\infty) + (+\infty) + \cdots]$, undefined for $[0 \le x \le +\infty)$ **... (9.1)**

And so $\prod\left(\frac{s}{2} - 1\right)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(s) = \int_0^{+\infty} \psi(x)(x)^{\left(\frac{s}{2}-1\right)}\,dx$

$$= \int_0^{+\infty}[(+\infty) + (+\infty) + \cdots](x)^{\left(\frac{s}{2}-1\right)}\,dx$$

$= [(+\infty) + (+\infty) + \cdots]$, undefined **...(9.2)**

Or $\Gamma(\frac{s}{2})(\pi)^{\left(-\frac{s}{2}\right)}\zeta(s) = \int_0^{+\infty} \psi(x)(x)^{\left(\frac{s}{2}-1\right)}\,dx$

$= [(+\infty) + (+\infty) + \cdots]$, undefined

See another method of finding the value of $\Gamma(\frac{s}{2})(\pi)^{-\left(\frac{s}{2}\right)}\zeta(s) =$

$\int_0^{+\infty}(\mathbf{e})^{(-\mathbf{nn}\pi\mathbf{x})}\ (\mathbf{x})^{(\frac{s}{2}-\mathbf{1})}$ **dx using integration by parts on appendix A.**

2.6.2 From $\Gamma(s) = \int_0^{+\infty}(e)^{(-u)}\,(u)^{(s-1)}\,du$

Thus $\Gamma(\frac{1-s}{2}) = \int_0^{+\infty}(e)^{(-u)}\,(u)^{(\frac{1-s}{2}-1)}\,du$

Let $u = nn\pi x$

$$\left(\frac{1}{n^{(1-s)}}\right)(\pi)^{-\left(\frac{1-s}{2}\right)}\Gamma\left(\frac{1-s}{2}\right) = \int_0^{+\infty}\frac{(e)^{(-nn\pi x)}}{(nn)^{\left(\frac{1-s}{2}\right)}(\pi)^{\left(\frac{1-s}{2}\right)}}(nn\pi x)^{(\frac{1-s}{2}-1)}(nn\pi)\,dx$$

$$= \int_0^{+\infty}(e)^{(-nn\pi x)}\,(x)^{(\frac{1-s}{2}-1)}\,dx$$

<u>Important</u>: 1. To make sure that the result of $(\mathbf{nn\pi x})^{(\frac{1-s}{2}-1)}$ devides by $(\mathbf{nn\pi})^{(\frac{1-s}{2}-1)}$ of equation above will exactly be $(\mathbf{x})^{(\frac{1-s}{2}-1)}$ without $(\mathbf{nn\pi})^{(\frac{1-s}{2}-1)}$left, the value of s from $(\frac{1}{n^s})$ of $\zeta(s)$ which $1 < \Re(s) \leq +\infty$, and from $(\mathbf{u})^{(\frac{1-s}{2}-1)}$ of $\prod(\frac{1-s}{2}-1)$ which $0 < \Re(s) \leq +\infty$ must be the same number or $1< \Re(s \leq +\infty$.

2. Next we have to prove that the values of all the real parts of s of the product $(\frac{1}{n^s})\prod(\frac{1-s}{2}-1)$ or new analytic function that will make the new function converge have to be only those numbers which are larger than 1 or $1< \Re(s \leq +\infty$ as it's original function $(\frac{1}{n^s})$ and $\prod(\frac{1-s}{2}-1)$ or not.

Then take infinite summation both sides

$$\sum_{n=1}^{+\infty}\left(\frac{1}{n^{(1-s)}}\right)(\pi)^{-\left(\frac{1-s}{2}\right)}\Gamma\left(\frac{1-s}{2}\right) = \sum_{n=1}^{+\infty}\int_0^{+\infty}(e)^{(-nn\pi x)}(x)^{(\frac{1-s}{2}-1)}\,dx$$

$$= \int_0^{+\infty}\sum_{n=1}^{+\infty}(e)^{(-nn\pi x)}(x)^{(\frac{1-s}{2}-1)}\,dx$$

But **Riemann** denoted $\sum_{n=1}^{+\infty}(e)^{(-nn\pi x)} = \psi(x)$

And from $\psi(x) = [(+\infty) + (+\infty) + \cdots]$ as proof above

So $\Gamma\left(\frac{1-s}{2}\right)(\pi)^{-\left(\frac{1-s}{2}\right)}\zeta(1-\mathrm{s}) = \int_0^{+\infty} \psi(\mathrm{x})(\mathrm{x})^{\left(\frac{1-s}{2}-1\right)}\,\mathrm{dx}$

$$= \int_0^{+\infty} [(+\infty)+(+\infty)+\cdots](\mathrm{x})^{\left(\frac{1-s}{2}-1\right)}\,\mathrm{dx}$$

$$= [(+\infty)+(+\infty)+\cdots]\text{, undefined}$$

See another method of finding the value of $\Gamma(\frac{1-s}{2})(\pi)^{-\left(\frac{1-s}{2}\right)}\zeta(1-\mathrm{s}) =$ $\int_0^{+\infty}(\mathrm{e})^{(-\mathrm{nn}\pi\mathrm{x})}\,(\mathrm{x})^{\left(\frac{1-s}{2}-1\right)}$ dx using integration by parts on appendix B.

2.6.3 From $\Gamma(\mathrm{s}) = \int_0^{+\infty}(\mathrm{e})^{(-\mathrm{u})}\,(\mathrm{u})^{(\mathrm{s}-1)}\,\mathrm{du}$, $\boldsymbol{\Re(\mathrm{s})>0}$

Thus $\Gamma(1-\mathrm{s}) = \int_0^{+\infty}(\mathrm{e})^{(-\mathrm{u})}\,(\mathrm{u})^{(1-s-1)}\mathrm{du}$

Let $\mathrm{u} = \mathrm{n}\pi\mathrm{x}$

$$\left(\frac{1}{n^{(1-s)}}\right)(\pi)^{-(1-s)}\Gamma(1-s) = \int_0^{+\infty}\frac{(\mathrm{e})^{(-n\pi x)}}{(n)^{(1-s)}(\pi)^{(1-s)}}(\mathrm{n}\pi\mathrm{x})^{(1-s-1)}\mathrm{n}\pi\,\mathrm{dx}$$

$$= \int_0^{+\infty}(\mathrm{e})^{(-\mathrm{n}\pi\mathrm{x})}\,(\mathrm{x})^{(1-s-1)}\,\mathrm{dx})$$

<u>Important</u>: 1. To make sure that the result of $(\mathbf{n}\boldsymbol{\pi}\mathbf{x})^{(1-s-1)}$ devides by $(\boldsymbol{n\pi})^{(1-s-1)}$ of equation above will exactly be $(\mathbf{x})^{(1-s-1)}$ without $(\boldsymbol{n\pi})^{(1-s-1)}$left, the value of s from $(\frac{1}{n^s})$ of $\zeta(\mathrm{s})$ which $1 < \Re(\mathrm{s}) \leq +\infty$, and from $(\mathbf{u})^{(1-s-1)}$ of $\prod(1-s-1)$ which $0 < \Re(\mathbf{s}) \leq +\infty$ must be the same number or 1< $\Re$(s $\leq +\infty$.

2. Next we have to prove that the values of all the real parts of s of the product $(\frac{1}{n^s})\prod(1-s-1)$ or new analytic function that will make the new function converge have to be only those numbers which are larger than 1 or 1< $\Re$(s $\leq +\infty$ as it's original function $(\frac{1}{n^s})$ and $\prod(1-s-1)$or not.

Then take infinite summation both sides

$$\sum_{n=1}^{+\infty}\left(\frac{1}{n^{(1-s)}}\right)(\pi)^{-(1-s)}\Gamma(1-s) = \sum_{n=1}^{+\infty}\int_0^{+\infty}(\mathrm{e})^{(-\mathrm{n}\pi\mathrm{x})}(\mathrm{x})^{(1-s-1)}\,\mathrm{dx}$$

$$= \int_0^{+\infty} \sum_{n=1}^{+\infty} (e)^{(-n\pi x)} (x)^{(1-s-1)}\, dx$$

denote $\sum_{n=1}^{+\infty} (e)^{(-n\pi x)} = \phi(x)$

So $(\pi)^{-(1-s)}\Gamma(1-s)\zeta(1-s) = \int_0^{+\infty} \phi(x)(x)^{(1-s-1)}\, dx$

Let's consider the value of $\phi(x)$.

Evaluate $\phi(x)$ by **Euler's Formula.**

$$\phi(x) = \sum_{n=1}^{+\infty} (e)^{(-n\pi x)}$$

$$= \sum_{n=1}^{+\infty} [(e)^{(-n\pi)}]^{(x)}$$

$$= \sum_{n=1}^{+\infty} [(e)^{(-1)(\pi)}]^{(nx)}$$

$$= \sum_{n=1}^{+\infty} [(e)^{(i)^2(\pi)}]^{(nx)}$$

$$= \sum_{n=1}^{+\infty} [(e)^{(i\pi)(i)}]^{(nx)}$$

$$= \sum_{n=1}^{+\infty} [\cos(\pi) + i\sin(\pi)]^{(i)(nx)}$$

$$= \sum_{n=1}^{+\infty} (-1)^{(nxi)}$$

From $(a)^{(ni)} = \cos(\text{Ln}(a)^n) + i\sin(\text{Ln}(a)^n)$

And $\text{Ln}(z) = \text{Ln}(|z|) + i\,\text{Arg}(z)$, for all complex z

$$Z = -1 + (0)i$$

So $\text{Ln}(-1) = \text{Ln}(|-1|) + i\,\text{Arg}(-1)$

$$= i\pi$$

And $\text{Ln}(1) = 0$

Then for $[0 \le x \le +\infty)$ if n = odd , x = odd then nx = odd

$$(-1)^{(nxi)} = \cos(\text{Ln}(-1)^{nx}) + i\sin(\text{Ln}(-1)^{nx})$$

And $\sum_{n=1}^{+\infty} (-1)^{(nxi)} = \sum_{n=1}^{+\infty} [\cos(\text{Ln}(-1)^{nx}) + i\sin(\text{Ln}(-1)^{nx})]$

$$= [\cos(\text{Ln}(-1)^{x}) + i\sin(\text{Ln}(-1)^{x})]$$

$$+ [\cos(\text{Ln}(-1)^{3x}) + i\sin(\text{Ln}(-1)^{3x})]$$

$$+ [\cos(\text{Ln}(-1)^{5x}) + i\sin(\text{Ln}(-1)^{5x})]$$

$$+ \ldots$$

$$= [\cos(\text{Ln}(-1)) + i\sin(\text{Ln}(-1))]$$

$$+ [\cos(\text{Ln}(-1)) + i\sin(\text{Ln}(-1))]$$

$$+ [\cos(\text{Ln}(-1)) + i\sin(\text{Ln}(-1))]$$

$$+ \ldots$$

$$= [\cos(i\pi) + i\sin(i\pi)]$$

$$+ [\cos(i\pi) + i\sin(i\pi)]$$

$$+ [\cos(i\pi) + i\sin(i\pi)]$$

$$+ \ldots$$

$$= (e)^{(ii\pi)} + (e)^{(ii\pi)} + (e)^{(ii\pi)} + \ldots$$

$$= (e)^{(-\pi)} + (e)^{(-\pi)} + (e)^{(-\pi)} + \ldots$$

$$= [\frac{1}{(e)^{(\pi)}} + \frac{1}{(e)^{(\pi)}} + \frac{1}{(e)^{(\pi)}} + \ldots]$$

$$+[\frac{1}{(e)^{(\pi)}} + \frac{1}{(e)^{(\pi)}} + \frac{1}{(e)^{(\pi)}} + \ldots]$$

$$+[\frac{1}{(e)^{(\pi)}} + \frac{1}{(e)^{(\pi)}} + \frac{1}{(e)^{(\pi)}} + \ldots]$$

$$+ \ldots$$

$$= [\approx 1] + [\approx 1] + [\approx 1] + \ldots$$

$$= (+\infty)$$

for $[0 \leq x \leq +\infty)$ if $n = \text{even}$, $x = \text{odd}$ then $nx = \text{even}$

$$(-1)^{(nxi)} = \cos(\text{Ln}(-1)^{nx}) + i\sin(\text{Ln}(-1)^{nx})$$

And $\sum_{n=1}^{+\infty} (-1)^{(nxi)} = \sum_{n=1}^{+\infty}[\cos((-1)^{nx}) + i\sin((-1)^{nx})]$

$$= [\cos((-1)^{2x}) + i\sin((-1)^{2x})]$$

$$+ [\cos((-1)^{4x}) + i\sin((-1)^{4x})]$$

$$+ [\cos((-1)^{6x}) + i\sin((-1)^{6x})]$$

$$+ \ ...$$

$$= [\cos(\text{Ln}(1)) + i\sin(\text{Ln}(1))]$$

$$+ [\cos(\text{Ln}(1)) + i\sin(\text{Ln}(1))]$$

$$+ [\cos(\text{Ln}(1)) + i\sin(\text{Ln}(1))]$$

$$+ \ ...$$

$$= [\cos(0) + i\sin(0)]$$

$$+ [\cos(0) + i\sin(0)]$$

$$+ [\cos(0) + i\sin(0)]$$

$$+ \ ...$$

$$= \frac{(+\infty)}{2}[1 + i(0)]$$

$$= (+\infty)$$

for $[0 \leq x \leq +\infty)$ if n = odd , x = zero or even then nx = zero or even

$$(-1)^{(nxi)} = \cos(\text{Ln}(-1)^{nx}) + i\sin(\text{Ln}(-1)^{nx})$$

And $\sum_{n=1}^{+\infty} (-1)^{(nxi)} = \sum_{n=1}^{+\infty}[\cos((-1)^{nx}) + i\sin((-1)^{nx})]$

$$= [\cos((-1)^{x}) + i\sin((-1)^{x})]$$

$$+ [\cos((-1)^{3x}) + i\sin((-1)^{3x})]$$

$$+ [\cos((-1)^{5x}) + i\sin((-1)^{5x})]$$

$$+ \ ...$$

$$= [\cos(\text{Ln}(1)) + i\sin(\text{Ln}(1))]$$

$$+ [\cos(\text{Ln}(1)) + i\sin(\text{Ln}(1))]$$

$$+ [\cos(\text{Ln}(1)) + i\sin(\text{Ln}(1))]$$

$$+ \ ...$$

$$= [\cos(0) + i\sin(0)]$$

$$+ [\cos(0) + i\sin(0)]$$

$$+ [\cos(0) + i\sin(0)]$$

$$+ \ldots$$

$$= \frac{(+\infty)}{2}[1 + i(0)]$$

$$= (+\infty)$$

for [0 ≤ x ≤ +∞) if n = even , x = zero or even then nx = zero or even

$$(-1)^{(nxi)} = \cos(\mathrm{Ln}(-1)^{nx}) + i\sin(\mathrm{Ln}(-1)^{nx})$$

And $\sum_{n=1}^{+\infty} (-1)^{(nxi)} = \sum_{n=1}^{+\infty}[\cos((-1)^{nx}) + i\sin((-1)^{nx})]$

$$= [\cos((-1)^{2x}) + i\sin((-1)^{2x})]$$

$$+ [\cos((-1)^{4x}) + i\sin((-1)^{4x})]$$

$$+ [\cos((-1)^{6x}) + i\sin((-1)^{6x})]$$

$$+ \ldots$$

$$= [\cos(\mathrm{Ln}(1)) + i\sin(\mathrm{Ln}(1))]$$

$$+ [\cos(\mathrm{Ln}(1)) + i\sin(\mathrm{Ln}(1))]$$

$$+ [\cos(\mathrm{Ln}(1)) + i\sin(\mathrm{Ln}(1))]$$

$$+ \ldots$$

$$= [\cos(0) + i\sin(0)]$$

$$+ [\cos(0) + i\sin(0)]$$

$$+ [\cos(0) + i\sin(0)]$$

$$+ \ldots$$

$$= \frac{(+\infty)}{2}[1 + i(0)]$$

$$= (+\infty)$$

So $\phi(x) = \sum_{n=1}^{+\infty} (-1)^{(nxi)}$ for all [0 ≤ x ≤ +∞) (zero, odd and even)

$= [(+\infty) + (+\infty) + \cdots]$, undefined for $[0 \leq x \leq +\infty)$

And so $(\pi)^{-(1-s)}\Gamma(1-s)\zeta(1-s) = \int_0^{+\infty} \phi(x)(x)^{(1-S-1)}\,dx$

$$= \int_0^{+\infty}[(+\infty) + (+\infty) + \cdots](x)^{(1-S-1)}\,dx$$

$$= [(+\infty) + (+\infty) + \cdots] \text{ , undefined}$$

See another method of finding the value of $\Gamma(1\text{–}s\text{–}1)(\pi)^{-(1-s-1)}\zeta(1\text{–}s)$ $= \int_0^{+\infty}(e)^{(-n\pi x)}\,(x)^{(1-S-1)}$ dx using integration by parts on appendix C.

2.6.4 From $\Gamma(s) = \int_0^{+\infty}(e)^{(-u)}\,(u)^{(s-1)}\,du$

Let $u = n\pi x$

$$\left(\frac{1}{n^{(s)}}\right)(\pi)^{-(s)}\Gamma(s) = \int_0^{+\infty}\frac{(e)^{(-n\pi x)}}{(n)^{(s)}(\pi)^{(s)}}(n\pi x)^{(s-1)}n\pi\,dx$$

$$= \int_0^{+\infty}(e)^{(-n\pi x)}\,(x)^{(s-1)}\,dx$$

<u>Important</u>: 1. To make sure that the result of $(n\pi x)^{(s-1)}$ devides by $(n\pi)^{(s-1)}$ of equation above will exactly be $(x)^{(s-1)}$ without $(n\pi)^{(s-1)}$left, the value of s from $(\frac{1}{n^s})$ of $\zeta(s)$ which $1 < \Re(s) \leq +\infty$, and from $(u)^{(s-1)}$ of $\prod(s-1)$ which $0 < \Re(s) \leq +\infty$ must be the same number or 1< $\Re(s \leq +\infty$.

2. Next we have to prove that the values of all the real parts of s of the product $(\frac{1}{n^s})\prod(s-1)$ or new analytic function that will make the new function converge have to be only those numbers which are larger than 1 or 1< $\Re(s \leq +\infty$ as it's original function $(\frac{1}{n^s})$ and $\prod(s-1)$ or not.

Then take infinite summation both sides

$$\sum_{n=1}^{+\infty}\left(\frac{1}{n^{(s)}}\right)(\pi)^{-(s)}\Gamma(s) = \sum_{n=1}^{+\infty}\int_0^{+\infty}(e)^{(-n\pi x)}(x)^{(s-1)}\,dx$$

$$= \int_0^{+\infty} \sum_{n=1}^{+\infty} (e)^{(-n\pi x)} (x)^{(s-1)}\, dx$$

Denote $\sum_{n=1}^{+\infty} (e)^{(-n\pi x)} = \phi(x)$

And from $\phi(x) = [(+\infty) + (+\infty) + \cdots]$ as proof above

So $(\pi)^{-(s)}\Gamma(s)\zeta(s) = \int_{0^+}^{+\infty} \phi(x)(x)^{(s-1)}\, dx$

$$= \int_0^{+\infty} [(+\infty) + (+\infty) + \cdots](x)^{(s-1)}\, dx$$

$$= [(+\infty) + (+\infty) + \cdots]\text{, undefined}$$

See another method of finding the value of $\Gamma(\mathbf{s-1})(\boldsymbol{\pi})^{-(S-1)}\boldsymbol{\zeta}(\mathbf{s})$ $= \int_0^{+\infty} (\mathbf{e})^{(-\mathbf{n}\pi\mathbf{x})} (\mathbf{x})^{(\mathbf{S-1})}$ dx using integration by parts on appendix D.

Next, from $(\pi)^{-(1-s)}\Gamma(1-s)\zeta(1-s) = \int_0^{+\infty} \phi(x)(x)^{(1-s-1)}\, dx$,

if we try to extend $\int_0^{+\infty} \phi(x)(x)^{(1-s-1)}\, dx$ to $\int_{+\infty}^{+\infty} \phi(x)(x)^{(1-s-1)}\, dx$ by taking integration along a closed curve C covered the domain $(+\infty, +\infty)$, then, by famous **Cauchy's theorem**, we will get

$$\int_{+\infty}^{+\infty} \phi(x)(x)^{(1-s-1)}\, dx = \int_0^{+\infty} \phi(x)(x)^{(1-s-1)}\, dx + \int_{+\infty}^{0} \phi(x)(x)^{(1-s-1)}\, dx$$

$$= \int_0^{+\infty} \phi(x)(x)^{(1-s-1)}\, dx - \int_0^{+\infty} \phi(x)(x)^{(1-s-1)}\, dx$$

$$= [[(+\infty) + (+\infty) + \cdots] - [(+\infty) + (+\infty) + \cdots]]$$

undefined , indeterminate form

$$= [\frac{(1)^{(1-s)}}{(1)} - \frac{(1)^{(1-s)}}{(1)}]\int_0^{+\infty} \phi(x)(x)^{(1-s-1)}\, dx$$

$$= [(0)\int_0^{+\infty} \phi(x)(x)^{(1-s-1)}\, dx]$$

$= [(0)[(+\infty) + (+\infty) + \cdots]]$, undefined (from law for infinite limits, but $= 0$ in measure theory)

From **Euler's Formula**

$$(e)^{\pm i\pi} = -1$$

$$(\cos\pi \pm i\sin\pi) = -1$$

$$\cos\pi = -1,\ \sin\pi = 0$$

So $[\frac{(1)^{(1-s)}}{(1)} - \frac{(1)^{(1-s)}}{(1)}]\int_0^{+\infty} \phi(\mathrm{x})(\mathrm{x})^{(1-s-1)}\,\mathrm{dx}$

$$= [\left(-e^{i\pi}\right)^{(1-s)} - \left(-e^{-i\pi}\right)^{(1-s)}]\int_0^{+\infty} \phi(\mathrm{x})(\mathrm{x})^{(1-s-1)}\,\mathrm{dx}$$

$$= [(-\cos\pi(1-s) - \mathrm{i}\sin\pi(1-s)$$

$$-(-\cos\pi(1-s) + \mathrm{i}\sin\pi(1-s)]\int_0^{+\infty} \phi(\mathrm{x})(\mathrm{x})^{(1-s-1)}\,\mathrm{dx}$$

$$= -2i\sin\pi(1-s)\int_0^{+\infty} \phi(\mathrm{x})(\mathrm{x})^{(1-s-1)}\,\mathrm{dx}$$

$$= [\,(0)\int_0^{+\infty} \phi(\mathrm{x})(\mathrm{x})^{(1-s-1)}\,\mathrm{dx}\,]$$

$=$ $[(0)$ $[(+\infty) + (+\infty) + \cdots]]$, undefined (from law for infinite limits , but = 0 in measure theory)

And from $\sin\pi(1-s) = (\sin\pi\cos\pi s - \cos\pi\sin\pi s)$

$$= \sin\pi s$$

$$= 0$$

And $\int_{+\infty}^{+\infty} \phi(\mathrm{x})(\mathrm{x})^{(1-s-1)}\,\mathrm{dx} = (0)\ [(+\infty) + (+\infty)]$

But $\sin\pi s = 2\sin\frac{\pi s}{2}\cos\frac{\pi s}{2}$

Then $\int_{+\infty}^{+\infty} \phi(\mathrm{x})(\mathrm{x})^{(1-s-1)}\,\mathrm{dx}$

$$= -2i\sin\pi(1-s)\int_0^{+\infty} \phi(\mathrm{x})(\mathrm{x})^{(1-s-1)}\,\mathrm{dx}$$

$$= -2i\sin\pi s\int_0^{+\infty} \phi(\mathrm{x})(\mathrm{x})^{(1-s-1)}\,\mathrm{dx}$$

$$= -2i2(\sin\frac{\pi s}{2}\cos\frac{\pi s}{2})\int_0^{+\infty} \phi(\mathrm{x})(\mathrm{x})^{(1-s-1)}\,\mathrm{dx}$$

$$= -2i2(\sin\frac{\pi s}{2}\cos\frac{\pi s}{2})\,\pi^{-(1-s)}\Gamma(1-s)\zeta(1-\mathrm{s}) \quad \textbf{... (9.3)}$$

$$= -2i2(0)\ \pi^{-(1-s)}\Gamma(1-s)\zeta(1-\mathrm{s})\,]$$

$= [(0)\ [(+\infty)+(+\infty)+\cdots]]$, **undefined (from law for infinite limits , but = 0 in measure theory)**

And from

$$(\pi)^{-(\frac{1-s}{2})}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s) = \int_0^{+\infty}\phi(x)(x)^{\left(\frac{1-s}{2}-1\right)}dx$$

$$= [(+\infty)+(+\infty)+\cdots]$$

Next, extend $\int_0^{+\infty}\phi(x)(x)^{\left(\frac{1-s}{2}-1\right)}dx$ to $\int_{+\infty}^{+\infty}\phi(x)(x)^{\left(\frac{1-s}{2}-1\right)}dx$ by taking integration along a closed curve C covered the domain $(+\infty , +\infty)$, then, by famous **Cauchy's theorem**, we will get

$$\int_{+\infty}^{+\infty}\phi(x)(x)^{\left(\frac{1-s}{2}-1\right)}dx = \int_0^{+\infty}\phi(x)(x)^{\left(\frac{1-s}{2}-1\right)}dx - \int_0^{+\infty}\phi(x)(x)^{\left(\frac{1-s}{2}-1\right)}dx$$

$$= [[(+\infty)+(+\infty)+\cdots]-[(+\infty)+(+\infty)+\cdots]]$$

undefined , indeterminate form

$$= [\frac{(1)^{(\frac{1-s}{2})}}{(1)} - \frac{(1)^{(\frac{1-s}{2})}}{(1)}]\int_0^{+\infty}\phi(x)(x)^{\left(\frac{1-s}{2}-1\right)}dx$$

$= [(0)[(+\infty)+(+\infty)+\cdots]]$, **undefined(from law for infinite limits, but = 0 in measure theory)**

$$= [\frac{(-e^{-\pi})^{(\frac{1-s}{2})}}{(1)} - \frac{(-e^{i\pi})^{(\frac{1-s}{2})}}{(1)}]\int_0^{+\infty}\phi(x)(x)^{\left(\frac{1-s}{2}-1\right)}dx$$

$$= [\frac{(-cos\pi\left(\frac{1-s}{2}\right)-i\sin\pi(\frac{1-s}{2}))}{(1)} - \frac{(-cos\pi\left(\frac{1-s}{2}\right)+i\sin\pi(\frac{1-s}{2}))}{(1)}]\int_0^{+\infty}\phi(x)(x)^{\left(\frac{1-s}{2}-1\right)}dx$$

$= -2i\sin\pi(\frac{1-s}{2})\int_0^{+\infty}\phi(x)(x)^{\left(\frac{1-s}{2}-1\right)}dx$, or

$$= -2i\sin\pi(\frac{1-s}{2})\ \pi^{-(\frac{1-s}{2})}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$$

$= [(0)\ [(+\infty)+(+\infty)+\cdots]]$, **undefined (from law for infinite limits , but = 0 in measure theory)**

From $\quad \sin\pi(\frac{1-s}{2}) \;=\; 0$

$$= (\sin\tfrac{\pi}{2}\cos\tfrac{\pi s}{2} - \cos\tfrac{\pi}{2}\sin\tfrac{\pi s}{2})$$

$$= \cos\frac{\pi s}{2}$$

Then $\int_{+\infty}^{+\infty} \phi(x)(x)^{\left(\frac{1-s}{2}-1\right)} dx \;=\; 0$

$$= -2i\sin\pi(\frac{1-s}{2})\,\pi^{-(\frac{1-s}{2})}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$$

$$= -2i\cos\frac{\pi s}{2}\pi^{-\left(\frac{1-s}{2}\right)}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s) \qquad \textbf{.... (9.4)}$$

= $[-2i(0)\,[(+\infty)+(+\infty)+\cdots]]$, undefined (from law for infinite limits , but = 0 in measure theory)

(As mention before, the aim of our work is to follow or prove all of Riemann's process of deriving equation even though it looks so strange . So we have to go on although the way to derive the desired equation may use doubtful or illegal mathematics. We will try to discuss about this later.)

Thus, from equations ...(9.3) and ...(9.4), if one do not care about the fact that they are undefined then

$$-2i2\sin\frac{\pi s}{2}\cos\frac{\pi s}{2}\pi^{-(1-s)}\Gamma(1-s)\zeta(1-s)$$

$$= -2i\cos\frac{\pi s}{2}\pi^{-\left(\frac{1-s}{2}\right)}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$$

$$\left[-2i2\sin\frac{\pi s}{2}(0)\,[(+\infty)+(+\infty)+\cdots]\right] = -2i\,(0)\,[(+\infty)+(+\infty)+\cdots]],$$

undefined (from law for infinite limits, but = 0 in measure theory)

If one do not care about $(0)\,[(+\infty)+(+\infty)+\cdots]$ or undefined term, then by Canceling term $-2i\cos\frac{\pi s}{2}$ which equals zero both sides, we will get

$$2\sin\frac{\pi s}{2}\pi^{-(1-s)}\Gamma(1-s)\zeta(1-s) \;=\; \pi^{-\left(\frac{1-s}{2}\right)}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$$

$$\left[\,2\sin\frac{\pi s}{2}\,[(+\infty)+(+\infty)+\cdots] = [(+\infty)+(+\infty)+\cdots]\right]$$

And from $(\pi)^{-(s)}\Gamma(s)\zeta(s) = \int_{0^+}^{+\infty} \phi(x)(x)^{(s-1)}\,dx$

$$[[(+\infty) + (+\infty) + \cdots] = [(+\infty) + (+\infty) + \cdots]]$$

It is true that $\Gamma(s)$ alone is undefined or has poles [$\Gamma(s) = \frac{\Gamma(s+k)}{s(s+1)(s+2)\ldots(s+k-1)}$, $s > -k$, $s \neq 0, -1, -2, \ldots, -(k-1)$ or $\Gamma(s) = \pm\infty$ in the Riemann sphere] only for some values of s ($s = 0, -1, -2, \ldots$) and $(\pi)^{-(s)}$alone is never equal to $(+\infty)$, so $\zeta(s)$ itself must always equal $[(+\infty) + (+\infty) + \cdots]$ to cause $(\pi)^{-(s)}\Gamma(s)\zeta(s) = [(+\infty) + (+\infty) + \cdots]$

$$\text{Or} \qquad \zeta(s) = [(+\infty) + (+\infty) + \cdots]$$

So $(\pi)^{-(s)}\Gamma(s)\zeta(s) = 2\sin\frac{\pi s}{2}\pi^{-(1-s)}\Gamma(1-s)\zeta(1-s)$ ... **(9.5)**

$$[[(+\infty) + (+\infty) + \cdots] = \mathbf{(2\sin\frac{\pi s}{2})}\, [(+\infty) + (+\infty) + \cdots]]$$

Because $\pi^{-(1-s)}\Gamma(1-s)\zeta(1-s)$ and $(\pi)^{-(s)}\Gamma(s)\zeta(s)$ are always equal to$[(+\infty) + (+\infty) + \cdots]$, and because $\sin\frac{\pi s}{2}\cos\frac{\pi s}{2} = 0$ while $\cos\frac{\pi s}{2}$ always$=$ 0, so $\sin\frac{\pi s}{2}$ should be 1.

Finally $\zeta(s) = 2\sin\frac{\pi s}{2}\pi^{(s-1)}\Gamma(1-s)\zeta(1-s)$ **... (9.6)**

$$[[(+\infty) + (+\infty) + \cdots] = \mathbf{2(1)}\, [(+\infty) + (+\infty) + \cdots]]$$

That means $\zeta(s)$ is always equal to $[(+\infty) + (+\infty) + \cdots]$ or undefined and $\sin\frac{\pi s}{2}$ should be 1.

If you need the exact $\zeta(s) = 2^s\sin\frac{\pi s}{2}\pi^{(s-1)}\Gamma(1-s)\zeta\,(1-s)$ instead of $2\sin\frac{\pi s}{2}\pi^{(s-1)}\Gamma(1-s)\zeta(1-s)$, you can get it by multiplying equation

$$(\pi)^{-(1-s)}\Gamma(1-s)\zeta(s) = \int_{0}^{+\infty} \phi(x)(x)^{(1-s-1)}\,dx$$

$$= [(+\infty) + (+\infty) + \cdots]$$

by $(2)^{-(1-s)}$ both sides. The above equation then becomes

$$(2\pi)^{-(1-s)}\Gamma(1-s)\zeta(\mathrm{s}) = \int_0^{+\infty} \frac{\phi(\mathrm{x})(\mathrm{x})^{(1-s-1)}}{(2)^{(1-s)}}\mathrm{dx}$$

$$= [(+\infty) + (+\infty) + \cdots]$$

And then extend $\int_0^{+\infty} \frac{\phi(\mathrm{x})(\mathrm{x})^{(1-s-1)}}{(2)^{(1-s)}}\mathrm{dx}$ to $\int_{+\infty}^{+\infty} \frac{\phi(\mathrm{x})(\mathrm{x})^{(1-s-1)}}{(2)^{(1-s)}}\mathrm{dx}$.

By famous **Cauchy's theorem**

$$\int_{+\infty}^{+\infty} \frac{\phi(\mathrm{x})(\mathrm{x})^{(1-s-1)}}{(2)^{(1-s)}}\mathrm{dx} = \int_0^{+\infty} \frac{\phi(\mathrm{x})(\mathrm{x})^{(1-s-1)}}{(2)^{(1-s)}}\mathrm{dx} + \int_{+\infty}^{0} \frac{\phi(\mathrm{x})(\mathrm{x})^{(1-s-1)}}{(2)^{(1-s)}}\mathrm{dx}$$

$$= \int_0^{+\infty} \frac{\phi(\mathrm{x})(\mathrm{x})^{(1-s-1)}}{(2)^{(1-s)}}\mathrm{dx} - \int_0^{+\infty} \frac{\phi(\mathrm{x})(\mathrm{x})^{(1-s-1)}}{(2)^{(1-s)}}\mathrm{dx}$$

$$= [[(+\infty) + (+\infty) + \cdots] - [(+\infty) + (+\infty) + \cdots]]$$

undefined , indeterminate form

$$= [\frac{(1)^{(1-s)}}{(1)} - \frac{(1)^{(1-s)}}{(1)}]\int_0^{+\infty} \frac{\phi(\mathrm{x})(\mathrm{x})^{(1-s-1)}}{(2)^{(1-s)}}\mathrm{dx}$$

$$= [0]\int_0^{+\infty} \frac{\phi(\mathrm{x})(\mathrm{x})^{(1-s-1)}}{(2)^{(1-s)}}\mathrm{dx}$$

$$= [(-\cos\pi(1-s) - \mathrm{i}\sin\pi(1-s)$$

$$-(-\cos\pi(1-s) + \mathrm{i}\sin\pi(1-s)] \int_{0^+}^{+\infty} \frac{(\mathrm{x})^{(1-s-1)}}{(2)^{(1-s)}}\mathrm{dx}$$

$$= -2i\sin\pi(1-s)(2\pi)^{-(1-s)}\Gamma(1-s)\zeta(1-\mathrm{s})$$

$= [\boldsymbol{-2i(0)}[(+\infty) + (+\infty) + \cdots]]$, **undefined (from law for infinite limits, but = 0 in measure theory)**

But $\sin\pi(1-s) = (\sin\pi\cos\pi s - \cos\pi\sin\pi s)$

$$= \sin\pi s$$

And $\sin\pi s \quad = 2\sin\frac{\pi s}{2}\cos\frac{\pi s}{2}$

so $\int_{+\infty}^{+\infty} \frac{\phi(\mathrm{x})(\mathrm{x})^{(1-s-1)}}{(2)^{(1-s)}}\mathrm{dx} = (0)[(+\infty) + (+\infty) + \cdots]$

$$= -2i\sin\pi(1-s)(2\pi)^{-(1-s)}\Gamma(1-s)\zeta(1-\mathrm{s})$$

$= -2i\sin\pi s(2\pi)^{-(1-s)}\Gamma(1-s)\zeta(1-\mathrm{s})$

$= -2i2(\sin\frac{\pi s}{2}\cos\frac{\pi s}{2})(2\pi)^{-(1-s)}\Gamma(1-s)\zeta(1-\mathrm{s})$ **...(9.7)**

$= [-2i2(0)[(+\infty)+(+\infty)+\cdots]]$, undefined (from law for infinite limits, but = 0 in measure theory)

Thus from equations ...(9.7) and ...(9.4)

$$-2i2\sin\frac{\pi s}{2}\cos\frac{\pi s}{2}(2\pi)^{-(1-s)}\Gamma(1-s)\zeta(1-\mathrm{s})$$

$$= -2i\cos\frac{\pi s}{2}\pi^{-\left(\frac{1-s}{2}\right)}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-\mathrm{s})$$

Cancel term $-2i\cos\frac{\pi s}{2}$ which equals zero both sides, we will get

$$2\sin\frac{\pi s}{2}(2\pi)^{-(1-s)}\Gamma(1-s)\zeta(1-\mathrm{s}) = \pi^{-\left(\frac{1-s}{2}\right)}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-\mathrm{s})$$

$$\left[\mathbf{2\sin\frac{\pi s}{2}(2)^{-(1-s)}}[(+\infty)+(+\infty)+\cdots] = [(+\infty)+(+\infty)+\cdots]\right]$$

And from $(\pi)^{-(s)}\Gamma(s)\zeta(\mathrm{s}) = \int_{0^+}^{+\infty}\phi(\mathrm{x})(\mathrm{x})^{(s-1)}\,\mathrm{dx}$

$$[[(+\infty)+(+\infty)+\cdots] = [(+\infty)+(+\infty)+\cdots]]$$

So $(\pi)^{-(s)}\Gamma(s)\zeta(\mathrm{s}) = 2\sin\frac{\pi s}{2}(2\pi)^{-(1-s)}\Gamma(1-s)\zeta(1-\mathrm{s})$

$$[[(+\infty)+(+\infty)+\cdots] = (\mathbf{2^s\sin\frac{\pi s}{2}})[(+\infty)+(+\infty)+\cdots]]$$

Because $\pi^{-(1-s)}\Gamma(1-s)\zeta(1-\mathrm{s})$ and $(\pi)^{-(s)}\Gamma(s)\zeta(\mathrm{s})$ are always equal to$[(+\infty)+(+\infty)+\cdots]$, and because $\sin\frac{\pi s}{2}\cos\frac{\pi s}{2} = 0$ while $\cos\frac{\pi s}{2}$ always= 0, so $\sin\frac{\pi s}{2}$ should be 1.

It is true that $\Gamma(s)$ alone is undefined or has poles [$\Gamma(s) = \frac{\Gamma(\mathrm{s}+\mathrm{k})}{s(s+1)(s+2)\ldots(s+k-1)}$, $\mathrm{s} > -k$, $\mathrm{s} \neq 0, -1, -2, \ldots, -(k-1)$ or $\Gamma(s) = \pm\infty$ in the Riemann sphere] only for some values of s ($\mathrm{s} = 0, -1, -2, \ldots$) and $(\pi)^{-(s)}$alone is never equal to $(+\infty)$, so $\zeta(\mathrm{s})$ itself must always equal

$[(+\infty) + (+\infty) + \cdots]$ to cause $(\pi)^{-(s)}\Gamma(s)\zeta(s)= [(+\infty) + (+\infty) + \cdots]$

Finally $\zeta(s) = 2^s \sin\frac{\pi s}{2}\pi^{(s-1)}\Gamma(1-s)\zeta(1-s)$ **... (9.8)**

$= 2^s(1)\ [(+\infty) + (+\infty) + \cdots]$

That means ζ(s) always = $(+\infty) + (+\infty) + \cdots]$ while $\sin\frac{\pi s}{2}$ should be 1.

It looks as if $\zeta(s) = 2^s \sin\frac{\pi s}{2}\pi^{(s-1)}\Gamma(1-s)\zeta(1-s)$ would be equal to zero when the value of $\sin\frac{\pi s}{2}$ were equal to zero (or the values of s (of $\sin\frac{\pi s}{2}$) were equal to $-2, -4, -6,$... called the trivial zeroes of ζ(s)). This is not true. Actually the process above shows that we can not correctly derive the functional equation = $2^s \sin\frac{\pi s}{2}\pi^{(s-1)}\Gamma(1-s)\zeta(1-s)$ because of the undefined terms occurred and used in between derivation. Or if we do not care about the undefined terms $[(0)[(+\infty) + (+\infty) + \cdots]]$, then the function $\zeta(s) = 2^s \sin\frac{\pi s}{2}\pi^{(s-1)}\Gamma(1-s)\zeta(1-s) = [(+\infty) + (+\infty) + \cdots]$ and $\sin\frac{\pi s}{2}$ should be 1. **There are no trivial zeroes (the values of s = $-2, -4,$ $-6,$...) of the Riemann zeta function $\zeta(s) = 2^s \sin\frac{\pi s}{2}\pi^{(s-1)}\Gamma(1-s)\zeta(1-s)$ at all!**

I would like to specify that the value of ζ(s) from equation $\zeta(s) = \sum_{n=1}^{+\infty}\left(\frac{1}{n^s}\right)$, which is up to the value of s and converges only when $\Re(s)>1$, is **<u>not</u>** the same as the value of ζ(s) from functional equation $\zeta(s) = 2^s \sin\frac{\pi s}{2}\pi^{(s-1)}\Gamma(1-s)\zeta(1-s)$ which is equal to $[(+\infty) + (+\infty) + \cdots]$, or undefined.

3. Integral of the remaining complex quantities

Next, Riemann tried to find the integral of the remaining complex quantities in negative sense around the domain. He mentioned that the integrand had discontinuities where x was equal to the whole multiple of $\pm 2\pi i$, if the real part of s was negative (integer). And the integral was thus equal to the sum of the integrals taken in negative sense around these values. The integral around the value $n2\pi i = (-2\pi n i)^{(s-1)}(-2\pi i)$, then

Riemann denoted

$$2\sin \pi s\ \zeta(s) \prod(s-1) = (2\pi)^{(s)} \sum(n)^{(s-1)} [(-i)^{(s-1)} + (i)^{(s-1)}]$$

Let us prove together,

Last time when Riemann talked about positive sense around a domain, he worked with values of x on (+∞, +∞). This time he talked about negative sense around that domain and worked with x which were imaginary numbers = $\pm n2\pi i$.

From ...(5) $\quad 2\sin \pi s\ \zeta(s)\prod(s-1) = i\int_{+\infty}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)} dx = 0$

$$0 = i\int_0^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)} dx + i\int_{+\infty}^{0} \frac{(x)^{(s-1)}}{(e^x-1)} dx$$

$$= i\int_0^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)} dx - i\int_0^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)} dx$$

$$= i\int_0^{+\infty} \frac{(x)^{(s-1)}}{(1-e^{(-x)})(e^x)} dx - i\int_0^{+\infty} \frac{(x)^{(s-1)}}{(1-e^{(-x)})(e^x)} dx$$

$$= i\int_0^{+\infty} \frac{(x)^{(s-1)}(e)^{(-x)}}{(1-e^{(-x)})} dx - i\int_0^{+\infty} \frac{(x)^{(s-1)}(e)^{(-x)}}{(1-e^{(-x)})} dx$$

For $\quad x = \pm x_n = \pm n2\pi i$

$$0 = i\int_0^{+\infty} (x_n)^{(s-1)}\left[\frac{(e^{-(x_n)})}{(1-e^{-(x_n)})}\right] dx_n - i\int_0^{+\infty} (-x_n)^{(s-1)}\left[\frac{(e^{-(-x_n)})}{(1-e^{-(-x_n)})}\right] d(-x_n)$$

From **Riemann Sum**

$\int_0^{+\infty} f(x)\, dx = \sum_{n=1}^{+\infty} f(s_n)\, \Delta x_n$ for $x_{n+1} \geq s_n \geq x_n$

$= \sum_{n=1}^{+\infty} f(x_n)\, \Delta x_n$ if $x_n = n2\pi i$ = right-hand end point on $[(x_n) - (x_{n+1})]$ of the interval [0 , +∞).

$\int_0^{+\infty} f(-x)\, dx = \sum_{n=1}^{+\infty} f(-s_n)\, \Delta(-x_n)$ for $(-x_{n-1}) \leq (-s_n) \leq (-x_n)$

$= \sum_{n=1}^{+\infty} f(-x_n)\, \Delta(-x_n)$ if $(-x_n) = (-n2\pi i)$ =

right-hand end point on

$[(-x_{n-1}) - (-x_n)]$ of the

interval $[0, +\infty)$

And from equation above

$$i\int_0^{+\infty}(x_n)^{(s-1)}[\frac{(e^{-(x_n)})}{(1-e^{-(x_n)})}]\,dx_n - i\int_0^{+\infty}(-x_n)^{(s-1)}[\frac{(e^{-(-x_n)})}{(1-e^{-(-x_n)})}]\,d(-x_n)$$

$$= 0$$

$$= i\int_0^{+\infty}(x_n)^{(s-1)}\sum_{n=1}^{+\infty}\left(e^{-(x_n)}\right)^{(n-1)} e^{-(x_n)}dx_n$$

$$-i\int_0^{+\infty}(-x_n)^{(s-1)}\sum_{n=1}^{+\infty}\left(e^{-(-x_n)}\right)^{(n-1)} e^{-(-x_n)}d(-x_n)$$

$$= i\int_0^{+\infty}(x_n)^{(s-1)}\sum_{n=1}^{+\infty}\left(e^{-(x_n)}\right)^{(n)} dx_n$$

$$-i\int_0^{+\infty}(-x_n)^{(s-1)}\sum_{n=1}^{+\infty}\left(e^{-(-x_n)}\right)^{(n)} d(-x_n)$$

$$= i\int_0^{+\infty}(n2\pi i)^{(s-1)}\sum_{n=1}^{+\infty}\left(e^{-(2\pi i)}\right)^{(nn)} dx_n$$

$$-i\int_0^{+\infty}(-n2\pi i)^{(s-1)}\sum_{n=1}^{+\infty}\left(e^{-(-2\pi i)}\right)^{(nn)} d(-x_n)$$

Apply **Riemann Sum**

$$0 = i\sum_{n=1}^{+\infty}(n2\pi i)^{(s-1)}\left(\sum_{n=1}^{+\infty}[\cos 2\pi - i\sin 2\pi]^{(nn)}\right)[2\pi i]$$

$$-i\sum_{n=1}^{+\infty}(-n2\pi i)^{(s-1)}\left(\sum_{n=1}^{+\infty}[\cos 2\pi - i\sin 2\pi]^{(nn)}\right)[-2\pi i]$$

$$= i\sum_{n=1}^{+\infty}(n2\pi i)^{(s-1)}\left(\sum_{n=1}^{+\infty}[1]^{(nn)}\right)[2\pi i]$$

$$-i\sum_{n=1}^{+\infty}(-n2\pi i)^{(s-1)}\left(\sum_{n=1}^{+\infty}[1]^{(nn)}\right)[-2\pi i]$$

$$= i\sum_{n=1}^{+\infty}(\mathrm{n}2\pi i)^{(s-1)}[2\pi i] - i\sum_{n=1}^{+\infty}(-\mathrm{n}2\pi i)^{(s-1)}[-2\pi i]$$

$$= i\sum_{n=1}^{+\infty}(i)^{(s)}(2\pi)^{(s)}(\mathrm{n})^{(s-1)} - i\sum_{n=1}^{+\infty}(-i)^{(s)}(2\pi)^{(s)}(\mathrm{n})^{(s-1)}$$

$$= i(i)^{(s)}(2\pi)^{(s)}\sum_{n=1}^{+\infty}(\mathrm{n})^{(s-1)} - i(-i)^{(s)}(2\pi)^{(s)}\sum_{n=1}^{+\infty}(\mathrm{n})^{(s-1)}$$

$$= i\frac{i}{i}(i)^{(s)}(2\pi)^{(s)}\sum_{n=1}^{+\infty}(\mathrm{n})^{(s-1)} + (-i)\frac{(-i)}{(-i)}(-i)^{(s)}(2\pi)^{(s)}\sum_{n=1}^{+\infty}(\mathrm{n})^{(s-1)}$$

$$= \frac{i^2}{i}(i)^{(s)}(2\pi)^{(s)}\sum_{n=1}^{+\infty}(\mathrm{n})^{(s-1)} + \frac{i^2}{(-i)}(-i)^{(s)}(2\pi)^{(s)}\sum_{n=1}^{+\infty}(\mathrm{n})^{(s-1)}$$

$= -1\ (i)^{(s-1)}(2\pi)^{(s)} \sum_{n=1}^{+\infty}(n)^{(s-1)} - 1(-i)^{(s-1)}(2\pi)^{(s)} \sum_{n=1}^{+\infty}(n)^{(s-1)}$

Multiply by (-1) both sides

$0 = (i)^{(s-1)}(2\pi)^{(s)} \sum_{n=1}^{+\infty}(n)^{(s-1)} + (-i)^{(s-1)}(2\pi)^{(s)} \sum_{n=1}^{+\infty}(n)^{(s-1)}$

$= (2\pi)^{(s)} \sum_{n=1}^{+\infty}(n)^{(s-1)} [(-i)^{(s-1)} + (i)^{(s-1)}]$

The result is exactly the same as that of Riemann

$$2\sin \pi s\ \zeta(s) \prod(s-1) = (2\pi)^{(s)} \sum(n)^{(s-1)} [(-i)^{(s-1)} + (i)^{(s-1)}]$$
$$= 0$$

4. Finding nontrivial zeroes on critical line ($s = \frac{1}{2} + ti$)

From ... (9.2) $\prod\left(\frac{s}{2}-1\right)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(s) = \int_0^{+\infty} \psi(x)(x)^{\left(\frac{s}{2}-1\right)} dx$

$= [(+\infty) + (+\infty) + \cdots]$

Actually we should not go on anymore with this functional equation $\prod\left(\frac{s}{2}-1\right)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(s) = \int_0^{+\infty} \psi(x)(x)^{\left(\frac{s}{2}-1\right)} dx = [(+\infty) + (+\infty) + \cdots]$, or undefined terms. We also have nothing to do with the equation $\prod\left(\frac{s}{2}\right)(s-1)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(s) = \xi(t) = [(+\infty) + (+\infty) + \cdots]$ denoted by Riemann too. If someone tries to continue studying this Riemann's Hypothesis, firstly he or she has to unavoidably solve the mysterious and doubtful equations below.

4.1. $\prod\left(\frac{s}{2}-1\right)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(s) = \frac{1}{s(s-1)} + \int_1^{+\infty} \psi(x)[(x)^{\left(\frac{s}{2}-1\right)} + (x)^{-\left(\frac{1+s}{2}\right)}]dx$

4.2. $\xi(t) = \prod\left(\frac{s}{2}\right)(s-1)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(s)$

Let's see what we can do with these two equations.

4.1. $\prod\left(\frac{s}{2}-1\right)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(s) = \frac{1}{s(s-1)} + \int_1^{+\infty} \psi(x)[(x)^{\left(\frac{s}{2}-1\right)} + (x)^{-\left(\frac{1+s}{2}\right)}]dx$

This equation is conditionally true if we do not care about the truth that of $\psi(x) = [(+\infty) + (+\infty) + \cdots]$ or $\int_0^{+\infty} (e)^{(-nn\pi x)} (x)^{(\frac{s}{2}-1)}\, dx =$ $\int_0^{+\infty} \psi(x)(x)^{(\frac{s}{2}-1)}\, dx = [(+\infty) + (+\infty) + \cdots]$, undefined.

Let us prove together.

From $\prod(\frac{s}{2}-1)(\pi)^{(-\frac{s}{2})}\zeta(s) = \int_0^{+\infty} \psi(x)(x)^{(\frac{s}{2}-1)}\, dx$ **...(9)**

$$= \int_1^{+\infty} \psi(x)(x)^{(\frac{s}{2}-1)}\, dx + \int_0^1 \psi(x)(x)^{(\frac{s}{2}-1)}\, dx$$

From $2\psi(x) + 1) = (x)^{(-\frac{1}{2})}(2\psi(\frac{1}{x}) + 1)$ **(Jacobi, Fund. S.184)**

$$\prod(\tfrac{s}{2}-1)(\pi)^{(-\frac{s}{2})}\zeta(s) = \int_1^{+\infty} \psi(x)(x)^{(\frac{s}{2}-1)}\, dx + \int_0^1 \psi\left(\frac{1}{x}\right)(x)^{(\frac{s-3}{2})}\, dx$$

$$+\frac{1}{2}\int_0^1 [(x)^{(\frac{s-3}{2})} - (x)^{(\frac{s}{2}-1)}]\, dx$$

$$= \int_1^{+\infty} \psi(x)(x)^{(\frac{s}{2}-1)}\, dx + \int_0^1 \psi\left(\frac{1}{x}\right)(x)^{(\frac{s-3}{2})}\, dx$$

$$+\frac{1}{2}\Big[\frac{(x)^{(\frac{s-1}{2})}}{(\frac{s-1}{2})}\Big]_{0^+}^1 - \frac{1}{2}\Big[\frac{(x)^{(\frac{s}{2})}}{(\frac{s}{2})}\Big]_{0^+}^1$$

$$= \int_1^{+\infty} \psi(x)(x)^{(\frac{s}{2}-1)}\, dx + \int_0^1 \psi\left(\frac{1}{x}\right)(x)^{(\frac{s-3}{2})}\, dx$$

$$+\frac{1}{2}\Big[\frac{(1-0)}{(\frac{s-1}{2})}\Big] - \frac{1}{2}\Big[\frac{(1-0)}{(\frac{s}{2})}\Big]$$

$$= \int_1^{+\infty} \psi(x)(x)^{(\frac{s}{2}-1)}\, dx + \int_0^1 \psi\left(\frac{1}{x}\right)(x)^{(\frac{s-3}{2})}\, dx$$

$$+\frac{1}{(s)(s-1)}$$

So we get

$\prod(\frac{s}{2}-1)(\pi)^{(-\frac{s}{2})}\zeta(s)$

$= \frac{1}{(s)(s-1)} + \int_1^{+\infty} \psi(x)(x)^{(\frac{s}{2}-1)}\, dx + \int_0^1 \psi\left(\frac{1}{x}\right)(x)^{(\frac{s-3}{2})}\, dx$ **... (10)**

Let's consider $\int_0^1 \psi\left(\frac{1}{x}\right)(x)^{(\frac{s-3}{2})}\, dx$

And let $u = \frac{1}{x}$ then $du = (-1)(x)^{-2}\, dx$, $dx = (-1)(u)^{-2}\, du$

Then $\int_0^1 \psi\left(\frac{1}{x}\right)(x)^{\left(\frac{s-3}{2}\right)} dx = \int_{+\infty}^1 \psi(u)(u)^{-\left(\frac{s-3}{2}\right)}(-1)(u)^{-2} du$

$$= \int_1^{+\infty} \psi(u)(u)^{-\left(\frac{1+s}{2}\right)} du$$

But $\int_1^{+\infty} \psi(u)(u)^{-\left(\frac{1+s}{2}\right)} du = \int_1^{+\infty} \psi(x)(x)^{-\left(\frac{1+s}{2}\right)} dx$

So $\int_0^1 \psi\left(\frac{1}{x}\right)(x)^{\left(\frac{s-3}{2}\right)} dx = \int_1^{+\infty} \psi(x)(x)^{-\left(\frac{1+s}{2}\right)} dx$

And then $\prod\left(\frac{s}{2}-1\right)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(s)$

$$= \frac{1}{(s)(s-1)} + \int_1^{+\infty} \psi(x)(x)^{\left(\frac{s}{2}-1\right)} dx + \int_0^1 \psi\left(\frac{1}{x}\right)(x)^{\left(\frac{s-3}{2}\right)} dx$$

$$= \frac{1}{(s)(s-1)} + \int_1^{+\infty} \psi(x)(x)^{\left(\frac{s}{2}-1\right)} dx + \int_1^{+\infty} \psi(x)(x)^{-\left(\frac{1+s}{2}\right)} dx$$

The same as that found in the original Riemann's papers (1859)

$\prod\left(\frac{s}{2}-1\right)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(s)$

$$= \int_1^{+\infty} \psi(x)(x)^{\left(\frac{s}{2}-1\right)} dx + \int_0^1 \psi\left(\frac{1}{x}\right)(x)^{\left(\frac{s-3}{2}\right)} dx + \frac{1}{2}\int_0^1 \left[(x)^{\left(\frac{s-3}{2}\right)} - (x)^{\left(\frac{s}{2}-1\right)}\right] dx$$

$$= \frac{1}{(s)(s-1)} + \int_1^{+\infty} \psi(x)\left[(x)^{\left(\frac{s}{2}-1\right)} + (x)^{-\left(\frac{1+s}{2}\right)}\right] dx \quad \text{... (11)}$$

4.2. $\xi(t) = \prod\left(\frac{s}{2}\right)(s-1)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(s)$

This equation is also true, but with confusion and restriction about the condition of the equation. Let's prove together.

Start from equations

$$\prod\left(\frac{s}{2}-1\right)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(s) = \int_0^{+\infty} \psi(x)(x)^{\left(\frac{s}{2}-1\right)} dx \quad \text{... (9)}$$

$\prod\left(\frac{s}{2}-1\right)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(s)$

$$= \frac{1}{(s)(s-1)} + \int_1^{+\infty} \psi(x)\left[(x)^{\left(\frac{s}{2}-1\right)} + (x)^{-\left(\frac{1+s}{2}\right)}\right] dx \quad \text{... (11)}$$

When we consider this equation of Riemann, what we want here is only to prove from how or from where his new functional equation was

derived. If it came from wrong sources (or former equations) or from wrong methods (of deriving equations), then it was a wrong equation and further using of it would be inappropriate.

To multiply equation ...(11) by$\left(\frac{s}{2}\right)(s-1)$ both sides and to set the value of $s = \left(\frac{1}{2} + it\right)$ (as Riemann did in the past), it needs to be proved That it will not be undefined or diverging. However, this analytic technique of Riemann could not overcome or change the truth that,

$$(\pi)^{-\left(\frac{1-s}{2}\right)}\prod\left(\frac{1-s}{2}-1\right)\zeta(1-s) = \int_0^{+\infty}\psi(x)(x)^{\left(\frac{1-s}{2}-1\right)}dx$$

$$= [(+\infty)+(+\infty)+\cdots] \text{ ,undefined}$$

$$(\pi)^{\left(-\frac{s}{2}\right)}\prod\left(\frac{s}{2}-1\right)\zeta(s) = \int_0^{+\infty}\psi(x)(x)^{\left(\frac{s}{2}-1\right)}dx$$

$$= [(+\infty)+(+\infty)+\cdots] \text{ ,undefined}$$

$$(\pi)^{-(1-s)}\prod(1-s-1)\zeta(1-s) = \int_0^{+\infty}\phi(x)(x)^{(1-s-1)}dx$$

$$= [(+\infty)+(+\infty)+\cdots] \text{ ,undefined}$$

$$(\pi)^{-(s)}\prod(s-1)\zeta(s) = \int_0^{+\infty}\phi(x)(x)^{(s-1)}dx$$

$$= [(+\infty)+(+\infty)+\cdots] \text{ ,undefined}$$

Let us follow the process of deriving the equation

Multiply equation ...(11) by$\left(\frac{s}{2}\right)(s-1)$ both sides

$$\prod\left(\frac{s}{2}-1\right)\left(\frac{s}{2}\right)(s-1)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(s)$$

$$= \frac{\left(\frac{s}{2}\right)(s-1)}{(s)(s-1)} + \left(\frac{s}{2}\right)(s-1)\int_1^{+\infty}\psi(x)\left[(x)^{\left(\frac{s}{2}-1\right)} + (x)^{-\left(\frac{1+s}{2}\right)}\right]dx$$

$$= \frac{1}{2} + \frac{\left(\frac{1}{2}+it\right)\left(\frac{1}{2}+it-1\right)}{2}\int_1^{+\infty}\psi(x)\left[(x)^{\left(\frac{1}{4}+\frac{it}{2}-1\right)} + (x)^{-\left(\frac{1+\frac{1}{2}+it}{2}\right)}\right]dx$$

$$= \frac{1}{2} - \frac{\left(tt+\frac{1}{4}\right)}{2}\int_1^{+\infty}\psi(x)\left[(x)^{\left(-\frac{3}{4}\right)}(x)^{\left(\frac{it}{2}\right)} + (x)^{\left(-\frac{3}{4}\right)}(x)^{\left(-\frac{it}{2}\right)}\right]dx$$

$$= \frac{1}{2} - \frac{\left(tt+\frac{1}{4}\right)}{2}\int_{1}^{+\infty} \psi(\mathrm{x})(\mathrm{x})^{\left(-\frac{3}{4}\right)}[(e)^{\left(\frac{it}{2}\log \mathrm{x}\right)} + (e)^{\left(-\frac{it}{2}\log \mathrm{x}\right)}]\, \mathrm{dx}$$

$$= \frac{1}{2} - \frac{\left(tt+\frac{1}{4}\right)}{2}\int_{1}^{+\infty} \psi(\mathrm{x})(\mathrm{x})^{\left(-\frac{3}{4}\right)}\left[\left(\cos(\frac{1}{2}\mathrm{t}\log \mathrm{x}) + i\sin(\frac{1}{2}\mathrm{t}\log \mathrm{x})\right)\right.$$

$$\left.+\left(\cos(\frac{1}{2}\mathrm{t}\log \mathrm{x}) - i\sin(\frac{1}{2}\mathrm{t}\log \mathrm{x})\right)\right] \mathrm{dx}$$

$$= \frac{1}{2} - \frac{\left(tt+\frac{1}{4}\right)}{2}\int_{1}^{+\infty} \psi(\mathrm{x})(\mathrm{x})^{\left(-\frac{3}{4}\right)}\left(2\cos(\frac{1}{2}\mathrm{t}\log \mathrm{x})\right) \mathrm{dx}$$

$$= \frac{1}{2} - \left(tt + \frac{1}{4}\right)\int_{1}^{+\infty} \psi(\mathrm{x})(\mathrm{x})^{\left(-\frac{3}{4}\right)} \cos(\frac{1}{2}\mathrm{t}\log \mathrm{x})\, \mathrm{dx}$$

$$= \xi(\mathrm{t})$$

The right hand side looks like that of Riemann, doesn't it? But the left hand side does not.

You can see, there are two doubtful equations of Riemann here that need explanations.

1. From the left hand side of the above equation $\prod(\frac{s}{2}-1)\left(\frac{s}{2}\right)(s-1)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(\mathrm{s}) = \xi(\mathrm{t})$ is different from the equation of Riemann $\prod\left(\frac{s}{2}\right)(s-1)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(\mathrm{s}) = \xi(\mathrm{t})$. Has he made a mistake to write $\prod\left(\frac{s}{2}\right)(s-1)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(\mathrm{s})$ instead of $\prod(\frac{s}{2}-1)\left(\frac{s}{2}\right)(s-1)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(\mathrm{s})$? The answer can be both yes or no. Let's prove from the facts that

$$\Gamma(\mathrm{s}) = \frac{\Gamma(\mathrm{s}+1)}{s} \quad ; \quad \text{converges if } \quad \boldsymbol{\Re(\mathrm{s}) > 0}$$

And $\quad \Gamma(\mathrm{s}) = \dfrac{\Gamma(\mathrm{s}+\mathrm{k})}{s(s+1)(s+2)\ldots(s+k-1)} \quad , \mathrm{s} > -k, \mathrm{s} \neq 0, -1, -2, \ldots, -(k-1)$

So $\quad \Gamma(1+\boldsymbol{s}) = \boldsymbol{s}\,\Gamma(\boldsymbol{s})$ for all s except $\mathrm{s} = 0, -1, -2, -3, \ldots$ which are poles of the function.

Then $\quad \Gamma\left(1+\frac{s}{2}\right) = \frac{s}{2}\Gamma\left(\frac{s}{2}\right)$

And $\quad \Gamma\left(\frac{s}{2}\right) = \prod(\frac{s}{2}-1)$

So $\prod\left(\frac{s}{2}\right) = \frac{s}{2}\prod\left(\frac{s}{2}-1\right)$ for all $\frac{s}{2}$ except $\frac{s}{2} = 0, -1, -2, -3, \dots$

which are poles of the function.

Then $\prod\left(\frac{s}{2}\right)(s-1)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(s) = \prod\left(\frac{s}{2}-1\right)\left(\frac{s}{2}\right)(s-1)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(s) = \xi(t)$

for all $\frac{s}{2}$ except $\frac{s}{2} = 0, -1, -2, -3, \dots$ which are poles of the function.

2.From $\prod\left(\frac{s}{2}-1\right)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(s) = \int_0^{+\infty} \psi(x)(x)^{\left(\frac{s}{2}-1\right)}\,dx$

$$= [(+\infty) + (+\infty) + \cdots] \qquad \dots (9.2)$$

Then $\prod\left(\frac{s}{2}-1\right)\left(\frac{s}{2}\right)(s-1)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(s)$ or $\prod\left(\frac{s}{2}\right)\left(\frac{s}{2}-1\right)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(s) =$

$[(+\infty) + (+\infty) + \cdots]$, undefined too.

And so $\prod\left(\frac{s}{2}-1\right)\left(\frac{s}{2}\right)(s-1)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(s)$, for $s = \frac{1}{2} + it$

$$= \frac{1}{2} - \left(tt + \frac{1}{4}\right)\int_1^{+\infty} \psi(x)(x)^{\left(-\frac{3}{4}\right)} \cos\left(\frac{1}{2}t\log x\right)dx$$

$$= \frac{1}{2} - \left(tt + \frac{1}{4}\right)\int_1^{+\infty}[(+\infty) + (+\infty) + \cdots](x)^{\left(-\frac{3}{4}\right)} \cos\left(\frac{1}{2}t\log x\right)dx$$

$$= \frac{1}{2} - \left(tt + \frac{1}{4}\right)[(+\infty) + (+\infty) + \cdots]\int_1^{+\infty}(x)^{\left(-\frac{3}{4}\right)} \cos\left(\frac{1}{2}t\log x\right)dx$$

$$= \xi(t)$$

$$= [(+\infty) + (+\infty) + \cdots], \text{ undefined}$$

Hence roots of the equation

$$\xi(t) = \frac{1}{2} - \left(tt + \frac{1}{4}\right)\int_1^{+\infty} \psi(x)(x)^{\left(-\frac{3}{4}\right)} \cos\left(\frac{1}{2}t\log x\right)dx$$

or the values of (t) that make $\xi(t) = 0$ do not exist. It is impossible to show that the number of roots of $\xi(t) = 0$, whose imaginary parts of t lie between $\frac{1}{2}i$ and $-\frac{1}{2}i$ and whose real parts lie between 0 and T, is approximately $= \left(\frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi}\right)$.

Next, let us consider the integral $\int d\log\xi(t) = \int d\log(\infty)$. It is impossible to show that the integral $\int d \log \xi(\mathrm{t})$, taken in a positive sense around the region consisting of the values of t whose imaginary parts lie between $\frac{1}{2} i$ and $-\frac{1}{2} i$ and whose real parts lie between 0 and T, is equal to $(\mathrm{T} \log \frac{\mathrm{T}}{2\pi} - \mathrm{T}) i$.

It is not right to denote that all $\propto$ from the complex number $\frac{1}{2} + i \propto$, which are called the non trivial zeroes of ζ(s), are roots of equation

$$\xi(t) = \frac{1}{2} - \left(tt + \frac{1}{4}\right) \int_1^{+\infty} \psi(\mathrm{x})(x)^{\left(-\frac{3}{4}\right)} \cos\left(\frac{1}{2} \mathrm{tlogx}\right) \mathrm{dx}$$

,and also it is not right to express log ξ(t) as $[\sum \log\left(1 - \frac{tt}{\propto\propto}\right) + \log \xi(0)]$. The reason is that $\xi(\mathrm{t}) = \frac{1}{2} - \left(tt + \frac{1}{4}\right) \int_1^{+\infty} \psi(\mathrm{x})(x)^{\left(-\frac{3}{4}\right)} \cos\left(\frac{1}{2} \mathrm{tlogx}\right) \mathrm{dx}$ is always equal to $[(+\infty) + (+\infty) + \cdots]$, or undefined as proved above.

5. Determination of the number of prime numbers that are smaller than x

Next, Riemann tried to determine the number of prime numbers that are smaller than x with the assistance of all the methods he had derived before.

From the **identity by Riemann**

$$\log \zeta(\mathrm{s}) = -\sum \log(1 - (p)^{-s})$$
$$= \sum p^{-s} + \frac{1}{2} \sum p^{-2s} + \frac{1}{3} \sum p^{-3s} + \cdots \qquad \textbf{... (12)}$$

Let's prove by using Maclaurin Series

$$-\frac{\mathrm{d}}{\mathrm{dx}}(\log(1-\mathrm{x})) = \frac{1}{(1-\mathrm{x})}$$
$$= \text{Geometric Series } (1+\mathrm{X}+\mathrm{X}^2+\mathrm{X}^3+\ldots) \quad \text{for } \mathrm{x} < 1$$

By integration

$$-\log(1-x) \quad = X + \tfrac{1}{2}X^2 + \tfrac{1}{3}X^3 + \tfrac{1}{4}X^4 + \ldots$$

Thus for $x = (p)^{-s} < 1$

$$-\log(1-(p)^{-s}) \quad = (p)^{-s} + \tfrac{1}{2}(p)^{-2s} + \tfrac{1}{3}(p)^{-3s} + \ldots$$

For p = prime numbers 2, 3, 5, …

$$-\log(1-(2)^{-s}) \quad = (2)^{-s} + \tfrac{1}{2}(2)^{-2s} + \tfrac{1}{3}(2)^{-3s} + \ldots$$

$$-\log(1-(3)^{-s}) \quad = (3)^{-s} + \tfrac{1}{2}(3)^{-2s} + \tfrac{1}{3}(3)^{-3s} + \ldots$$

$$-\log(1-(5)^{-s}) \quad = (5)^{-s} + \tfrac{1}{2}(5)^{-2s} + \tfrac{1}{3}(5)^{-3s} + \ldots$$

Then $-[\log(1-(2)^{-s}) + \log(1-(3)^{-s}) + \log(1-(5)^{-s}) + \ldots]$

$$= (2)^{-s} + \tfrac{1}{2}(2)^{-2s} + \tfrac{1}{3}(2)^{-3s} + \ldots$$

$$+(3)^{-s} + \tfrac{1}{2}(3)^{-2s} + \tfrac{1}{3}(3)^{-3s} + \ldots$$

$$+(5)^{-s} + \tfrac{1}{2}(5)^{-2s} + \tfrac{1}{3}(5)^{-3s} + \ldots$$

$$+\ldots$$

Or $-\sum \log(1-(p)^{-s}) = \sum p^{-s} + \frac{1}{2}\sum p^{-2s} + \frac{1}{3}\sum p^{-3s} + \cdots$

For p = prime numbers = 2, 3, 5,…

n = all whole numbers = 1, 2, 3,…,∞

Riemann denoted that

$$\zeta(s) = \prod_{p\ prime} \left(1 - \frac{1}{p^s}\right)^{-1} = \sum_{n=1}^{+\infty} \left(\frac{1}{n^s}\right)$$

$$\log \zeta(s) = \log \prod_{p\ prime} \left(1 - \frac{1}{p^s}\right)^{-1}$$

$$= \log\,[(1-(2)^{-s})^{-1}.\,(1-(3)^{-s})^{-1}.\,(1-(5)^{-s})^{-1} \ldots.]$$

$$= \log(1-(2)^{-s})^{-1} + \log(1-(3)^{-s})^{-1} + \log(1-(5)^{-s})^{-1} + \ldots$$

$$= -[\log(1-(2)^{-s}) + \log(1-(3)^{-s}) + \log(1-(5)^{-s}) + \cdots]$$

$$= -\sum \log\left(1-(p)^{(-s)}\right)$$

So $\log \zeta(s) = -\sum \log\left(1-(p)^{(-s)}\right)$

$$= \quad \sum p^{-s} + \tfrac{1}{2}\sum p^{-2s} + \tfrac{1}{3}\sum p^{-3s} + \cdots$$

One can replace $(p^{-s})^n$ by $s\int_{p^n}^{+\infty}(\mathrm{x})^{-(s+1)}\mathrm{dx}$.

Let's prove together

$$s\int_{p^n}^{+\infty} (x)^{-(s+1)}\,\mathrm{dx} = \frac{(\mathrm{s}\,(\mathrm{x})^{(-\mathrm{s})})}{(-s)}\Big]_{p^n}^{+\infty}$$

$$= -\frac{1}{(\mathrm{x})^{(\mathrm{s})}}\Big]_{p^n}^{+\infty}$$

$$= -\left(\frac{1}{+\infty} - \frac{1}{(p^n)^s}\right)$$

$$= (p)^{(-s)n} \qquad \textbf{.... (13)}$$

I think it is useless to go on proving the rest of Riemann's paper. I hope that my paper is clear enough to point out the mistakes or give disproof of the original **Riemann's Hypothesis.** I feel good if my paper can give warning to people who are trying to apply the Riemann Hypothesis to explain physical phenomena which may be very dangerous (in many cases). At least I wish my paper will give answers or proofs of the following sentences of somebody.

1. "All zeroes of the function ξ(t) are real". This is not true because

$$\xi(\mathrm{t}) = \prod\left(\tfrac{s}{2}-1\right)\left(\frac{s}{2}\right)(s-1)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(\mathrm{s}) \quad \text{, for s} = \frac{1}{2} + it$$

$$= \frac{1}{2} - \left(tt + \frac{1}{4}\right)\int_1^{+\infty} \psi(\mathrm{x})(\mathrm{x})^{\left(-\frac{3}{4}\right)} \cos\left(\frac{1}{2}\mathrm{tlogx}\right)\mathrm{dx}$$

$$= [(+\infty) + (+\infty) + \cdots] \text{ , undefined}$$

So there are no roots (all zeroes) of equation

$$\xi(\mathrm{t}) = \prod\left(\tfrac{s}{2}-1\right)\left(\frac{s}{2}\right)(s-1)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(\mathrm{s}), \text{ for s} = \frac{1}{2} + it$$

$$= \frac{1}{2} - \left(tt + \frac{1}{4}\right)\int_1^{+\infty} \psi(\mathrm{x})(\mathrm{x})^{\left(-\frac{3}{4}\right)} \cos\left(\frac{1}{2}\mathrm{tlogx}\right)\mathrm{dx}$$

$$= [(+\infty) + (+\infty) + \cdots] \text{ , undefined}$$

2. "The function (functional equation) ζ(s) has zeroes at the negative

even integers $-2, -4, -6, \ldots$ and one refers to them as the trivial zeroes".

This is not true , actually there are no trivial zeroes of ζ(s) because ζ(s)

always = $[(+\infty) + (+\infty) + \cdots]$, undefined as proof above.

3. "The nontrivial zeroes of ζ(s) have real part equal to $\frac{1}{2}$ or the nontrivial zeroes are complex numbers $= \frac{1}{2} + i \propto$ where ∝ are zeroes of ξ(t)". This is not true because $\xi(t) = \prod\left(\frac{s}{2} - 1\right)\left(\frac{s}{2}\right)(s-1)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(s)$ for $s = \frac{1}{2} + it$, or $\xi(t) = \frac{1}{2} - \left(tt + \frac{1}{4}\right)\int_1^{+\infty} \psi(x)(x)^{\left(-\frac{3}{4}\right)} \cos\left(\frac{1}{2}t\log x\right) dx$, is always equal to $[(+\infty) + (+\infty) + \cdots]$, or undefined for any values of s (or t). So $\propto$ = zeroes of equation $\xi(t) = \prod\left(\frac{s}{2} - 1\right)\left(\frac{s}{2}\right)(s-1)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(s)$ can not be found anyway and the nontrivial zeroes of ζ(s) (or $\frac{1}{2} + i \propto$) can not be found by this way too.

References

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Appendix A

Another method to find the value of $\sum_{n=1}^{+\infty} \int_0^{+\infty} (e)^{(-nn\pi x)}(x)^{\left(\frac{s}{2}-1\right)} dx$

using integration by parts method.

For s = −2

$$\sum_{n=1}^{+\infty} \int_0^{+\infty} (\mathrm{e})^{(-\mathrm{nn}\pi\mathrm{x})} (\mathrm{x})^{\left(\frac{s}{2}-1\right)} \mathrm{dx}$$

$$= \sum_{n=1}^{+\infty} \lim_{s\to-2} [(\mathrm{e})^{(-\mathrm{nn}\pi\mathrm{x})} \frac{(\mathrm{x})^{\left(\frac{s}{2}\right)}}{\left(\frac{s}{2}\right)}]_0^{+\infty}$$

$$- \sum_{n=1}^{+\infty} \lim_{s\to-2} \int_0^{+\infty} \left(-\,\mathrm{nn}\pi\right) (\mathrm{e})^{(-\mathrm{nn}\pi\mathrm{x})} \frac{(\mathrm{x})^{\left(\frac{s}{2}\right)}}{\left(\frac{s}{2}\right)} \mathrm{dx}$$

$$= \sum_{n=1}^{+\infty} [\lim_{x\to+\infty} [(\mathrm{e})^{(-\mathrm{nn}\pi\mathrm{x})} \frac{(\mathrm{x})^{\left(\frac{-2}{2}\right)}}{\left(\frac{-2}{2}\right)}] - \lim_{x\to0} [(\mathrm{e})^{(-\mathrm{nn}\pi\mathrm{x})} \frac{(\mathrm{x})^{\left(\frac{-2}{2}\right)}}{\left(\frac{-2}{2}\right)}]]$$

$$- \sum_{n=1}^{+\infty} \int_0^{+\infty} \left(-\,\mathrm{nn}\pi\right) (\mathrm{e})^{(-\mathrm{nn}\pi\mathrm{x})} \frac{(\mathrm{x})^{\left(\frac{-2}{2}\right)}}{\left(\frac{-2}{2}\right)} \mathrm{dx}$$

$$= \sum_{n=1}^{+\infty} [[\frac{(+\infty)^{(-1)}}{(\mathrm{e})^{(+\infty)(nn\pi)}(-1)}] - [\frac{(0)^{(-1)}}{(\mathrm{e})^{(0)(nn\pi)}(-1)}]]$$

$$- \sum_{n=1}^{+\infty} \int_0^{+\infty} \left(-\,\mathrm{nn}\pi\right) (\mathrm{e})^{(-\mathrm{nn}\pi\mathrm{x})} \frac{(\mathrm{x})^{(-1)}}{(-1)} \mathrm{dx}$$

$$= \sum_{n=1}^{+\infty} [\frac{(1)}{(\mathrm{e})^{(+\infty)(nn\pi)}(+\infty)(-1)} - \frac{(1)}{(\mathrm{e})^{(0)(nn\pi)}(0)(-1)}]$$

$$- \sum_{n=1}^{+\infty} \int_0^{+\infty} \left(-\,\mathrm{nn}\pi\right) (\mathrm{e})^{(-\mathrm{nn}\pi\mathrm{x})} \frac{(\mathrm{x})^{(-1)}}{(-1)} \mathrm{dx}$$

$$= [\frac{1}{(\mathrm{e})^{(+\infty)(\pi)}(+\infty)(-1)} + \frac{1}{(\mathrm{e})^{(+\infty)(4\pi)}(+\infty)(-1)} + \cdots]$$

$$-[\frac{1}{(\mathrm{e})^{(0)(\pi)}(0)(-1)} + \frac{1}{(\mathrm{e})^{(0)(4\pi)}(0)(-1)} + \cdots]$$

$$- \sum_{n=1}^{+\infty} \int_0^{+\infty} \left(-\,\mathrm{nn}\pi\right) (\mathrm{e})^{(-\mathrm{nn}\pi\mathrm{x})} \frac{(\mathrm{x})^{(-1)}}{(-1)} \mathrm{dx}$$

$= [(-0) + (-0) + \cdots] + [\frac{1}{(0)} + \frac{1}{(0)} + \ldots]$, $\frac{1}{0} = \infty$ in the Riemann sphere

$$- \sum_{n=1}^{+\infty} \int_0^{+\infty} \left(-\,\mathrm{nn}\pi\right) (\mathrm{e})^{(-\mathrm{nn}\pi\mathrm{x})} \frac{(\mathrm{x})^{(-1)}}{(-1)} \mathrm{dx}$$

$= \text{undefined } [\infty + \infty + \ldots] - \sum_{n=1}^{+\infty} \int_0^{+\infty} \left(-\,\mathrm{nn}\pi\right) (\mathrm{e})^{(-\mathrm{nn}\pi\mathrm{x})} \frac{(\mathrm{x})^{(-1)}}{(-1)} \mathrm{dx}$

So $\sum_{n=1}^{+\infty} \int_0^{+\infty} (\mathrm{e})^{(-\mathrm{nn}\pi\mathrm{x})}(\mathrm{x})^{\left(\frac{s}{2}-1\right)}\,\mathrm{dx}$ for $s=-2$

$$=\text{undefined }[\infty+\infty+\ldots]-\sum_{n=1}^{+\infty} \int_0^{+\infty}(-\mathrm{nn}\pi)(\mathrm{e})^{(-\mathrm{nn}\pi\mathrm{x})}\frac{(\mathrm{x})^{(-1)}}{(-1)}\,\mathrm{dx}$$

For s = −3

$$\sum_{n=1}^{+\infty} \int_0^{+\infty} (\mathrm{e})^{(-\mathrm{nn}\pi\mathrm{x})}(\mathrm{x})^{\left(\frac{s}{2}-1\right)}\,\mathrm{dx}$$

$$=\text{undefined}[\infty+\infty+\cdots]-\sum_{n=1}^{+\infty} \int_0^{+\infty}(-\mathrm{nn}\pi)(\mathrm{e})^{(-\mathrm{nn}\pi\mathrm{x})}\frac{(\mathrm{x})^{\left(\frac{-3}{2}\right)}}{\left(\frac{-3}{2}\right)}\,\mathrm{dx}$$

For s $=-4$

$$\sum_{n=1}^{+\infty} \int_0^{+\infty} (\mathrm{e})^{(-\mathrm{nn}\pi\mathrm{x})}(\mathrm{x})^{\left(\frac{s}{2}-1\right)}\,\mathrm{dx}$$

$$=\text{undefined}[\infty+\infty+\ \ldots]-\sum_{n=1}^{+\infty} \int_0^{+\infty}(-\mathrm{nn}\pi)(\mathrm{e})^{(-\mathrm{nn}\pi\mathrm{x})}\frac{(\mathrm{x})^{(-2)}}{(-2)}\,\mathrm{dx}$$

For s = −5

$$\sum_{n=1}^{+\infty} \int_0^{+\infty} (\mathrm{e})^{(-\mathrm{nn}\pi\mathrm{x})}(\mathrm{x})^{\left(\frac{s}{2}-1\right)}\,\mathrm{dx}$$

$$=\text{undefined}[\infty+\infty+\cdots]-\sum_{n=1}^{+\infty} \int_0^{+\infty}(-\mathrm{nn}\pi)(\mathrm{e})^{(-\mathrm{nn}\pi\mathrm{x})}\frac{(\mathrm{x})^{\left(\frac{-5}{2}\right)}}{\left(\frac{-5}{2}\right)}\,\mathrm{dx}$$

For s = $\mathbf{\frac{1}{2}}$

$$\sum_{n=1}^{+\infty} \int_0^{+\infty} (\mathrm{e})^{(-\mathrm{nn}\pi\mathrm{x})}(\mathrm{x})^{\left(\frac{s}{2}-1\right)}\,\mathrm{dx}$$

$$=\sum_{n=1}^{+\infty}\left[\frac{(\mathrm{e})^{(-\mathrm{nn}\pi\mathrm{x})}}{(-\mathrm{nn}\pi)}(\mathrm{x})^{\left(\frac{s}{2}-1\right)}\right]_0^{+\infty}-\sum_{n=1}^{+\infty} \int_0^{+\infty}\frac{(\mathrm{e})^{(-\mathrm{nn}\pi\mathrm{x})}}{(-\mathrm{nn}\pi)}\left(\frac{s}{2}-1\right)(\mathrm{x})^{\left(\frac{s}{2}-2\right)}\,\mathrm{dx}$$

$$=\sum_{n=1}^{+\infty}\lim_{s\to\frac{1}{2}}\left[\frac{(\mathrm{e})^{(-\mathrm{nn}\pi\mathrm{x})}}{(-\mathrm{nn}\pi)}(\mathrm{x})^{\left(\frac{s}{2}-1\right)}\right]_0^{+\infty}$$

$$-\sum_{n=1}^{+\infty}\lim_{s\to\frac{1}{2}}\int_0^{+\infty}\frac{(\mathrm{e})^{(-\mathrm{nn}\pi\mathrm{x})}}{(-\mathrm{nn}\pi)}\left(\frac{s}{2}-1\right)(\mathrm{x})^{\left(\frac{s}{2}-2\right)}\,\mathrm{dx}$$

$$=\sum_{n=1}^{+\infty}\left[\lim_{x\to+\infty}\left[\frac{(\mathrm{e})^{(-\mathrm{nn}\pi\mathrm{x})}}{(-\mathrm{nn}\pi)}(\mathrm{x})^{\left(-\frac{3}{4}\right)}\right]-\lim_{x\to0}\left[\frac{(\mathrm{e})^{(-\mathrm{nn}\pi\mathrm{x})}}{(-\mathrm{nn}\pi)}(\mathrm{x})^{\left(-\frac{3}{4}\right)}\right]\right]$$

$$-\sum_{n=1}^{+\infty}\int_0^{+\infty}\frac{(\mathrm{e})^{(-\mathrm{nn}\pi\mathrm{x})}}{(-\mathrm{nn}\pi)}\left(-\frac{3}{4}\right)(\mathrm{x})^{\left(-\frac{7}{4}\right)}\,\mathrm{dx}$$

$$= \sum_{n=1}^{+\infty} \left[\frac{1}{(-\text{nn}\pi)(\text{e})^{(+\infty)(nn\pi)}(+\infty)^{\left(\frac{3}{4}\right)}} - \frac{1}{(-\text{nn}\pi)(\text{e})^{(0)(nn\pi)}(0)^{\left(\frac{3}{4}\right)}}\right]$$

$$- \sum_{n=1}^{+\infty} \int_0^{+\infty} \frac{(\text{e})^{(-\text{nn}\pi\text{x})}}{(-\text{nn}\pi)} (-\tfrac{3}{4})(\text{x})^{(-\frac{7}{4})} \text{ dx}$$

$$= [\frac{-1}{(+\infty)} + \frac{-1}{(+\infty)} + \cdots] - \left[\frac{-1}{(0)} + \frac{-1}{(0)} + \cdots\right], \ \frac{1}{0} = \infty \text{ in the Riemann sphere}$$

$$= \text{undefined } [\infty + \infty + \cdots] - \sum_{n=1}^{+\infty} \int_0^{+\infty} \frac{(\text{e})^{(-\text{nn}\pi\text{x})}}{(-\text{nn}\pi)} (-\tfrac{3}{4})(\text{x})^{(-\frac{7}{4})} \text{ dx}$$

So $\sum_{n=1}^{+\infty} \int_0^{+\infty} (\text{e})^{(-\text{nn}\pi\text{x})}(\text{x})^{(\frac{s}{2}-1)} \text{ dx}$ for $s = \frac{1}{2}$

$$= \text{undefined } [\infty + \infty + \cdots] - \sum_{n=1}^{+\infty} \int_0^{+\infty} \frac{(\text{e})^{(-\text{nn}\pi\text{x})}}{(-\text{nn}\pi)} (-\tfrac{3}{4})(\text{x})^{(-\frac{7}{4})} \text{ dx}$$

For s = 2

$$\sum_{n=1}^{+\infty} \int_0^{+\infty} (\text{e})^{(-\text{nn}\pi\text{x})}(\text{x})^{(\frac{s}{2}-1)} \text{ dx}$$

$$= \sum_{n=1}^{+\infty} \left[\frac{(\text{e})^{(-\text{nn}\pi\text{x})}}{(-\text{nn}\pi)}(\text{x})^{(\frac{s}{2}-1)}\right]_0^{+\infty} - \sum_{n=1}^{+\infty} \int_0^{+\infty} \frac{(\text{e})^{(-\text{nn}\pi\text{x})}}{(-\text{nn}\pi)} \left(\frac{s}{2}-1\right)(\text{x})^{(\frac{s}{2}-2)} \text{ dx}$$

$$= \sum_{n=1}^{+\infty} \lim_{s\to 2}\left[\frac{(\text{e})^{(-\text{nn}\pi\text{x})}}{(-\text{nn}\pi)}(\text{x})^{(\frac{s}{2}-1)}\right]_0^{+\infty}$$

$$- \sum_{n=1}^{+\infty} \lim_{s\to 2} \int_0^{+\infty} \frac{(\text{e})^{(-\text{nn}\pi\text{x})}}{(-\text{nn}\pi)} \left(\frac{s}{2}-1\right)(\text{x})^{(\frac{s}{2}-2)} \text{ dx}$$

$$= \sum_{n=1}^{+\infty} \left[\lim_{x\to+\infty}\left[\frac{(\text{e})^{(-\text{nn}\pi\text{x})}}{(-\text{nn}\pi)}(\text{x})^{(0)}\right] - \lim_{x\to 0}\left[\frac{(\text{e})^{(-\text{nn}\pi\text{x})}}{(-\text{nn}\pi)}(\text{x})^{(0)}\right]\right]$$

$$- \sum_{n=1}^{+\infty} \int_0^{+\infty} \frac{(\text{e})^{(-\text{nn}\pi\text{x})}}{(-\text{nn}\pi)} (0)(\text{x})^{(-1)} \text{ dx}$$

$$= \sum_{n=1}^{+\infty} \left[\frac{1}{(-\text{nn}\pi)(\text{e})^{(+\infty)(nn\pi)}} - \frac{1}{(-\text{nn}\pi)(\text{e})^{(0)(nn\pi)}}\right]$$

$$-(0)$$

$$= [\frac{1}{(-\pi)(+\infty)} + \frac{1}{(-4\pi)(+\infty)} + \cdots] - [\frac{1}{(-\pi)} + \frac{1}{(-4\pi)} + \cdots]$$

$$= [\frac{1}{(\pi)} + \frac{1}{(4\pi)} + \cdots]$$

So $\sum_{n=1}^{+\infty} \int_0^{+\infty} (\text{e})^{(-\text{nn}\pi\text{x})}(\text{x})^{(\frac{s}{2}-1)} \text{ dx}$ for $s = 2$

$$= [\frac{1}{(\pi)} + \frac{1}{(4\pi)} + \cdots]$$

For s = 3

$$\sum_{n=1}^{+\infty} \int_0^{+\infty} (e)^{(-nn\pi x)} (x)^{(\frac{s}{2}-1)} \, dx$$

$$= \sum_{n=1}^{+\infty} \lim_{s \to 3} [\frac{(e)^{(-nn\pi x)}}{(-nn\pi)} (x)^{(\frac{s}{2}-1)}]_0^{+\infty}$$

$$- \sum_{n=1}^{+\infty} \lim_{s \to 3} \int_0^{+\infty} \frac{(e)^{(-nn\pi x)}}{(-nn\pi)} \left(\frac{s}{2} - 1\right) (x)^{(\frac{s}{2}-2)} \, dx$$

$$= \sum_{n=1}^{+\infty} [\lim_{x \to +\infty} [\frac{(e)^{(-nn\pi x)}}{(-nn\pi)} (x)^{\left(\frac{1}{2}\right)}] - \lim_{x \to 0} [\frac{(e)^{(-nn\pi x)}}{(-nn\pi)} (x)^{\left(\frac{1}{2}\right)}]]$$

$$- \sum_{n=1}^{+\infty} \int_0^{+\infty} \frac{(e)^{(-nn\pi x)}}{(-nn\pi)} (\tfrac{1}{2}) (x)^{(-\frac{1}{2})} \, dx$$

$$= \sum_{n=1}^{+\infty} [\frac{(+\infty)^{(\frac{1}{2})}}{(-nn\pi)(e)^{(+\infty)(nn\pi)}} - \frac{(0)^{(\frac{1}{2})}}{(-nn\pi)(e)^{(0)(nn\pi)}}]$$

$$- \sum_{n=1}^{+\infty} \int_0^{+\infty} \frac{(e)^{(-nn\pi x)}}{(-nn\pi)} (\tfrac{1}{2}) (x)^{(-\frac{1}{2})} \, dx$$

$$= [\frac{+\infty}{(-\pi)(+\infty)} + \frac{+\infty}{(-4\pi)(+\infty)} + \cdots] - [\frac{0}{(-\pi)} + \frac{0}{(-4\pi)} + \cdots]$$

$$- \sum_{n=1}^{+\infty} \int_0^{+\infty} \frac{(e)^{(-nn\pi x)}}{(-nn\pi)} (\tfrac{1}{2}) (x)^{(-\frac{1}{2})} \, dx$$

$$= - [\frac{+\infty}{+\infty} + \frac{+\infty}{+\infty} + \cdots] + [0 + 0 + \cdots] - \sum_{n=1}^{+\infty} \int_0^{+\infty} \frac{(e)^{(-nn\pi x)}}{(-nn\pi)} (\tfrac{1}{2}) (x)^{(-\frac{1}{2})} \, dx$$

$$= - [\frac{+\infty}{+\infty} + \frac{+\infty}{+\infty} + \cdots] - \sum_{n=1}^{+\infty} \int_0^{+\infty} \frac{(e)^{(-nn\pi x)}}{(-nn\pi)} (\tfrac{1}{2}) (x)^{(-\frac{1}{2})} \, dx$$

Find the values of $\frac{+\infty}{+\infty}$ which is indeterminate form using L'Hospital's Rule

$$\sum_{n=1}^{+\infty} \lim_{s \to 3} [\frac{(e)^{(-nn\pi x)}}{(-nn\pi)} (x)^{(\frac{s}{2}-1)}]_0^{+\infty}$$

$$= \sum_{n=1}^{+\infty} [\frac{(x)^{\frac{1}{2}}}{(-nn\pi)(e)^{(nn\pi x)}}]_0^{+\infty}$$

Apply L'Hospital's Rule

$$= \sum_{n=1}^{+\infty} \left[\frac{(x)^{\left(-\frac{1}{2}\right)}}{(-nn\pi)(nn\pi)(e)^{(nn\pi x)}}\right]_{0}^{+\infty}$$

$$= \sum_{n=1}^{+\infty} [\lim_{x\to+\infty}[\frac{1}{(-nn\pi)(nn\pi)(e)^{(nn\pi x)}(x)^{\left(\frac{1}{2}\right)}}]$$

$$-\lim_{x\to 0}[\frac{1}{(-nn\pi)(nn\pi)(e)^{(nn\pi x)}(x)^{\left(\frac{1}{2}\right)}}]]$$

$$= [\frac{1}{(-\pi)(\pi)(e)^{(\pi)(+\infty)}(+\infty)^{\left(\frac{1}{2}\right)}} + \frac{1}{(-4\pi)(4\pi)\ (e)^{(4\pi)(+\infty)}(+\infty)^{\left(\frac{1}{2}\right)}} + ...]$$

$$-[\frac{1}{(-\pi)(\pi)(e)^{(\pi)(0)}(0)^{\left(\frac{1}{2}\right)}} + \frac{1}{(-4\pi)(4\pi)(e)^{(4\pi)(0)}(0)^{\left(\frac{1}{2}\right)}} + ...]$$

$= -\ [0+0+...]+[\frac{1}{0} + \frac{1}{0}+...], \frac{1}{0} = \infty$ in the Riemann sphere

$=$ undefined $[\infty + \infty+...]$

So $\sum_{n=1}^{+\infty} \int_{0}^{+\infty} (e)^{(-nn\pi x)}(x)^{\left(\frac{s}{2}-1\right)}\,dx$ for $s = 3$

$=$ undefined $[\infty + \infty+...] - \sum_{n=1}^{+\infty} \int_{0}^{+\infty} \frac{(e)^{(-nn\pi x)}}{(-nn\pi)} \left(\frac{1}{2}\right)(x)^{\left(-\frac{1}{2}\right)}\,dx$

For $s = 4$

$$\sum_{n=1}^{+\infty} \int_{0}^{+\infty} (e)^{(-nn\pi x)}(x)^{\left(\frac{s}{2}-1\right)}\,dx$$

$$= [\frac{1}{(\pi)^{(2)}} + \frac{1}{(4\pi)^{(2)}} + ...]$$

For $s = 5$

$$\sum_{n=1}^{+\infty} \int_{0}^{+\infty} (e)^{(-nn\pi x)}(x)^{\left(\frac{s}{2}-1\right)}\,dx$$

$=$ undefined $[\infty + \infty+...] - \sum_{n=1}^{+\infty} \int_{0}^{+\infty} \frac{(e)^{(-nn\pi x)}}{(-nn\pi)}(1)(x)^{\left(\frac{1}{2}\right)}\,dx$

Appendix B

Another method to find the value of $\sum_{n=1}^{+\infty} \int_{0^+}^{+\infty} (e)^{(-nn\pi x)}(x)^{\left(\frac{1-s}{2}-1\right)}\,dx$ using integration by parts.

For $s = -2$

$$\sum_{n=1}^{+\infty} \int_0^{+\infty} (e)^{(-nn\pi x)} (x)^{\left(\frac{1-s}{2}-1\right)} dx$$

$$= \sum_{n=1}^{+\infty} \lim_{s\to -2} \left[\frac{(e)^{(-nn\pi x)}}{(-nn\pi)} (x)^{\left(\frac{1-s}{2}-1\right)}\right]_{0^+}^{+\infty}$$

$$- \sum_{n=1}^{+\infty} \lim_{s\to -2} \int_0^{+\infty} \frac{(e)^{(-nn\pi x)}}{(-nn\pi)} \left(\frac{1-s}{2}-1\right)(x)^{\left(\frac{1-s}{2}-2\right)} dx$$

$$= \sum_{n=1}^{+\infty} \left[\lim_{x\to +\infty} \left[\frac{(e)^{(-nn\pi x)}}{(-nn\pi)} (x)^{\left(\frac{1+2}{2}-1\right)}\right] - \lim_{x\to 0} \left[\frac{(e)^{(-nn\pi x)}}{(-nn\pi)} (x)^{\left(\frac{1+2}{2}-1\right)}\right]\right]$$

$$- \sum_{n=1}^{+\infty} \int_0^{+\infty} \frac{(e)^{(-nn\pi x)}}{(-nn\pi)} \left(\frac{1+2}{2}-1\right)(x)^{\left(\frac{1+2}{2}-2\right)} dx$$

$$= \sum_{n=1}^{+\infty} \left[\lim_{x\to +\infty} \left[\frac{(e)^{(-nn\pi x)}}{(-nn\pi)} (x)^{\left(\frac{1}{2}\right)}\right] - \lim_{x\to 0} \left[\frac{(e)^{(-nn\pi x)}}{(-nn\pi)} (x)^{\left(\frac{1}{2}\right)}\right]\right]$$

$$- \sum_{n=1}^{+\infty} \int_0^{+\infty} \frac{(e)^{(-nn\pi x)}}{(-nn\pi)} \left(\frac{1}{2}\right)(x)^{\left(-\frac{1}{2}\right)} dx$$

$$= \sum_{n=1}^{+\infty} \left[\frac{(+\infty)^{\left(\frac{1}{2}\right)}}{(-nn\pi)(e)^{(+\infty)(nn\pi)}} - \frac{(0)^{\left(\frac{1}{2}\right)}}{(-nn\pi)(e)^{(0)(nn\pi)}}\right]$$

$$- \sum_{n=1}^{+\infty} \int_0^{+\infty} \frac{(e)^{(-nn\pi x)}}{(-nn\pi)} \left(\frac{1}{2}\right)(x)^{\left(-\frac{1}{2}\right)} dx$$

$$= \left[\frac{+\infty}{(-\pi)(e)^{(+\infty)(\pi)}} + \frac{+\infty}{(-4\pi)(e)^{(+\infty)(4\pi)}} + \cdots\right] - \left[\frac{(0)}{(-\pi)(e)^{(0)(\pi)}} + \frac{(0)}{(-4\pi)(e)^{(0)(4\pi)}} + \cdots\right]$$

$$= -\left[\frac{+\infty}{+\infty} + \frac{+\infty}{+\infty} + \cdots\right] + [0 + 0 + \cdots]$$

$$- \sum_{n=1}^{+\infty} \int_0^{+\infty} \frac{(e)^{(-nn\pi x)}}{(-nn\pi)} \left(\frac{1}{2}\right)(x)^{\left(-\frac{1}{2}\right)} dx$$

Find the values of $\frac{+\infty}{+\infty}$ which is indeterminate forms using L'Hospital's Rule

From $\sum_{n=1}^{+\infty} \lim_{x\to +\infty} \left[\frac{(x)^{\left(\frac{1}{2}\right)}}{(-nn\pi)(e)^{(nn\pi x)}}\right]$

Apply L'Hospital's Rule

$$= \sum_{n=1}^{+\infty} \lim_{x\to +\infty} \left[\frac{\left(\frac{1}{2}\right)(x)^{\left(-\frac{1}{2}\right)}}{2(-nn\pi)(nn\pi)(e)^{(nn\pi x)}}\right]$$

$= \sum_{n=1}^{+\infty} \lim_{x\to+\infty}[\frac{(\frac{1}{2})(1)}{(-\text{nn}\pi)(\text{nn}\pi)(\text{e})^{(\text{nn}\pi\text{x})}(\text{x})^{(\frac{1}{2})}}]$

$= [\frac{(\frac{1}{2})(1)}{(-\pi)(\pi)(\text{e})^{(\pi)(+\infty)}(+\infty)} + \frac{(\frac{1}{2})(1)}{(-4\pi)(4\pi)\ (\text{e})^{(4\pi)(+\infty)}(+\infty)} + ...]$

$= [0+0+...]$

Find the value of $\sum_{n=1}^{+\infty} \int_0^{+\infty} \frac{(\text{e})^{(-\text{nn}\pi\text{x})}}{(-\text{nn}\pi)}(\frac{1}{2})(\text{x})^{(-\frac{1}{2})}\ \text{dx}$

$= \sum_{n=1}^{+\infty} [\frac{(\text{e})^{(-\text{nn}\pi\text{x})}}{(-\text{nn}\pi)(-\text{nn}\pi)}(\frac{1}{2})(\text{x})^{(-\frac{1}{2})}]_0^{+\infty}$

$- \sum_{n=1}^{+\infty} \int_0^{+\infty} \frac{(\text{e})^{(-\text{nn}\pi\text{x})}}{(-\text{nn}\pi)(-\text{nn}\pi)}(-\frac{1}{2})(\frac{1}{2})(\text{x})^{(-\frac{3}{2})}\ \text{dx}$

$= \sum_{n=1}^{+\infty} [\lim_{x\to+\infty}[\frac{(\frac{1}{2})}{(\text{e})^{(\text{nn}\pi\text{x})}(-\text{nn}\pi)(-\text{nn}\pi)(\text{x})^{(\frac{1}{2})}}]$

$-\lim_{x\to 0}[\frac{(\frac{1}{2})}{(\text{e})^{(\text{nn}\pi\text{x})}(-\text{nn}\pi)(-\text{nn}\pi)(\text{x})^{(\frac{1}{2})}}]]$

$- \sum_{n=1}^{+\infty} \int_0^{+\infty} \frac{(\text{e})^{(-\text{nn}\pi\text{x})}}{(-\text{nn}\pi)(-\text{nn}\pi)}(-\frac{1}{2})(\frac{1}{2})(\text{x})^{(-\frac{3}{2})}\ \text{dx}$

$= [\frac{(\frac{1}{2})}{(\text{e})^{(\pi)(+\infty)}(-\pi)(-\pi)(+\infty)^{(\frac{1}{2})}} + \frac{(\frac{1}{2})}{(\text{e})^{(4\pi)(+\infty)}(-4\pi)(-4\pi)(+\infty)^{(\frac{1}{2})}} + \cdots]$

$-[\frac{(\frac{1}{2})}{(\text{e})^{(\pi)(0)}(-\pi)(-\pi)(0)^{(\frac{1}{2})}} + \frac{(\frac{1}{2})}{(\text{e})^{(4\pi)(0)}(-4\pi)(-4\pi)(0)^{(\frac{1}{2})}} + \cdots]$

$- \sum_{n=1}^{+\infty} \int_0^{+\infty} \frac{(\text{e})^{(-\text{nn}\pi\text{x})}}{(-\text{nn}\pi)(-\text{nn}\pi)}(-\frac{1}{2})(\frac{1}{2})(\text{x})^{(-\frac{3}{2})}\ \text{dx}$

$= [\frac{1}{+\infty} + \frac{1}{+\infty} + ...] - \left[\frac{1}{0} + \frac{1}{0} + \cdots\right], \frac{1}{0} = \infty$ in the Riemann sphere

$- \sum_{n=1}^{+\infty} \int_0^{+\infty} \frac{(\text{e})^{(-\text{nn}\pi\text{x})}}{(-\text{nn}\pi)(-\text{nn}\pi)}(-\frac{1}{2})(\frac{1}{2})(\text{x})^{(-\frac{3}{2})}\ \text{dx}$

$= -$ undefined$[\infty + \infty + ...] - \sum_{n=1}^{+\infty} \int_0^{+\infty} \frac{(\text{e})^{(-\text{nn}\pi\text{x})}}{(-\text{nn}\pi)(-\text{nn}\pi)}(-\frac{1}{2})(\frac{1}{2})(\text{x})^{(-\frac{3}{2})}\ \text{dx}$

So $\sum_{n=1}^{+\infty} \int_0^{+\infty} (\text{e})^{(-\text{nn}\pi\text{x})}(\text{x})^{(\frac{1-s}{2}-1)}\ \text{dx}$ for s $=-2$

$= \text{undefined}[\infty + \infty + ...] + \sum_{n=1}^{+\infty} \int_0^{+\infty} \frac{(e)^{(-nn\pi x)}}{(-nn\pi)(-nn\pi)} (-\frac{1}{2})(\frac{1}{2})(x)^{(-\frac{3}{2})} \, dx$

For s = −3

$\sum_{n=1}^{+\infty} \int_0^{+\infty} (e)^{(-nn\pi x)} (x)^{(\frac{1-s}{2}-1)} \, dx$

$= \sum_{n=1}^{+\infty} \lim_{s \to -3} [\frac{(e)^{(-nn\pi x)}}{(-nn\pi)} (x)^{(\frac{1-s}{2}-1)}]_0^{+\infty}$

$- \sum_{n=1}^{+\infty} \lim_{s \to -3} \int_0^{+\infty} \frac{(e)^{(-nn\pi x)}}{(-nn\pi)} (\frac{1-s}{2} - 1)(x)^{(\frac{1-s}{2}-2)} \, dx$

$= \sum_{n=1}^{+\infty} [\lim_{x \to +\infty} [\frac{(e)^{(-nn\pi x)}}{(-nn\pi)} (x)^{(\frac{1-s}{2}-1)}] - \lim_{x \to 0} [\frac{(e)^{(-nn\pi x)}}{(-nn\pi)} (x)^{(\frac{1-s}{2}-1)}]]$

$- \sum_{n=1}^{+\infty} \int_0^{+\infty} \frac{(e)^{(-nn\pi x)}}{(-nn\pi)} (\frac{1-s}{2} - 1)(x)^{(\frac{1-s}{2}-2)} \, dx$

$= \sum_{n=1}^{+\infty} [\lim_{x \to +\infty} [\frac{(e)^{(-nn\pi x)}}{(-nn\pi)} (x)^{(1)}] - \lim_{x \to 0} [\frac{(e)^{(-nn\pi x)}}{(-nn\pi)} (x)^{(1)}]]$

$- \sum_{n=1}^{+\infty} \int_0^{+\infty} \frac{(e)^{(-nn\pi x)}}{(-nn\pi)} (1)(x)^{(0)} \, dx$

$= \sum_{n=1}^{+\infty} [\frac{(+\infty)^{(1)}}{(-nn\pi)(e)^{(+\infty)(nn\pi)}} - \frac{(0)^{(1)}}{(-nn\pi)(e)^{(0)(nn\pi)}}]$

$- \sum_{n=1}^{+\infty} \int_0^{+\infty} \frac{(e)^{(-nn\pi x)}}{(-nn\pi)} (1)(1) \, dx$

$= [\frac{+\infty}{(-\pi)(e)^{(+\infty)(\pi)}} + \frac{+\infty}{(-4\pi)(e)^{(+\infty)(4\pi)}} + \cdots] - [\frac{(0)}{(-\pi)(e)^{(0)(\pi)}} + \frac{(0)}{(-4\pi)(e)^{(0)(4\pi)}} + \cdots]$

$= -[\frac{+\infty}{+\infty} + \frac{+\infty}{+\infty} + \cdots] + [0 + 0 + \cdots]$

$- \sum_{n=1}^{+\infty} \int_0^{+\infty} \frac{(e)^{(-nn\pi x)}}{(-nn\pi)} \, dx$

Find the values of $\frac{+\infty}{+\infty}$ which is indeterminate form using L'Hospital's Rule

$\sum_{n=1}^{+\infty} [\lim_{x \to +\infty} [\frac{(x)^{(1)}}{(-nn\pi)(e)^{(nn\pi x)}}]$

Apply L'Hospital's Rule

$= \sum_{n=1}^{+\infty}\ [\lim_{x\to+\infty}[\frac{1}{(-\text{nn}\pi)(\text{nn}\pi)(\text{e})^{(\text{nn}\pi\text{x})}}]$

$= [\frac{1}{(-\pi)(\pi)(\text{e})^{(\pi)(+\infty)}} + \frac{1}{(-4\pi)(4\pi)\ (\text{e})^{(4\pi)(+\infty)}} + ...]$

$= [0+0+...]$

Find the value of $\sum_{n=1}^{+\infty}\ \int_0^{+\infty} \frac{(\text{e})^{(-\text{nn}\pi\text{x})}}{(-\text{nn}\pi)}\ \text{dx}$

$= \sum_{n=1}^{+\infty}\ [\frac{(\text{e})^{(-\text{nn}\pi\text{x})}}{(-\text{nn}\pi)(-\text{nn}\pi)}]_0^{+\infty}$

$= \sum_{n=1}^{+\infty}\ [\lim_{x\to+\infty}[\frac{(\text{e})^{(-\text{nn}\pi\text{x})}}{(-\text{nn}\pi)(-\text{nn}\pi)}]\ -\lim_{x\to0}[\frac{(\text{e})^{(-\text{nn}\pi\text{x})}}{(-\text{nn}\pi)(-\text{nn}\pi)}]]$

$= [\frac{(\text{e})^{(-\pi)(+\infty)}}{(-\pi)(-\pi)} + \frac{(\text{e})^{(-4\pi)(+\infty)}}{(-4\pi)(-4\pi)} + \cdots] - [\frac{(\text{e})^{(-\pi)(0)}}{(-\pi)(-\pi)} + \frac{(\text{e})^{(-4\pi)(0)}}{(-4\pi)(-4\pi)} + \cdots]$

$= [\frac{1}{\infty} + \frac{1}{\infty} + ...] - [\frac{1}{(-\pi)(-\pi)} + \frac{1}{(-4\pi)(-4\pi)} + \cdots]$

$= -\ [\frac{1}{(-\pi)(-\pi)} + \frac{1}{(-4\pi)(-4\pi)} + ...]$

So $\sum_{n=1}^{+\infty}\ \int_0^{+\infty}\ (\text{e})^{(-\text{nn}\pi\text{x})}(\text{x})^{(\frac{1-s}{2}-1)}\ \text{dx}$ for s =−3

$= [\frac{1}{(\pi)^2} + \frac{1}{(4\pi)^2} + ...]$

For s = −4

$\sum_{n=1}^{+\infty}\ \int_0^{+\infty}\ (\text{e})^{(-\text{nn}\pi\text{x})}(\text{x})^{(\frac{1-s}{2}-1)}\ \text{dx}$

$= \text{undefined}\ [\infty + \infty + ...] - \sum_{n=1}^{+\infty}\ \int_0^{+\infty} \frac{(\text{e})^{(-\text{nn}\pi\text{x})}}{(-\text{nn}\pi)(-\text{nn}\pi)} (\frac{3}{2})(\frac{1}{2})(-\frac{1}{2})(\text{x})^{(-\frac{3}{2})}\ \text{dx}$

For s = −5

$\sum_{n=1}^{+\infty}\ \int_0^{+\infty}\ (\text{e})^{(-\text{nn}\pi\text{x})}(\text{x})^{(\frac{1-s}{2}-1)}\ \text{dx}$

$= [\frac{2}{(\pi)^3} + \frac{2}{(4\pi)^3} + \cdots]$

For s = $\frac{1}{2}$

$\sum_{n=1}^{+\infty}\ \int_0^{+\infty}\ (\text{e})^{(-\text{nn}\pi\text{x})}(\text{x})^{(\frac{1-s}{2}-1)}\ \text{dx}$

$$= \sum_{n=1}^{+\infty} \lim_{s\to 2}[\frac{(e)^{(-nn\pi x)}}{(-nn\pi)}(x)^{(\frac{1-s}{2}-1)}]_0^{+\infty}$$

$$-\sum_{n=1}^{+\infty} \lim_{s\to 2}\int_0^{+\infty}\frac{(e)^{(-nn\pi x)}}{(-nn\pi)}\left(\frac{1-s}{2}-1\right)(x)^{(\frac{1-s}{2}-2)}\ dx$$

$$= \sum_{n=1}^{+\infty}\ [\lim_{x\to +\infty}[\frac{(e)^{(-nn\pi x)}}{(-nn\pi)}(x)^{\left(-\frac{3}{4}\right)}] - \lim_{x\to 0}[\frac{(e)^{(-nn\pi x)}}{(-nn\pi)}(x)^{\left(-\frac{3}{4}\right)}]]$$

$$-\sum_{n=1}^{+\infty}\ \int_0^{+\infty}\frac{(e)^{(-nn\pi x)}}{(-nn\pi)}(-\frac{3}{4})(x)^{(-\frac{7}{4})}\ dx$$

$$= \sum_{n=1}^{+\infty}\ [\frac{1}{(-nn\pi)(e)^{(+\infty)(nn\pi)}(+\infty)^{\left(\frac{3}{4}\right)}} - \frac{1}{(-nn\pi)(e)^{(0)(nn\pi)}(0)^{\left(\frac{3}{4}\right)}}]$$

$$-\sum_{n=1}^{+\infty}\ \int_0^{+\infty}\frac{(e)^{(-nn\pi x)}}{(-nn\pi)}(-\frac{3}{4})(x)^{(-\frac{7}{4})}\ dx$$

$$= [\frac{1}{(-nn\pi)(e)^{(+\infty)(nn\pi)}(+\infty)^{\left(\frac{3}{4}\right)}} + \frac{1}{(-nn\pi)(e)^{(+\infty)(nn\pi)}(+\infty)^{\left(\frac{3}{4}\right)}} + ..]$$

$$-[\frac{1}{(-nn\pi)(e)^{(0)(nn\pi)}(0)^{\left(\frac{3}{4}\right)}} + \frac{1}{(-nn\pi)(e)^{(0)(nn\pi)}(0)^{\left(\frac{3}{4}\right)}} + ..]$$

$$-\sum_{n=1}^{+\infty}\ \int_0^{+\infty}\frac{(e)^{(-nn\pi x)}}{(-nn\pi)}(-\frac{3}{4})(x)^{(-\frac{7}{4})}\ dx$$

$$= -[\frac{1}{(+\infty)} + \frac{1}{(+\infty)} + ..] + [\frac{1}{0} + \frac{1}{0} + ..]\ , \frac{1}{0} = \infty \text{ in the Riemann sphere}$$

$$-\sum_{n=1}^{+\infty}\ \int_0^{+\infty}\frac{(e)^{(-nn\pi x)}}{(-nn\pi)}(-\frac{3}{4})(x)^{(-\frac{7}{4})}\ dx$$

$$= \text{undefined}[\infty + \infty + \cdots] - \sum_{n=1}^{+\infty}\ \int_0^{+\infty}\frac{(e)^{(-nn\pi x)}}{(-nn\pi)}(-\frac{3}{4})(x)^{(-\frac{7}{4})}\ dx$$

So $\sum_{n=1}^{+\infty}\ \int_0^{+\infty}\ (e)^{(-nn\pi x)}(x)^{(\frac{1-s}{2}-1)}\, d$ for $s = \frac{1}{2}$

$$= \text{undefined}[\infty + \infty + \cdots] - \sum_{n=1}^{+\infty}\ \int_0^{+\infty}\frac{(e)^{(-nn\pi x)}}{(-nn\pi)}(-\frac{3}{4})(x)^{(-\frac{7}{4})}\ dx$$

For s = 2

$$\sum_{n=1}^{+\infty}\ \int_0^{+\infty}\ (e)^{(-nn\pi x)}(x)^{(\frac{1-s}{2}-1)}\,dx$$

$$= \sum_{n=1}^{+\infty}\ \lim_{s\to 2}[\frac{(e)^{(-nn\pi x)}}{(-nn\pi)}(x)^{(\frac{1-s}{2}-1)}]_0^{+\infty}$$

$$-\sum_{n=1}^{+\infty}\ \lim_{s\to 2}\int_0^{+\infty}\frac{(e)^{(-nn\pi x)}}{(-nn\pi)}\left(\frac{1-s}{2}-1\right)(x)^{\left(\frac{1-s}{2}-2\right)}\ dx$$

$$=\sum_{n=1}^{+\infty}\ [\lim_{x\to+\infty}[\frac{(e)^{(-nn\pi x)}}{(-nn\pi)}(x)^{\left(\frac{1-2}{2}-1\right)}]-\lim_{x\to 0}[\frac{(e)^{(-nn\pi x)}}{(-nn\pi)}(x)^{\left(\frac{1-2}{2}-1\right)}]]$$

$$-\sum_{n=1}^{+\infty}\ \int_0^{+\infty}\frac{(e)^{(-nn\pi x)}}{(-nn\pi)}\left(\frac{1-2}{2}-1\right)(x)^{\left(\frac{1-2}{2}-2\right)}\ dx$$

$$=\sum_{n=1}^{+\infty}\ [\lim_{x\to+\infty}[\frac{(e)^{(-nn\pi x)}}{(-nn\pi)}(x)^{\left(-\frac{3}{2}\right)}]-\lim_{x\to 0}[\frac{(e)^{(-nn\pi x)}}{(-nn\pi)}(x)^{\left(-\frac{3}{2}\right)}]]$$

$$-\sum_{n=1}^{+\infty}\ \int_0^{+\infty}\frac{(e)^{(-nn\pi x)}}{(-nn\pi)}(-\frac{3}{2})(x)^{(-\frac{5}{2})}\ dx$$

$$=\sum_{n=1}^{+\infty}\ [\frac{1}{(-nn\pi)(e)^{(+\infty)(nn\pi)}(+\infty)^{\left(\frac{3}{2}\right)}}-\frac{1}{(-nn\pi)(e)^{(0)(nn\pi)}(0)^{\left(\frac{3}{2}\right)}}]$$

$$-\sum_{n=1}^{+\infty}\ \int_0^{+\infty}\frac{(e)^{(-nn\pi x)}}{(-nn\pi)}(-\frac{3}{2})(x)^{(-\frac{5}{2})}\ dx$$

$$=[\frac{1}{(-nn\pi)(e)^{(+\infty)(nn\pi)}(+\infty)^{\left(\frac{3}{2}\right)}}+\frac{1}{(-nn\pi)(e)^{(+\infty)(nn\pi)}(+\infty)^{\left(\frac{3}{2}\right)}}+..]$$

$$-[\frac{1}{(-nn\pi)(e)^{(0)(nn\pi)}(0)^{\left(\frac{3}{2}\right)}}+\frac{1}{(-nn\pi)(e)^{(0)(nn\pi)}(0)^{\left(\frac{3}{2}\right)}}+..]$$

$$-\sum_{n=1}^{+\infty}\ \int_0^{+\infty}\frac{(e)^{(-nn\pi x)}}{(-nn\pi)}(-\frac{3}{2})(x)^{(-\frac{5}{2})}\ dx$$

$$=-[\frac{1}{(+\infty)}+\frac{1}{(+\infty)}+..]+[\frac{1}{0}+\frac{1}{0}+..]\ ,\ \frac{1}{0}=\infty \text{ in the Riemann sphere}$$

$$-\sum_{n=1}^{+\infty}\ \int_0^{+\infty}\frac{(e)^{(-nn\pi x)}}{(-nn\pi)}(-\frac{3}{2})(x)^{(-\frac{5}{2})}\ dx$$

$$=\text{undefined}[\infty+\infty+\cdots]-\sum_{n=1}^{+\infty}\ \int_0^{+\infty}\frac{(e)^{(-nn\pi x)}}{(-nn\pi)}(-\frac{3}{2})(x)^{(-\frac{5}{2})}\ dx$$

So $\sum_{n=1}^{+\infty}\ \int_0^{+\infty}\ (e)^{(-nn\pi x)}(x)^{\left(\frac{1-s}{2}-1\right)}\,d$ for $s=2$

$$=\text{undefined}[\infty+\infty+\cdots]-\sum_{n=1}^{+\infty}\ \int_0^{+\infty}\frac{(e)^{(-nn\pi x)}}{(-nn\pi)}(-\frac{3}{2})(x)^{(-\frac{5}{2})}\ dx$$

For $s = 3$

$$\sum_{n=1}^{+\infty}\ \int_0^{+\infty}\ (e)^{(-nn\pi x)}(x)^{\left(\frac{1-s}{2}-1\right)}\,dx$$

$$= \text{undefined}[\infty + \infty + \cdots] - \sum_{n=1}^{+\infty} \int_0^{+\infty} \frac{(e)^{(-nn\pi x)}}{(-nn\pi)} (-2)(x)^{(-3)} \, dx$$

For s = 4

$$\sum_{n=1}^{+\infty} \int_0^{+\infty} (e)^{(-nn\pi x)} (x)^{(\frac{1-s}{2}-1)} \, dx$$

$$= \text{undefined}[\infty + \infty + \cdots] - \sum_{n=1}^{+\infty} \int_0^{+\infty} \frac{(e)^{(-nn\pi x)}}{(-nn\pi)(-nn\pi)} (-\frac{5}{2})(x)^{(-\frac{7}{2})} \, dx$$

For s = 5

$$\sum_{n=1}^{+\infty} \int_0^{+\infty} (e)^{(-nn\pi x)} (x)^{(\frac{1-s}{2}-1)} \, dx$$

$$= \text{undefined}[\infty + \infty + \cdots] - \sum_{n=1}^{+\infty} \int_0^{+\infty} \frac{(e)^{(-nn\pi x)}}{(-nn\pi)} (-3)(x)^{(-4)} \, dx$$

Appendix C

Another method to find the value of $\sum_{n=1}^{+\infty} \int_0^{+\infty} (e)^{(-n\pi x)} (x)^{(s-1)} \, dx$ using integration by part

For s = −2

$$\sum_{n=1}^{+\infty} \int_0^{+\infty} (e)^{(-n\pi x)} (x)^{(1-S-1)} \, dx$$

$$= \sum_{n=1}^{+\infty} \lim_{s \to -2} [\frac{(e)^{(-n\pi x)}}{(-n\pi)} (x)^{(1-S-1)}]_{0^+}^{+\infty}$$

$$- \sum_{n=1}^{+\infty} \lim_{s \to -2} \int_0^{+\infty} \frac{(e)^{(-n\pi x)} (1-S-1)}{(-n\pi)} (x)^{(1-S-2)} \, dx$$

$$= \sum_{n=1}^{+\infty} [\lim_{x \to +\infty} [\frac{(e)^{(-n\pi x)}}{(-n\pi)} (x)^{(3-1)}] - \lim_{x \to 0} [\frac{(e)^{(-n\pi x)}}{(-n\pi)} (x)^{(3-1)}]]$$

$$- \sum_{n=1}^{+\infty} \int_0^{+\infty} \frac{(e)^{(-n\pi x)}}{(-n\pi)} (3-1)(x)^{(3-2)} \, dx$$

$$= \sum_{n=1}^{+\infty} [\lim_{x \to +\infty} [\frac{(e)^{(-n\pi x)}}{(-n\pi)} (x)^{(2)}] - \lim_{x \to 0} [\frac{(e)^{(-n\pi x)}}{(-n\pi)} (x)^{(2)}]]$$

$$-\sum_{n=1}^{+\infty}\int_{0}^{+\infty}\frac{(\mathrm{e})^{(-\mathrm{n}\pi\mathrm{x})}}{(-\mathrm{n}\pi)}(2)(\mathrm{x})^{(1)}\ \mathrm{dx}$$

$$=\sum_{n=1}^{+\infty}\left[\frac{(+\infty)^{(2)}}{(-\mathrm{n}\pi)(\mathrm{e})^{(+\infty)(n\pi)}}-\frac{(0)^{(2)}}{(-\mathrm{n}\pi)(\mathrm{e})^{(0)(n\pi)}}\right]$$

$$-\sum_{n=1}^{+\infty}\int_{0}^{+\infty}\frac{(\mathrm{e})^{(-\mathrm{n}\pi\mathrm{x})}}{(-\mathrm{n}\pi)}(2)(\mathrm{x})^{(1)}\ \mathrm{dx}$$

$$=\left[\frac{+\infty}{(-\pi)(\mathrm{e})^{(+\infty)(\pi)}}+\frac{+\infty}{(-2\pi)(\mathrm{e})^{(+\infty)(2\pi)}}+\cdots\right]-\left[\frac{(0)}{(-\pi)(\mathrm{e})^{(0)(\pi)}}+\frac{(0)}{(-2\pi)(\mathrm{e})^{(0)(2\pi)}}+\cdots\right]$$

$$=-\left[\frac{+\infty}{+\infty}+\frac{+\infty}{+\infty}+\cdots\right]+[0+0+\cdots]$$

$$-\sum_{n=1}^{+\infty}\int_{0}^{+\infty}\frac{(\mathrm{e})^{(-\mathrm{n}\pi\mathrm{x})}}{(-\mathrm{n}\pi)}(2)(\mathrm{x})^{(1)}\ \mathrm{dx}$$

Find the values of $\frac{+\infty}{+\infty}$ which is indeterminate forms using L'Hospital's Rule

From $\sum_{n=1}^{+\infty}\lim_{x\to+\infty}\left[\frac{(\mathrm{x})^{(2)}}{(-\mathrm{n}\pi)(\mathrm{e})^{(\mathrm{n}\pi\mathrm{x})}}\right]$

Apply L'Hospital's Rule

$$=\sum_{n=1}^{+\infty}\lim_{x\to+\infty}\left[\frac{(2)(\mathrm{x})^{(1)}}{2(-\mathrm{n}\pi)(\mathrm{n}\pi)(\mathrm{e})^{(\mathrm{n}\pi\mathrm{x})}}\right]$$

$$=\sum_{n=1}^{+\infty}\lim_{x\to+\infty}\left[\frac{(1)(2)(\mathrm{x})^{(0)}}{(-\mathrm{n}\pi)(\mathrm{n}\pi)(\mathrm{n}\pi)(\mathrm{e})^{(\mathrm{n}\pi\mathrm{x})}}\right]$$

$$=\left[\frac{1}{(-\pi)(\pi)(\pi)(\mathrm{e})^{(\pi)(+\infty)}}+\frac{1}{(-2\pi)(2\pi)(2\pi)\ (\mathrm{e})^{(2\pi)(+\infty)}}\right.$$

Find the value of $\sum_{n=1}^{+\infty}\int_{0}^{+\infty}\frac{(\mathrm{e})^{(-n\pi\mathrm{x})}}{(-\mathrm{n}\pi)}(2)(\mathrm{x})^{(1)}\ \mathrm{dx}$

$$=\sum_{n=1}^{+\infty}\left[\frac{(\mathrm{e})^{(-\mathrm{n}\pi\mathrm{x})}}{(-\mathrm{n}\pi)(-\mathrm{n}\pi)}(2)(\mathrm{x})^{(1)}\right]_{0}^{+\infty}$$

$$-\sum_{n=1}^{+\infty}\int_{0}^{+\infty}\frac{(\mathrm{e})^{(-\mathrm{n}\pi\mathrm{x})}}{(-\mathrm{n}\pi)(-\mathrm{n}\pi)}(2)(\mathrm{x})^{(0)}\ \mathrm{dx}$$

$$=\sum_{n=1}^{+\infty}\left[\lim_{x\to+\infty}\left[\frac{(2)(\mathrm{x})}{(\mathrm{e})^{(\mathrm{n}\pi\mathrm{x})}(-\mathrm{n}\pi)(-\mathrm{n}\pi)}\right]-\lim_{x\to0}\left[\frac{(2)(\mathrm{x})}{(\mathrm{e})^{(\mathrm{n}\pi\mathrm{x})}(-\mathrm{n}\pi)(-\mathrm{n}\pi)}\right]\right]$$

$$-\sum_{n=1}^{+\infty}\int_{0}^{+\infty}\frac{(\mathrm{e})^{(-\mathrm{n}\pi\mathrm{x})}}{(-\mathrm{n}\pi)(-\mathrm{n}\pi)}(2)\ \mathrm{dx}$$

$$= [\frac{(2)(+\infty)}{(e)^{(\pi)(+\infty)}(-\pi)(-\pi)} + \frac{(2)(+\infty)}{(e)^{(2\pi)(+\infty)}(-2\pi)(-2\pi)} + \cdots]$$

$$-[\frac{(2)(0)}{(e)^{(\pi)(0)}(-\pi)(-\pi)} + \frac{(2)(0)}{(e)^{(2\pi)(0)}(-2\pi)(-2\pi)} + \cdots]$$

$$-\sum_{n=1}^{+\infty} \int_0^{+\infty} \frac{(e)^{(-n\pi x)}}{(-n\pi)(-n\pi)}(2)\ dx$$

$$= [\frac{+\infty}{+\infty} + \frac{+\infty}{+\infty} + \ldots] - [0 + 0 + \cdots]$$

$$-\sum_{n=1}^{+\infty} \int_0^{+\infty} \frac{(e)^{(-n\pi x)}}{(-n\pi)(-n\pi)}(2)\ dx$$

Find the values of $\frac{+\infty}{+\infty}$ which is indeterminate form using L'Hospital's Rule

From $\sum_{n=1}^{+\infty}\ [\lim_{x\to+\infty}[\frac{(2)(x)}{(-n\pi)(-n\pi)(e)^{(n\pi x)}}]$

Apply L'Hospital's Rule

$$= \sum_{n=1}^{+\infty}\ [\lim_{x\to+\infty}[\frac{2}{(-n\pi)(-n\pi)(n\pi)(e)^{(n\pi x)}}]$$

$$= [\frac{2}{(-\pi)(-\pi)(\pi)(e)^{(\pi)(+\infty)}} + \frac{2}{(-2\pi)(-2\pi)(2\pi)\ (e)^{(2\pi)(+\infty)}} + \ldots]$$

$$= [0+0+\ldots]$$

Find the value of $\sum_{n=1}^{+\infty} \int_0^{+\infty} \frac{(e)^{(-n\pi x)}(2)}{(-n\pi)(-n\pi)}\ dx$

$$= \sum_{n=1}^{+\infty}\ [\frac{(e)^{(-n\pi x)}(2)}{(-n\pi)^3}]_0^{+\infty}$$

$$= \sum_{n=1}^{+\infty}\ [\lim_{x\to+\infty}[\frac{(e)^{(-n\pi x)}(2)}{(-n\pi)^3}] - \lim_{x\to 0}[\frac{(e)^{(-n\pi x)}(2)}{(-n\pi)^3}]]$$

$$= [\frac{(e)^{(-\pi)(+\infty)}(2)}{(-\pi)^3} + \frac{(e)^{(-2\pi)(+\infty)}(2)}{(-2\pi)^3} + \cdots] - [\frac{(e)^{(-\pi)(0)}(2)}{(-\pi)^3} + \frac{(e)^{(-2\pi)(0)}(2)}{(-2\pi)^3} + \cdots]$$

$$= [\frac{2}{+\infty} + \frac{2}{+\infty} + \ldots] - [\frac{2}{(-\pi)^3} + \frac{2}{(-2\pi)^3} + \cdots]$$

$$= [\frac{2}{(\pi)^3} + \frac{2}{(2\pi)^3} + \ldots]$$

So $\sum_{n=1}^{+\infty} \int_0^{+\infty} (e)^{(-n\pi x)}(x)^{(1-s-1)}\, dx$ for s =−2

$= [\frac{2}{(\pi)^3}+\frac{2}{(2\pi)^3}+...]$

For s = −3

$\sum_{n=1}^{+\infty} \int_0^{+\infty} (e)^{(-n\pi x)}(x)^{(1-s-1)}\,dx$

$= [\frac{6}{(\pi)^3}+\frac{6}{(2\pi)^{(3)}}+...]$

For s = −4

$\sum_{n=1}^{+\infty} \int_0^{+\infty} (e)^{(-n\pi x)}(x)^{(1-s-1)}\,dx$

$= +[\frac{24}{(\pi)^{(5)}}+\frac{24}{(2\pi)^{(5)}}+\cdots]$

For s = −5

$\sum_{n=1}^{+\infty} \int_0^{+\infty} (e)^{(-n\pi x)}(x)^{(1-s-1)}\,dx$

$= [\frac{120}{(\pi)^{(6)}}+\frac{120}{(2\pi)^{(6)}}+\cdots]$

For s = $\frac{1}{2}$

$\sum_{n=1}^{+\infty} \int_0^{+\infty} (e)^{(-n\pi x)}(x)^{(1-s-1)}\,dx$

$= \sum_{n=1}^{+\infty} \lim_{s\to 2}[\frac{(e)^{(-n\pi x)}}{(-n\pi)}(x)^{(1-s-1)}]_0^{+\infty}$

$-\sum_{n=1}^{+\infty} \lim_{s\to 2}\int_0^{+\infty}\frac{(e)^{(-n\pi x)}}{(-n\pi)}(1-s-1)(x)^{(1-s-2)}\,dx$

$= \sum_{n=1}^{+\infty} [\lim_{x\to+\infty}[\frac{(e)^{(-n\pi x)}}{(-n\pi)}(x)^{(-\frac{1}{2})}] - \lim_{x\to 0}[\frac{(e)^{(-n\pi x)}}{(-n\pi)}(x)^{(-\frac{1}{2})}]]$

$-\sum_{n=1}^{+\infty} \int_0^{+\infty}\frac{(e)^{(-n\pi x)}}{(-n\pi)}(-\frac{1}{2})(x)^{(-\frac{3}{2})}\,dx$

$= \sum_{n=1}^{+\infty} [\frac{1}{(-n\pi)(e)^{(+\infty)(n\pi)}(+\infty)^{(\frac{1}{2})}} - \frac{1}{(-n\pi)(e)^{(0)(n\pi)}(0)^{(\frac{1}{2})}}]$

$-\sum_{n=1}^{+\infty} \int_0^{+\infty}\frac{(e)^{(-n\pi x)}}{(-n\pi)}(-\frac{1}{2})(x)^{(-\frac{3}{2})}\,dx$

$= [\frac{1}{(-\pi)(e)^{(+\infty)(\pi)}(+\infty)^{(\frac{1}{2})}} + \frac{1}{(-2\pi)(e)^{(+\infty)(2\pi)}(+\infty)^{(\frac{1}{2})}} +..]$

$$-\left[\frac{1}{(-\pi)(e)^{(0)(\pi)}(0)^{\left(\frac{1}{2}\right)}}+\frac{1}{(-2\pi)(e)^{(0)(2\pi)}(0)^{\left(\frac{1}{2}\right)}}+..\right]$$

$$-\sum_{n=1}^{+\infty}\int_0^{+\infty}\frac{(e)^{(-n\pi x)}}{(-n\pi)}\left(-\frac{1}{2}\right)(x)^{\left(-\frac{3}{2}\right)}\ dx$$

$$=-\left[\frac{1}{(+\infty)}+\frac{1}{(+\infty)}+..\right]+\left[\frac{1}{0}+\frac{1}{0}+..\right],\ \frac{1}{0}=\infty \text{ in the Riemann sphere}$$

$$-\sum_{n=1}^{+\infty}\int_0^{+\infty}\frac{(e)^{(-n\pi x)}}{(-n\pi)}\left(-\frac{1}{2}\right)(x)^{\left(-\frac{3}{2}\right)}\ dx$$

$$=\text{undefined}\ [\infty+\infty+\cdots]-\sum_{n=1}^{+\infty}\int_0^{+\infty}\frac{(e)^{(-n\pi x)}}{(-n\pi)}\left(-\frac{1}{2}\right)(x)^{\left(-\frac{3}{2}\right)}\ dx$$

So $\sum_{n=1}^{+\infty}\int_0^{+\infty}(e)^{(-n\pi x)}(x)^{(1-s-1)}\,d$ for $s=\frac{1}{2}$

$$=\text{undefined}\ [\infty+\infty+\cdots]-\sum_{n=1}^{+\infty}\int_0^{+\infty}\frac{(e)^{(-n\pi x)}}{(-n\pi)}\left(-\frac{1}{2}\right)(x)^{\left(-\frac{3}{2}\right)}\ dx$$

For s = 2

$$\sum_{n=1}^{+\infty}\int_0^{+\infty}(e)^{(-n\pi x)}(x)^{(1-s-1)}\,dx$$

$$=\sum_{n=1}^{+\infty}\lim_{s\to 2}\left[\frac{(e)^{(-n\pi x)}}{(-n\pi)}(x)^{(1-s-1)}\right]_0^{+\infty}$$

$$-\sum_{n=1}^{+\infty}\lim_{s\to 2}\int_0^{+\infty}\frac{(e)^{(-n\pi x)}}{(-n\pi)}(1-s-1)(x)^{(1-s-2)}\ dx$$

$$=\sum_{n=1}^{+\infty}\left[\lim_{x\to+\infty}\left[\frac{(e)^{(-n\pi x)}}{(-n\pi)}(x)^{(-2)}\right]-\lim_{x\to 0}\left[\frac{(e)^{(-n\pi x)}}{(-n\pi)}(x)^{(-2)}\right]\right]$$

$$-\sum_{n=1}^{+\infty}\int_0^{+\infty}\frac{(e)^{(-n\pi x)}}{(-n\pi)}(-2)(x)^{(-3)}\ dx$$

$$=\sum_{n=1}^{+\infty}\left[\frac{1}{(-n\pi)(e)^{(+\infty)(n\pi)}(+\infty)^{(2)}}-\frac{1}{(-n\pi)(e)^{(0)(n\pi)}(0)^{(2)}}\right]$$

$$-\sum_{n=1}^{+\infty}\int_0^{+\infty}\frac{(e)^{(-n\pi x)}}{(-n\pi)}(-2)(x)^{(-3)}\ dx$$

$$=\left[\frac{1}{(-\pi)(e)^{(+\infty)(\pi)}(+\infty)^{(2)}}+\frac{1}{(-2\pi)(e)^{(+\infty)(2\pi)}(+\infty)^{(2)}}+..\right]$$

$$-\left[\frac{1}{(-\pi)(e)^{(0)(\pi)}(0)^{(2)}}+\frac{1}{(-2\pi)(e)^{(0)(2\pi)}(0)^{(2)}}+..\right]$$

$$-\sum_{n=1}^{+\infty} \int_0^{+\infty} \frac{(e)^{(-n\pi x)}}{(-n\pi)}(-2)(x)^{(-3)}\ dx$$

$$= -\left[\frac{1}{(+\infty)}+\frac{1}{(+\infty)}+..\right]+\left[\frac{1}{0}+\frac{1}{0}+..\right], \frac{1}{0} = \infty \text{ in the Riemann sphere}$$

$$-\sum_{n=1}^{+\infty} \int_0^{+\infty} \frac{(e)^{(-n\pi x)}}{(-n\pi)}(-2)(x)^{(-3)}\ dx$$

$$= \text{undefined } [\infty + \infty + \cdots] - \sum_{n=1}^{+\infty} \int_0^{+\infty} \frac{(e)^{(-n\pi x)}}{(-n\pi)}(-2)(x)^{(-3)}\ dx$$

So $\sum_{n=1}^{+\infty} \int_0^{+\infty} (e)^{(-n\pi x)}(x)^{(1-s-1)}\ d$ for s = 2

$$= \text{undefined } [\infty + \infty + \cdots] - \sum_{n=1}^{+\infty} \int_0^{+\infty} \frac{(e)^{(-n\pi x)}}{(-n\pi)}(-2)(x)^{(-3)}\ dx$$

For s = 3

$$\sum_{n=1}^{+\infty} \int_0^{+\infty} (e)^{(-n\pi x)}(x)^{(1-s-1)}\ dx$$

$$= \text{undefined } [\infty + \infty + \cdots] - \sum_{n=1}^{+\infty} \int_0^{+\infty} \frac{(e)^{(-n\pi x)}}{(-n\pi)}(-3)(x)^{(-4)}\ dx$$

For s = 4

$$\sum_{n=1}^{+\infty} \int_0^{+\infty} (e)^{(-n\pi x)}(x)^{(1-s-1)}\ dx$$

$$= \text{undefined } [\infty + \infty + \cdots] - \sum_{n=1}^{+\infty} \int_0^{+\infty} \frac{(e)^{(-n\pi x)}}{(-n\pi)(-n\pi)}(-4)(x)^{(-5)}\ dx$$

For s = 5

$$\sum_{n=1}^{+\infty} \int_0^{+\infty} (e)^{(-n\pi x)}(x)^{(1-s-1)}\ dx$$

$$= \text{undefined } [\infty + \infty + \cdots] - \sum_{n=1}^{+\infty} \int_0^{+\infty} \frac{(e)^{(-n\pi x)}}{(-n\pi)}(-5)(x)^{(-6)}\ dx$$

Appendix D

Another method to find the value of $\sum_{n=1}^{+\infty} \int_0^{+\infty} (e)^{(-n\pi x)}(x)^{(s-1)}\ dx$ using integration by part

For s = −2

$\sum_{n=1}^{+\infty} \int_0^{+\infty} (\mathrm{e})^{(-\mathrm{n}\pi\mathrm{x})} (\mathrm{x})^{(s-1)}\, \mathrm{dx}$

$= \sum_{n=1}^{+\infty} \lim_{s\to-2} [(\mathrm{e})^{(-\mathrm{n}\pi\mathrm{x})} \frac{(\mathrm{x})^{(s)}}{(s)}]_0^{+\infty}$

$- \sum_{n=1}^{+\infty} \lim_{s\to-2} \int_0^{+\infty} (-\mathrm{n}\pi)(\mathrm{e})^{(-\mathrm{n}\pi\mathrm{x})} \frac{(\mathrm{x})^{(s)}}{(s)}\, \mathrm{dx}$

$= \sum_{n=1}^{+\infty} [\lim_{x\to+\infty}[(\mathrm{e})^{(-\mathrm{n}\pi\mathrm{x})} \frac{(\mathrm{x})^{(-2)}}{(-2)}] - \lim_{x\to0}[(\mathrm{e})^{(-\mathrm{n}\pi\mathrm{x})} \frac{(\mathrm{x})^{(-2)}}{(-2)}]]$

$- \sum_{n=1}^{+\infty} \int_0^{+\infty} (-\mathrm{n}\pi)(\mathrm{e})^{(-\mathrm{n}\pi\mathrm{x})} \frac{(\mathrm{x})^{(-2)}}{(-2)} \mathrm{dx}$

$= \sum_{n=1}^{+\infty} [[\frac{(+\infty)^{(-2)}}{(\mathrm{e})^{(+\infty)(n\pi)}(-2)}] - [\frac{(0)^{(-2)}}{(\mathrm{e})^{(0)(n\pi)}(-2)}]]$

$- \sum_{n=1}^{+\infty} \int_0^{+\infty} (-\mathrm{n}\pi)(\mathrm{e})^{(-\mathrm{n}\pi\mathrm{x})} \frac{(\mathrm{x})^{(-2)}}{(-2)}\, \mathrm{dx}$

$= \sum_{n=1}^{+\infty} [\frac{(1)}{(\mathrm{e})^{(+\infty)(n\pi)}(+\infty)(-2)} - \frac{(1)}{(\mathrm{e})^{(0)(n\pi)}(0)(-2)}]$

$- \sum_{n=1}^{+\infty} \int_0^{+\infty} (-\mathrm{n}\pi)(\mathrm{e})^{(-\mathrm{n}\pi\mathrm{x})} \frac{(\mathrm{x})^{(-2)}}{(-2)}\, \mathrm{dx}$

$= [\frac{1}{(\mathrm{e})^{(+\infty)(\pi)}(+\infty)(-2)} + \frac{1}{(\mathrm{e})^{(+\infty)(2\pi)}(+\infty)(-2)} + \cdots]$

$-[\frac{1}{(\mathrm{e})^{(0)(\pi)}(0)(-2)} + \frac{1}{(\mathrm{e})^{(0)(2\pi)}(0)(-2)} + \cdots]$

$- \sum_{n=1}^{+\infty} \int_0^{+\infty} (-\mathrm{n}\pi)(\mathrm{e})^{(-\mathrm{n}\pi\mathrm{x})} \frac{(\mathrm{x})^{(-2)}}{(-2)}\, \mathrm{dx}$

$= [(-0) + (-0) + \cdots] + [\frac{1}{(0)} + \frac{1}{(0)} + \ldots]$, $\frac{1}{0} = \infty$ in the Riemann sphere

$- \sum_{n=1}^{+\infty} \int_0^{+\infty} (-\mathrm{n}\pi)(\mathrm{e})^{(-\mathrm{n}\pi\mathrm{x})} \frac{(\mathrm{x})^{(-2)}}{(-2)}\, \mathrm{dx}$

$=$ undefined $[\infty+\infty+\ldots] - \sum_{n=1}^{+\infty} \int_0^{+\infty} (-\mathrm{n}\pi)(\mathrm{e})^{(-\mathrm{n}\pi\mathrm{x})} \frac{(\mathrm{x})^{(-2)}}{(-2)}\, \mathrm{dx}$

So $\sum_{n=1}^{+\infty} \int_0^{+\infty} (\mathrm{e})^{(-\mathrm{n}\pi\mathrm{x})} (\mathrm{x})^{(s-1)}\, \mathrm{dx}$ for $s = -2$

$=$ undefined $[\infty+\infty+\ldots] - \sum_{n=1}^{+\infty} \int_0^{+\infty} (-\mathrm{n}\pi)(\mathrm{e})^{(-\mathrm{n}\pi\mathrm{x})} \frac{(\mathrm{x})^{(-2)}}{(-2)}\, \mathrm{dx}$

For $s = -3$

$$\sum_{n=1}^{+\infty}\ \int_0^{+\infty}\ (e)^{(-n\pi x)}(x)^{(s-1)}\,dx$$

$$= \text{undefined}\ [\infty+\infty+\ldots] - \sum_{n=1}^{+\infty}\ \int_0^{+\infty}(-n\pi)(e)^{(-n\pi x)}\,\frac{(x)^{(-3)}}{(-3)}\ dx$$

For s = −4

$$\sum_{n=1}^{+\infty}\ \int_0^{+\infty}\ (e)^{(-n\pi x)}(x)^{(s-1)}\,dx$$

$$= \text{undefined}\ [\infty+\infty+\ldots] - \sum_{n=1}^{+\infty}\ \int_0^{+\infty}(-n\pi)(e)^{(-n\pi x)}\,\frac{(x)^{(-4)}}{(-4)}\ dx$$

For s = −5

$$\sum_{n=1}^{+\infty}\ \int_0^{+\infty}\ (e)^{(-n\pi x)}(x)^{(s-1)}\,dx$$

$$= \text{undefined}\ [\infty+\infty+\ldots] - \sum_{n=1}^{+\infty}\ \int_0^{+\infty}(-n\pi)(e)^{(-n\pi x)}\,\frac{(x)^{(-5)}}{(-5)}\ dx$$

For s = $\frac{1}{2}$

$$\sum_{n=1}^{+\infty}\ \int_0^{+\infty}\ (e)^{(-n\pi x)}(x)^{(s-1)}\,dx$$

$$= \sum_{n=1}^{+\infty}\ \lim_{s\to 2}[\frac{(e)^{(-n\pi x)}}{(-n\pi)}\,(x)^{(s-1)}]_0^{+\infty}$$

$$- \sum_{n=1}^{+\infty}\ \lim_{s\to 2}\int_0^{+\infty}\frac{(e)^{(-n\pi x)}}{(-n\pi)}\,(s-1)(x)^{(s-2)}\ dx$$

$$= \sum_{n=1}^{+\infty}\ [\lim_{x\to+\infty}[\frac{(e)^{(-n\pi x)}}{(-n\pi)}\,(x)^{\left(-\frac{1}{2}\right)}] - \lim_{x\to 0}[\frac{(e)^{(-n\pi x)}}{(-n\pi)}\,(x)^{\left(-\frac{1}{2}\right)}]]$$

$$- \sum_{n=1}^{+\infty}\ \int_0^{+\infty}\frac{(e)^{(-n\pi x)}}{(-n\pi)}\,(-\tfrac{1}{2})(x)^{(-\frac{3}{2})}\ dx$$

$$= \sum_{n=1}^{+\infty}\ [\frac{1}{(-n\pi)(e)^{(+\infty)(n\pi)}(+\infty)^{\left(\frac{1}{2}\right)}} - \frac{1}{(-n\pi)(e)^{(0)(n\pi)}(0)^{\left(\frac{1}{2}\right)}}]$$

$$- \sum_{n=1}^{+\infty}\ \int_0^{+\infty}\frac{(e)^{(-n\pi x)}}{(-n\pi)}\,(-\tfrac{1}{2})(x)^{(-\frac{3}{2})}\ dx$$

$$= [\frac{1}{(-\pi)(e)^{(+\infty)(\pi)}(+\infty)^{\left(\frac{1}{2}\right)}} + \frac{1}{(-n\pi)(e)^{(+\infty)(2\pi)}(+\infty)^{\left(\frac{1}{2}\right)}} +..]$$

$$- [\frac{1}{(-n\pi)(e)^{(0)(\pi)}(0)^{\left(\frac{1}{2}\right)}} + \frac{1}{(-n\pi)(e)^{(0)(2\pi)}(0)^{\left(\frac{1}{2}\right)}} +..]$$

$$-\sum_{n=1}^{+\infty} \int_0^{+\infty} \frac{(e)^{(-n\pi x)}}{(-n\pi)} (-\tfrac{1}{2})(x)^{(-\frac{3}{2})} \, dx$$

$$= -\left[\frac{1}{(+\infty)} + \frac{1}{(+\infty)} + ..\right] + \left[\frac{1}{0} + \frac{1}{0} + ..\right], \frac{1}{0} = \infty \text{ in the Riemann sphere}$$

$$-\sum_{n=1}^{+\infty} \int_0^{+\infty} \frac{(e)^{(-n\pi x)}}{(-n\pi)} (-\tfrac{1}{2})(x)^{(-\frac{3}{2})} \, dx$$

$$= \text{undefined } [\infty + \infty + \cdots] - \sum_{n=1}^{+\infty} \int_0^{+\infty} \frac{(e)^{(-n\pi x)}}{(-n\pi)} (-\tfrac{1}{2})(x)^{(-\frac{3}{2})} \, dx$$

So $\sum_{n=1}^{+\infty} \int_0^{+\infty} (e)^{(-n\pi x)}(x)^{(1-s-1)} \, d$ for $s = \frac{1}{2}$

$$= \text{undefined } [\infty + \infty + \cdots] - \sum_{n=1}^{+\infty} \int_0^{+\infty} \frac{(e)^{(-n\pi x)}}{(-n\pi)} (-\tfrac{1}{2})(x)^{(-\frac{3}{2})} \, dx$$

For s = 2

$$\sum_{n=1}^{+\infty} \int_0^{+\infty} (e)^{(-n\pi x)}(x)^{(s-1)} \, dx$$

$$= \sum_{n=1}^{+\infty} \lim_{s \to 2} \left[\frac{(e)^{(-n\pi x)}}{(-n\pi)} (x)^{(s-1)}\right]_0^{+\infty}$$

$$-\sum_{n=1}^{+\infty} \lim_{s \to 2} \int_0^{+\infty} \frac{(e)^{(-n\pi x)}}{(-n\pi)} (s-1)(x)^{(s-2)} \, dx$$

$$= \sum_{n=1}^{+\infty} \left[\lim_{x \to +\infty} \left[\frac{(e)^{(-n\pi x)}}{(-n\pi)} (x)^{(1)}\right] - \lim_{x \to 0} \left[\frac{(e)^{(-n\pi x)}}{(-n\pi)} (x)^{(1)}\right]\right]$$

$$-\sum_{n=1}^{+\infty} \int_0^{+\infty} \frac{(e)^{(-n\pi x)}}{(-n\pi)} (1)(x)^{(0)} \, dx$$

$$= \sum_{n=1}^{+\infty} \left[\frac{(+\infty)^{(1)}}{(-n\pi)(e)^{(+\infty)(n\pi)}} - \frac{(0)^{(1)}}{(-n\pi)(e)^{(0)(n\pi)}}\right]$$

$$-\sum_{n=1}^{+\infty} \int_0^{+\infty} \frac{(e)^{(-n\pi x)}}{(-n\pi)} (1)(x)^{(0)} \, dx$$

$$= \left[\frac{+\infty}{(-\pi)(+\infty)} + \frac{+\infty}{(-2\pi)(+\infty)} + \cdots\right] - \left[\frac{0}{(-\pi)} + \frac{0}{(-2\pi)} + \cdots\right]$$

$$-\sum_{n=1}^{+\infty} \int_0^{+\infty} \frac{(e)^{(-n\pi x)}}{(-n\pi)} (1)(x)^{(0)} \, dx$$

$$= -\left[\frac{+\infty}{+\infty} + \frac{+\infty}{+\infty} + \cdots\right] + [0 + 0 + \cdots]$$

Find the values of $\frac{+\infty}{+\infty}$ which is indeterminate form using L'Hospital's Rule

From $\sum_{n=1}^{+\infty} \lim_{x\to+\infty}[\frac{(x)^{(1)}}{(-n\pi)(e)^{(n\pi x)}}]$

Apply L'Hospital's Rule

$= \sum_{n=1}^{+\infty} \lim_{x\to+\infty}[\frac{(x)^{(0)}}{(-n\pi)(n\pi)(e)^{(n\pi x)}}]$

$= \sum_{n=1}^{+\infty} \lim_{x\to+\infty}[\frac{1}{(-n\pi)(n\pi)(e)^{(n\pi)(+\infty)}}]$

$= [\frac{1}{(-\pi)(\pi)(e)^{(\pi)(+\infty)}(+\infty)} + \frac{1}{(-2\pi)\,(2\pi)(e)^{(2\pi)(+\infty)}(+\infty)} + \ldots]$

$= [0+0+\ldots]$

Find the value of $\sum_{n=1}^{+\infty} \int_0^{+\infty} \frac{(e)^{(-n\pi x)}}{(-n\pi)}\, dx$

$= \sum_{n=1}^{+\infty} [\frac{(e)^{(-n\pi x)}}{(-n\pi)(-n\pi)}]_0^{+\infty}$

$= \sum_{n=1}^{+\infty} [\lim_{x\to+\infty} [\frac{(e)^{(-n\pi x)}}{(-n\pi)(-n\pi)}] - \lim_{x\to 0}[\frac{(e)^{(-n\pi x)}}{(-n\pi)(-n\pi)}]]$

$= [\frac{1}{(e)^{(\pi)(+\infty)}(-\pi)(-\pi)} + \frac{1}{(e)^{(2\pi)(+\infty)}(-2\pi)(-2\pi)} + \cdots]$

$-[\frac{1}{(e)^{(\pi)(0)}(-\pi)(-\pi)} + \frac{1}{(e)^{(2\pi)(0)}(-2\pi)(-2\pi)} + \cdots]$

$= [0+0+\ldots] - \left[\frac{1}{(-\pi)^{(2)}} + \frac{1}{(-2\pi)^{(2)}} + \cdots\right]$

$= -[\frac{1}{(\pi)^{(2)}} + \frac{1}{(2\pi)^{(2)}} + \ldots]$

So $\sum_{n=1}^{+\infty} \int_0^{+\infty} (e)^{(-n\pi x)}(x)^{(s-1)}\, dx$ for $s = 2$

$= [\frac{1}{(\pi)^{(2)}} + \frac{1}{(2\pi)^{(2)}} + \ldots]$

For s = 3

$\sum_{n=1}^{+\infty} \int_0^{+\infty} (e)^{(-n\pi x)}(x)^{(s-1)}\, dx$

$$= [\frac{2}{(\pi)^{(3)}} + \frac{2}{(2\pi)^{(3)}} + \ldots]$$

For s = 4

$$\sum_{n=1}^{+\infty} \int_0^{+\infty} (e)^{(-n\pi x)} (x)^{(s-1)}\, dx$$

$$= [\frac{(6)}{(\pi)^{(4)}} + \frac{(6)}{(2\pi)^{(4)}} + \cdots]$$

For s = 5

$$\sum_{n=1}^{+\infty} \int_0^{+\infty} (e)^{(-n\pi x)} (x)^{(s-1)}\, dx$$

$$= [\frac{(24)}{(\pi)^{(5)}} + \frac{(24)}{(2\pi)^{(5)}} + \cdots]$$