DISPROOF OF THE RIEMANN ZETA FUNCTION AND RIEMANN HYPOTHESIS (REVISION 3)

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Abstract

Bernhard Riemann has written down a very mysterious work "Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse" since 1859. This paper of Riemann tried to show some functional equations related to prime numbers without proof. Let us investigate those functional equations together about how and where they came from. And at the same time let us find out whether or not the Riemann Zeta Function $\zeta(s) = 2^{s}(\pi)^{(s-1)}\sin(\pi\frac{s}{2})\Gamma(1-s)\zeta(1-s)$ really has zeroes at negative even integers $(-2, -4, -6 \ldots)$, which are called the trivial zeroes, and the nontrivial zeroes of Riemann Zeta Function which are in the critical strip $(0 < \Re(s) < 1)$ all lie on the critical line $(\boldsymbol{\Re(s) = \frac{1}{2}})$ (or the nontrivial zeroes of Riemann Zeta Function are complex numbers of the form $(\frac{1}{2} + \propto i)$). Step by step, you will not believe your eyes to see that Riemann has made such unbelievable mistakes in his work. Finally, you can easily find out that there are no trivial and nontrivial zeroes of Riemann zeta function at all.

1.Introduction

Prime numbers are the most interesting and useful numbers. Many great mathematicians try to work with them in several ways. One of them, Bernhard Riemann, has written down a very famous work "Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse" since 1859 showing a functional equation ζ(s) or Riemann Zeta Function without proof. He believed that with the assistance of his functional equation and all of the methods shown in his paper, the number of prime numbers that are

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smaller than x can be determined.

Someone believes that by using analytic continuation technique, he or she can extend a domain of a powerful analytic function, derived from two or more ordinary expressions or equations, which can help him or her reach the shore he or she tries to. One of them, Riemann, might has thought for about 150 years ago that he could extend the domain of his new analytic function, which was the composition of Riemann Zeta Function and Pi or Gamma function, to the entire complex plane by using this technique. But this technique, just like others, needs to be checked or proved for the essential conditions of the former equations and of the new functional equation itself. Until now usages of Riemann Hypothesis in mathematics and physics are still found more and more, despite the truth that it is just a “hard to solve” problem, not a proven one!

2. $\zeta(s)$, $2\sin \pi s\, \zeta(s)\prod(s-1)$, $\zeta(s) = 2^s\pi^{(s-1)}\sin\frac{\pi s}{2}\Gamma(1-s)\zeta(1-s)$ derivations, and trivial zero solution of $\zeta(s)$

2.1 Let’s start from the great observation **“The Euler Product”**

$$\prod\left(1-\frac{1}{P^s}\right)^{-1} = \sum_{n=1}^{+\infty}\left(\frac{1}{n^s}\right)$$

For p = all prime numbers

n = all whole numbers = 1,2,3,…,∞

Leonard Euler proved this “Euler Product Formula” in 1737.

Let us follow the proof from the series

$$\sum_{n=1}^{+\infty}\left(\frac{1}{n^s}\right) = 1+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\frac{1}{5^s}+\ldots \qquad \ldots(A)$$

Multiply …(A) by $\frac{1}{2^s}$ bothsides

$$\frac{1}{2^s}\sum_{n=1}^{+\infty}\left(\frac{1}{n^s}\right) = \frac{1}{2^s}+\frac{1}{4^s}+\frac{1}{6^s}+\frac{1}{8^s}+\frac{1}{10^s}+\ldots \qquad \ldots(B)$$

Subtract… (A) by …(B) to remove all elements that have factors of 2

$$\left(1-\frac{1}{2^s}\right)\sum_{n=1}^{+\infty}\left(\frac{1}{n^s}\right) = 1+\frac{1}{3^s}+\frac{1}{5^s}+\frac{1}{7^s}+\frac{1}{9^s}+\frac{1}{11^s}+\ldots \qquad \ldots(C)$$

Multiply…(C) by $\frac{1}{3^s}$ bothsides

$$\frac{1}{3^s}(1-\frac{1}{2^s})\sum_{n=1}^{+\infty}\ (\frac{1}{n^s}) \ = \ \frac{1}{3^s}+\frac{1}{9^s}+\ \frac{1}{15^s}+\frac{1}{21^s}+\frac{1}{27^s}+\ldots \qquad \ldots(D)$$

Subtract …(C) by… (D) to remove all elements that have factors of 3 or 2 or both

$$(1-\frac{1}{3^s})(1-\frac{1}{2^s})\sum_{n=1}^{+\infty}\ (\frac{1}{n^s}) \ = \ 1+\frac{1}{5^s}+\frac{1}{7^s}+\ \frac{1}{11^s}+\frac{1}{13^s}+\frac{1}{17^s}+\ldots \qquad \ldots(E)$$

Repeat the process infinitely yields

$$\ldots\ (1-\frac{1}{11^s})(1-\frac{1}{7^s})(1-\frac{1}{5^s})\ (1-\frac{1}{3^s})\ (1-\frac{1}{2^s})\sum_{n=1}^{+\infty}\ (\frac{1}{n^s}) \ = \ 1$$

$$\text{Or } \sum_{n=1}^{+\infty}\ (\frac{1}{n^s}) \ = \ \frac{1}{\left(1-\frac{1}{2^s}\right)\left(1-\frac{1}{3^s}\right)\left(1-\frac{1}{5^s}\right)\left(1-\frac{1}{7^s}\right)\left(1-\frac{1}{11^s}\right)\ldots}$$

$$\sum_{n=1}^{+\infty}\ (\frac{1}{n^s}) \ = \ \prod_{p\ prime}\left[\frac{1}{(1-\frac{1}{p^s})}\right]$$

$$\sum_{n=1}^{+\infty}\ (\frac{1}{n^s}) \ = \ \prod_{p\ prime}(1-\frac{1}{p^s})^{-1}$$

Riemann denoted this relation $\zeta(s) = \sum_{n=1}^{+\infty}\ (\frac{1}{n^s}) = \prod_{p\ prime}(1-\frac{1}{p^s})^{-1}$ **would converge only when real part of s was greater than 1($\Re(s)>1$)** in his paper **"**Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse**"** since 1859.

Riemann Zeta Function ζ(s) will diverge for all s ≤ 1, for example

$$\text{If } \Re(s) = 1,\ \zeta(s) \ = \ \sum_{n=1}^{+\infty}\ (\frac{1}{n^s}) \ = \ \frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\ldots$$

$$\zeta(1) \ = \ \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$$

$$\text{By comparison test } \frac{1}{1}\ +\frac{1}{2}\ \ +\frac{1}{3}+\frac{1}{4}\ \ +\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\ \ +\frac{1}{9}+\cdots$$

$$>\frac{1}{1}\ +\frac{1}{2}\ \ +\frac{1}{4}+\frac{1}{4}\ \ +\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\ \ +\frac{1}{16}+\cdots$$

$$\text{but } \frac{1}{1}\ +\frac{1}{2}\ \ +(\frac{1}{4}+\frac{1}{4})\ \ +(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8})\ \ +(\frac{1}{16}+\cdots)$$

$$= \frac{1}{1} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$$

$$= +\infty$$

so $\zeta(1) = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = +\infty$

finally diverges to ∞

If $\mathfrak{R}(s) = 0$, $\zeta(s) = \sum_{n=1}^{+\infty} (\frac{1}{n^s}) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \ldots$

$$\zeta(0) = \frac{1}{1^0} + \frac{1}{2^0} + \frac{1}{3^0} + \frac{1}{4^0} + \ldots$$

$$= 1 + 1 + 1 + 1 + \ldots$$

finally diverges to ∞

If $\mathfrak{R}(s) = -1$, $\zeta(s) = \sum_{n=1}^{+\infty} (\frac{1}{n^s}) = \frac{1}{1^{-1}} + \frac{1}{2^{-1}} + \frac{1}{3^{-1}} + \frac{1}{4^{-1}} + \ldots$

$$\zeta(-1) = 1 + 2 + 3 + 4 + \ldots$$

finally diverges to ∞

2.2 Next, let us consider $\Gamma(s)$ = Gamma function

2.2.1 $\Gamma(s)$ when $s > 0$

Gamma function was first introduced by Leonhard Euler (1707-1783) in his goal to generalize the factorial to non integer values, and was studied more by Adrien-Marie Legendre (1752-1833)

$$\Gamma(s) = \int_0^\infty (e)^{(-u)} (u)^{(s-1)} du$$

Which will converge if real part of s is greater than 0 ($\mathfrak{R}(s) > 0$)

And can be rewritten as

$$\Gamma(s) = \frac{\Gamma(s+1)}{s}$$

or $\Gamma(s+1) = (s)\Gamma(s)$ converges if $\mathfrak{R}(s) > 0$

Let us prove using integration by parts

$$\Gamma(s+1) = \int_0^\infty (e)^{(-u)} (u)^{(s)} du \qquad \text{for } \mathfrak{R}(s) > 0$$

$$= -(u)^{(s)}(e)^{(-u)}]_0^\infty + \int_0^\infty (e)^{(-u)} (s)(u)^{(s-1)} du$$

$$= [\lim_{u\to\infty} -(u)^{(s)}(e)^{(-u)} - \lim_{u\to 0} -(u)^{(s)}(e)^{(-u)}]$$

$$+ \int_0^\infty (e)^{(-u)} (s)(u)^{(s-1)} du$$

$$= [\frac{-\infty}{\infty} + \frac{0}{1}] + s\int_0^\infty (e)^{(-u)} (u)^{(s-1)} du$$

Use **L'Hospital'Rule** to find $\frac{-\infty}{\infty}$ (indeterminate form)

$$\lim_{u\to\infty} \frac{-(u)^{(s)}}{(e)^{(u)}} = \lim_{u\to\infty} \frac{(-1)(s)(u)^{(s-1)}}{(e)^{(u)}}$$

Repeat differentiation until $(u)^{(s)} \to (u)^{(0)}$

Then $\lim_{u\to\infty} \frac{-(u)^{(s)}}{(e)^{(u)}} = \lim_{u\to\infty} \frac{(-1)(s)(s-1)\ldots(u)^{(0)}}{(e)^{(u)}}$

$$= \frac{(-1)(s)(s-1)\ldots(1)}{(\infty)}$$

$$= 0$$

Thus $\Gamma(s+1) = 0 + s\int_0^\infty (e)^{(-u)} (u)^{(s-1)} du$

So $\Gamma(s+1) = s\,\Gamma(s) \qquad \mathfrak{R}(s) > 0$

Find $\Gamma(\frac{1}{2})$

From $\Gamma(s) = \int_0^\infty (e)^{(-u)} (u)^{(s-1)} du$

$$\Gamma(\tfrac{1}{2}) = \int_0^\infty (e)^{(-u)} (u)^{(\frac{1}{2}-1)} du$$

$$= -(u)^{(-\frac{1}{2})}(e)^{(-u)}]_0^\infty + \int_0^\infty (e)^{(-u)} (-\tfrac{1}{2})(u)^{(-\frac{1}{2}-1)} du$$

$$= [\lim_{u\to\infty} -(u)^{(-\frac{1}{2})}(e)^{(-u)} - \lim_{u\to 0} -(u)^{(-\frac{1}{2})}(e)^{(-u)}]$$

$$+ \int_0^\infty (e)^{(-u)} (-\tfrac{1}{2})(u)^{(-\frac{1}{2}-1)} du$$

$$= [-0+0]+(-\tfrac{1}{2})\int_0^{\infty}(e)^{(-u)}.(u)^{(-\frac{1}{2}-1)}\,du$$

$$= (-\tfrac{1}{2})\Gamma(-\tfrac{1}{2})$$

From **Euler Reflection Formula**

$$\Gamma(s)\,\Gamma(1-s) = \frac{\pi}{\sin\pi s} \quad , \quad 0 < s < 1$$

$$\Gamma(\tfrac{1}{2})\,\Gamma(1-\tfrac{1}{2}) = \frac{\pi}{\sin\frac{\pi}{2}} = \pi$$

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

$$= 1.772$$

Find $\Gamma(1)$

$$\Gamma(s+1) = \int_0^{\infty}(e)^{(-u)}\,(u)^{(s)}\,du$$

$$\Gamma(0+1) = \int_0^{\infty}(e)^{(-u)}\,(u)^{(0)}\,du$$

$$= -(e)^{(-u)}\Big]_0^{\infty}$$

$$= \lim_{u\to\infty} -(e)^{(-u)} - \lim_{u\to 0} -(e)^{(-u)}$$

$$= -0+1$$

$$\Gamma(1) = 1$$

Find $\Gamma(2)$

From $\quad \Gamma(s+1) = s\,\Gamma(s)$

$$\Gamma(1+1) = 1\,\Gamma(1)$$

$$\Gamma(2) = 1$$

And for s = positive integers = 1, 2, 3.....

The relation between gamma function and factorial can be found from

$$\Gamma(s+1) = s\,\Gamma(s) \quad , \quad \mathfrak{R}(s) > 0$$

$$= s(s-1)\Gamma(s-1)$$

$$= s(s-1)(s-2)\ldots(1)\,\Gamma(1)$$

$$= s! \quad \text{for } s = \text{positive integers}$$

2.2.2 $\Gamma(s)$ when $s = 0$

Find $\Gamma(0)$

From **Euler Reflection Formula**

$$\Gamma(s)\,\Gamma(1-s) = \frac{\pi}{\sin\pi s}$$

$$\lim_{s\to 0}\Gamma(s)\,\Gamma(1-s) = \lim_{s\to 0}\frac{\pi}{\sin\pi s}$$

$$\Gamma(0)\,\Gamma(1) = \frac{\pi}{\approx\sin 0} = \infty$$

$$\text{And } \Gamma(1) = 1$$

$$\text{so } \Gamma(0) = \infty$$

2.2.3 $\Gamma(s)$ when $s < 0$

By substitution $\Re(s) < 0$ into equation above yields $\Gamma(s+1)$ which will equal $(s)\Gamma(s)$ for every $\Re(s) < 0$ (negative integers, or negative non integers).

Let us proof using integration by parts

$$\Gamma(s) = \int_0^\infty (u)^{(s-1)}.(e)^{(-u)}\,du \quad , \quad \boldsymbol{\Re(s)} < 0\,, \quad s = -a$$

$$\Gamma(s+1) = \int_0^\infty (u)^{(s)}.(e)^{(-u)}\,du$$

$$= \left[-(u)^{(-a)}(e)^{(-u)}\right]_0^\infty + \int_0^\infty (e)^{(-u)}\,(-a)(u)^{(-a-1)}\,du$$

$$= \left[\lim_{u\to\infty} -(u)^{(-a)}(e)^{(-u)} - \lim_{u\to 0} -(u)^{(-a)}(e)^{(-u)}\right]$$

$$+ \int_0^\infty (e)^{(-u)}\,(-a)(u)^{(-a-1)}\,du$$

$$= [-0+0] + s\int_0^\infty (e)^{(-u)}\,(u)^{(s-1)}\,du$$

$$= (s)\Gamma(s) \quad , \quad \boldsymbol{\Re(s)} < 0$$

Find $\Gamma(-\frac{1}{2})$

From $\quad \Gamma(-\frac{1}{2}+1) = (-\frac{1}{2})\Gamma(-\frac{1}{2})$

$\Gamma(-\frac{1}{2}) = (-2)\,\Gamma(\frac{1}{2})$

$= (-2)\sqrt{\pi}$

$= -3.545$

Find $\Gamma(-1)$

From $\quad \Gamma(0) = \int_0^\infty (\mathrm{u})^{(-1)}(\mathrm{e})^{(-\mathrm{u})}\,\mathrm{du}$

$= -(\mathrm{u})^{(-1)}(\mathrm{e})^{(-\mathrm{u})}]\,_0^\infty + \int_0^\infty (\mathrm{e})^{(-\mathrm{u})}(-1)(\mathrm{u})^{(-2)}\,\mathrm{du}$

$= [\lim_{u\to\infty} -(\mathrm{u})^{(-1)}(\mathrm{e})^{(-\mathrm{u})} - \lim_{u\to 0} -(\mathrm{u})^{(-1)}(\mathrm{e})^{(-\mathrm{u})}]$

$+ \int_0^\infty (\mathrm{e})^{(-\mathrm{u})}(-1)(\mathrm{u})^{(-2)}\,\mathrm{du}$

$= [-0+0] + (-1)\int_0^\infty (\mathrm{e})^{(-\mathrm{u})}(\mathrm{u})^{(-2)}\,\mathrm{du}$

$= (-1)\int_0^\infty (\mathrm{e})^{(-\mathrm{u})}(\mathrm{u})^{(-2)}\,\mathrm{du}$

$\infty = (-1)\Gamma(-1)$

$\Gamma(-1) = -\infty$

Find $\Gamma(-\frac{3}{2})$

From $\quad \Gamma(-\frac{1}{2}) = \int_0^\infty (\mathrm{u})^{\left(-\frac{1}{2}-1\right)}(\mathrm{e})^{(-\mathrm{u})}\,\mathrm{du}$

$= -(\mathrm{u})^{\left(-\frac{1}{2}-1\right)}(\mathrm{e})^{(-\mathrm{u})}]\,_0^\infty + \int_0^\infty (\mathrm{e})^{(-\mathrm{u})}\left(-\frac{1}{2}-1\right)(\mathrm{u})^{(\left(-\frac{1}{2}-1\right)-1)}\,\mathrm{du}$

$= [\lim_{u\to\infty} -(\mathrm{u})^{(-\frac{1}{2}-1)}(\mathrm{e})^{(-\mathrm{u})} - \lim_{u\to 0} -(\mathrm{u})^{(-\frac{1}{2}-1)}(\mathrm{e})^{(-\mathrm{u})}]$

$+ \int_0^\infty (\mathrm{e})^{(-\mathrm{u})}\left(-\frac{1}{2}-1\right)(\mathrm{u})^{(\left(-\frac{1}{2}-1\right)-1)}\,\mathrm{du}$

$$= [-0+0] + \left(-\tfrac{3}{2}\right)\int_0^{\infty} (\mathrm{e})^{(-\mathrm{u})}(\mathrm{u})^{\left(\left(-\frac{3}{2}\right)-1\right)}\,\mathrm{du}$$

$$= \left(-\tfrac{3}{2}\right)\Gamma\left(-\tfrac{3}{2}\right)$$

$$\Gamma\left(-\tfrac{3}{2}\right) = \left(-\tfrac{2}{3}\right)\Gamma(-\tfrac{1}{2})$$

$$= \left(-\tfrac{2}{3}\right)(-2)\sqrt{\pi}$$

$$= 2.363$$

Find $\Gamma(-2)$

From $\Gamma(-1) = \int_0^{\infty}(\mathrm{u})^{(-2)}(\mathrm{e})^{(-\mathrm{u})}\,\mathrm{du}$

$$= -(\mathrm{u})^{(-2)}\,(\mathrm{e})^{(-\mathrm{u})}\Big]_0^{\infty} + \int_0^{\infty}(\mathrm{e})^{(-\mathrm{u})}(-2)(\mathrm{u})^{(-3)}\,\mathrm{du}$$

$$= [\lim_{u\to\infty} -(\mathrm{u})^{(-2)}(\mathrm{e})^{(-\mathrm{u})} - \lim_{u\to 0} -(\mathrm{u})^{(-2)}(\mathrm{e})^{(-\mathrm{u})}]$$

$$+ \int_0^{\infty}(\mathrm{e})^{(-\mathrm{u})}(-2)(\mathrm{u})^{(-3)}\,\mathrm{du}$$

$$= [-0+0] + (-2)\int_0^{\infty}(\mathrm{e})^{(-\mathrm{u})}(\mathrm{u})^{(-3)}\,\mathrm{du}$$

$$-\infty = (-2)\Gamma(-2)$$

$$\Gamma(-2) = \infty$$

Next for s = zero, positive, negative integers or non integers

From $\Gamma(\mathrm{s}) = \int_0^{\infty}(\mathrm{u})^{(\mathrm{s}-1)}(\mathrm{e})^{(-\mathrm{u})}\,\mathrm{du}$

$$\Gamma(1-\mathrm{s}) = \int_0^{\infty}(\mathrm{u})^{((1-s)-1)}(\mathrm{e})^{(-\mathrm{u})}\,\mathrm{du}$$

$$= -(\mathrm{u})^{(-s)}(\mathrm{e})^{(-\mathrm{u})}\Big]_0^{\infty} + \int_0^{\infty}(\mathrm{e})^{(-\mathrm{u})}((1-s)-1)(\mathrm{u})^{((1-\mathrm{s})-2)}\,\mathrm{du}$$

$$= [\lim_{u\to\infty} -(\mathrm{u})^{(-s)}(\mathrm{e})^{(-\mathrm{u})} - \lim_{u\to 0} -(\mathrm{u})^{(-s)}(\mathrm{e})^{(-\mathrm{u})}]$$

$$+ \int_0^{\infty}(\mathrm{e})^{(-\mathrm{u})}((1-s)-1)(\mathrm{u})^{((1-\mathrm{s})-2)}\,\mathrm{du}$$

$$= [-0+0] + ((1-s)-1)\Gamma((1-s)-1)$$

$= [(1-s)-1]\Gamma[(1-s)-1]$

So $\Gamma(1-s) = (-s)\Gamma(-s)$

(s = zero, positive, negative integers or non integers)

And $\Gamma(1+s) = (s)\Gamma(s)$

(s = zero, positive, negative integers or non integers)

2.3 Consider $\prod(s)$ = Pi function

Pi function has been denoted by Carl Friedrich Gauss since 1813

$$\prod(s) = \int_{0^+}^{+\infty} (e)^{(-u)} (u)^{(s)} du$$

The relation between Pi and Gamma functions are

$$\prod(s-1) = \int_{0}^{\infty} (e)^{(-u)} (u)^{(s-1)} du \qquad \dots (1)$$

$$= \Gamma(s)$$

Which will converge if real part of s is greater than 0, ($\Re(s) > 0$)

2.4 How to find the product of $\zeta(s)\prod(s-1)$ and corresponding value of $\Re(s)$

From equation ... (1)

$+\infty \geq u \geq 0$

Let $u = nx$

Then $+\infty \geq x \geq 0$

Multiply equation ... (1) by $\frac{1}{n^s}$ both sides

$$\left(\frac{1}{n^s}\right)\prod(s-1) = \left(\frac{1}{n^s}\right) \int_{0^+}^{+\infty} (e)^{(-u)} (u)^{(s-1)} du$$

$$= \int_{0^+}^{+\infty} (e)^{(-nx)} (nx)^{(s-1)} (n)^{(-s)} dnx$$

$$= \int_{0^+}^{+\infty} (e)^{(-nx)} (nx)^{(s-1)} (n)^{(-s)} ndx$$

$$= \int_{0^+}^{+\infty} (e)^{(-nx)} (nx)^{(s-1)} (n)^{-(s-1)} dx$$

$$= \int_{0^+}^{+\infty} (e)^{(-nx)} (x)^{(s-1)} dx \qquad \text{...(1.1)}$$

To make sure that the result of $(nx)^{(s-1)}$ multiplies by $(n)^{-(s-1)}$ of equation ...(1.1) will exactly be $(x)^{(s-1)}$ without $(n)^{(s-1)}$left, the values of s from $(\frac{1}{n^s})$ of ζ(s) (which $1 < \Re(s) \le +\infty$) , and from $(u)^{(s-1)}$ of $\prod(s-1)$ (which $0 < \Re(s) \le +\infty$) must be the same. So the values of all the real parts of s of the product $(\frac{1}{n^s})\prod(s-1)$ must be those numbers which are larger than 1 or ($1< \Re(s) \le +\infty$).

Then try to make infinite summation of $(\frac{1}{n^s})\prod(s-1)$

for $\boldsymbol{\Re(s)} > 1$

$$\sum_{n=1}^{+\infty} (\frac{1}{n^s})\prod(s-1) = \sum_{n=1}^{+\infty} \int_{0^+}^{+\infty} (e)^{(-nx)} (x)^{(s-1)} dx \qquad \text{... (1.2)}$$

Or
$$\zeta(s)\prod(s-1) = \sum_{n=1}^{+\infty} \int_{0^+}^{+\infty} (e)^{(-nx)} (x)^{(s-1)} dx \qquad \text{... (1.3)}$$

And from
$$(e)^{(-nx)} = (e^{(-x)})^{(n)}$$

$$= (e^{(-x)})^{(n)} [\frac{(e)^{(-x)}}{(e)^{(-x)}}]$$

$$= (e^{(-x)})^{(n-1)}(e)^{(-x)}$$

Then
$$\zeta(s)\prod(s-1) = \sum_{n=1}^{+\infty} \int_{0^+}^{+\infty} (e^{(-x)})^{(n-1)}(e)^{(-x)} (x)^{(s-1)}dx$$

$$= \int_{0^+}^{+\infty} \sum_{n=1}^{+\infty} (e^{(-x)})^{(n-1)}(e)^{(-x)} (x)^{(s-1)}dx \qquad \text{... (1.4)}$$

, for $\boldsymbol{\Re(s)} > 1$

Let
$$\sum_{n=1}^{+\infty} (e^{(-x)})^{(n-1)} = \sum_{n=1}^{+\infty} ar^{(n-1)}$$

From **Geometric Series**

$$\sum_{n=1}^{+\infty} ar^{(n-1)} = \lim_{n\to\infty} Sn$$

$$= \lim_{n\to\infty} \frac{(a-ar^{n})}{(1-r)}$$

$$= \lim_{n\to\infty} [\frac{a}{(1-r)} - \frac{ar^{n}}{(1-r)}] \quad , a = 1 \quad , r = (e)^{(-x)} \ (<1)$$

$$, 0 \leq x \leq +\infty$$

But $\lim_{n\to\infty} \frac{ar^{n}}{(1-r)} = \lim_{n\to\infty} \frac{(e^{(-x)})^{n}}{(1-e^{(-x)})} = 0 \quad , 0 \leq x \leq +\infty$

So $\sum_{n=1}^{+\infty} ar^{(n-1)} = \frac{a}{(1-r)}$, $a = 1$, $r = (e)^{(-x)}$ (<1) ,$0 \leq x \leq +\infty$

And then $\sum_{n=1}^{+\infty} (e^{(-x)})^{(n-1)} = \frac{1}{(1-e^{(-x)})}$

Thus $\zeta(s)\prod(s-1) = \int_{0^{+}}^{+\infty} \frac{(e)^{(-x)}(x)^{(s-1)}}{(1-e^{(-x)})} dx$

$$= \int_{0^{+}}^{+\infty} \frac{(x)^{(s-1)}}{(1-e^{(-x)})(e^{x})} dx$$

or $\zeta(s)\prod(s-1) = \int_{0^{+}}^{+\infty} \frac{(x)^{(s-1)}}{(e^{x}-1)} dx$, $\Re(\mathbf{s}) > 1$ ... (2)

2.5 Riemann's attempt to extend the analytic equation $\zeta(s)\prod(s-1)$ to the negative side of real axis, the formation of the equation

$$2\sin \pi s \ \zeta(s)\prod(s-1) = i \int_{+\infty}^{+\infty} \frac{(-x)^{(s-1)}}{(e^{x}-1)} dx$$

Riemann substituted $(-x)$ into $(x)^{(s-1)}$ of integral ... (2), and took consideration in positive sense around a domain $(+\infty,+\infty)$, then by **Cauchy's theorem** " if two different paths connect the same two points, and a function is holomorphic everywhere "in between" the two paths, then the two path integrals of the function will be the same." And briefly, "the path integral along a Jordan curve of a function, holomorphic in the interior of the curve, is zero."

$$\oint_c f(u)du \ = \ 0$$

if a and b are two points on Jordan curve (simple closed curve)c

then $\oint_c f(u)du = \int_a^b f(u)du + \int_b^a f(u)du$

$= 0$

And let us consider improper integral when b→ +∞ , a = 0^+

Then $\oint_c f(u)du = \lim_{b\to+\infty} \int_{0^+}^{b} f(u)du + \lim_{b\to+\infty} \int_b^{0^+} f(u)du$

Or $\oint_c f(u)du = \int_{0^+}^{\infty} f(u)du + \int_{\infty}^{0^+} f(u)du$

$= 0$

And for $f(u)du = \frac{(-x)^{(s-1)}}{(e^x-1)} dx$

Then $\int_{+\infty}^{+\infty} \frac{(-x)^{(s-1)}}{(e^x-1)} dx = \int_{0^+}^{+\infty} \frac{(-x)^{(s-1)}}{(e^x-1)} dx + \int_{+\infty}^{0^+} \frac{(-x)^{(s-1)}}{(e^x-1)} dx = 0$

or $\int_{+\infty}^{+\infty} \frac{(-x)^{(s-1)}}{(e^x-1)} dx = \int_{0^+}^{+\infty} \frac{(-x)^{(s-1)}}{(e^x-1)} dx - \int_{0^+}^{+\infty} \frac{(-x)^{(s-1)}}{(e^x-1)} dx = 0$

That means the value of the equation $\zeta(s)\prod(s-1) = \int_{0^+}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)} dx$ after extending to $\int_{+\infty}^{+\infty} \frac{(-x)^{(s-1)}}{(e^x-1)} dx$ is always equal to zero independent from the values of s of $\zeta(s)$ or $\prod(s-1)$.

Now, let us go further from the above equation

$$\int_{+\infty}^{+\infty} \frac{(-x)^{(s-1)}}{(e^x-1)} dx = 0$$

$$= \int_{0^+}^{+\infty} \frac{(-1)^{(s-1)}\ (x)^{(s-1)}}{(e^x-1)} dx - \int_{0^+}^{+\infty} \frac{(-1)^{(s-1)}\ (x)^{(s-1)}}{(e^x-1)} dx$$

$$= \frac{(-1)^{(s)}}{(-1)} \int_{0^+}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)} dx - \frac{(-1)^{(s)}}{(-1)} \int_{0^+}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)} dx$$

From **Euler's Formula**

$$(e)^{\pm i\pi} = -1$$

$$(\cos\pi \pm i\sin\pi) = -1$$

$\cos\pi = -1$, $\sin\pi = 0$

Hence $\int_{+\infty}^{+\infty} \frac{(-x)^{(s-1)}}{(e^x-1)} dx = 0$

$$= \frac{(-1)^{(s)}}{(-1)} \int_{0^+}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)} dx - \frac{(-1)^{(s)}}{(-1)} \int_{0^+}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)} dx$$

$$= \frac{(e^{i\pi})^{(s)}}{(-1)} \int_{0^+}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)} dx - \frac{(e^{-i\pi})^{(s)}}{(-1)} \int_{0^+}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)} dx$$

$$= [\frac{(e^{i\pi})^{(s)}}{(-1)} - \frac{(e^{-i\pi})^{(s)}}{(-1)}] \int_{0^+}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)} dx$$

$$= [-(e^{i\pi})^{(s)} + (e^{-i\pi})^{(s)}] \int_{0^+}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)} dx$$

$$= [(e^{-i\pi})^{(s)} - (e^{i\pi})^{(s)}] \int_{0^+}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)} dx \qquad \dots (3)$$

$$= [(\cos\pi s - i\sin\pi s) - (\cos\pi s + i\sin\pi s)] \int_{0^+}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)} dx$$

$$= -2i\sin\pi s \int_{0^+}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)} dx$$

$$= -2i\sin\pi s\ \zeta(s) \prod(s-1) \qquad \dots (4)$$

$$= (0)\ \zeta(s) \prod(s-1)$$

Or $\int_{+\infty}^{+\infty} \frac{(-x)^{(s-1)}}{(e^x-1)} dx = -2i\sin\pi s\ \zeta(s) \prod(s-1)$

$$0 = (0)\ \zeta(s) \prod(s-1)$$

Multiply by i both sides

$$i \int_{+\infty}^{+\infty} \frac{(-x)^{(s-1)}}{(e^x-1)} dx = -2(i)^2 \sin\pi s\ \zeta(s) \prod(s-1)$$

$$= -2(-1)\sin\pi s\ \zeta(s)\prod(s-1)$$

Or $2\sin\pi s\ (\zeta s)\prod(s-1) = i \int_{+\infty}^{+\infty} \frac{(-x)^{(s-1)}}{(e^x-1)} dx = 0 \qquad \dots (5)$

$$(0)(\zeta s)\prod(s-1) = 0$$

That means the value of the equation $2\sin\pi s\ \zeta(s)\prod(s-1) = i \int_{+\infty}^{+\infty} \frac{(-x)^{(s-1)}}{(e^x-1)} dx$ must always equal zero.

Riemann observed the many valued function from above equation

$$(-x)^{(s-1)} = \quad (e)^{(s-1)\mathrm{Log}(-x)}$$

and said that the logarithm of $(-x)$ was determined to be real only when x was negative. Therefore Riemann tried to show that the integral $\int_{+\infty}^{+\infty} \frac{(-x)^{(s-1)}}{(e^x-1)}\,dx$ would be valuable if $x < 0$ in contrary with the domain $(+\infty\,,+\infty)$ of the integral, this looked strange and confused.

Another confusion was that Riemann did not change (x) of the denominator $(e^x - 1)$ of his equation $\int \frac{(-x)^{(s-1)}}{(e^x-1)}\,dx$ to $(-x)$ simultaneously while he changed (x) of the numerator $(x)^{(s-1)}$ to $(-x)$. Actually (x) of both denominator and numerator come from the same function $\prod(s-1)$, so they have to be changed to $(-x)$ at the same time.

I do not really know what was in his mind, but if one looks carefully at the first page of his original paper **"Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse",** you can see the traces of confusion and hesitation which caused him to change the boundary of the integral $\int \frac{(-x)^{(s-1)}}{(e^x-1)}\,dx$ from $(+\infty, +\infty)$ to $(-\infty\,, +\infty)$ and back to $(+\infty\,, +\infty)$ again.

2.5.1 Firstly, he might try to extend the functional equation $\zeta(s)\prod(s-1)$ to the negative values along the x-axis (which means that he was trying to consider the integral on the domain $(-\infty, +\infty)$.

From equation ... (1.4)

$$\zeta(s)\prod(s-1) \;=\; \int_0^{+\infty} \sum_{n=1}^{+\infty} (e^{(-x)})^{(n-1)}(e)^{(-x)}\,(x)^{(s-1)}dx$$

Riemann extended it to negative values along x-axis

$$\zeta(s)\prod(s-1) = \int_0^{+\infty} \sum_{n=1}^{+\infty} (e^{(-x)})^{(n-1)} (e)^{(-x)} (x)^{(s-1)} dx$$

$$+ \int_{-\infty}^{0} \sum_{n=1}^{+\infty} (e^{(-x)})^{(n-1)} (e)^{(-x)} (x)^{(s-1)} dx$$

Consider $\int_{-\infty}^{0} \sum_{n=1}^{+\infty} (e^{(-x)})^{(n-1)} (e)^{(-x)} (x)^{(s-1)} dx$

Let $\sum_{n=1}^{+\infty} (e^{(-x)})^{(n-1)} = \sum_{n=1}^{+\infty} ar^{(n-1)}$

From **Geometric Series**

$$\sum_{n=1}^{+\infty} ar^{(n-1)} = \lim_{n\to\infty} Sn$$

$$= \lim_{n\to\infty} \frac{(a-ar^{n})}{(1-r)}$$

$$= \lim_{n\to\infty} [\frac{a}{(1-r)} - \frac{ar^{n}}{(1-r)}] \quad , a = 1 \ , r = (e)^{(-x)} (>1)$$

$$= \lim_{n\to\infty} [\frac{1}{(1-r)} - \frac{r^{n}}{(1-r)}] \quad , x \le 0 \text{ or } (-x)=\text{positive}$$

$$= \lim_{n\to\infty} [\frac{1-r^{n}}{(1-r)}]$$

Then from **Factorization**, let us consider the numerator $(1 - r^{n})$

$$(a^{n}-b^{n}) = (a-b)(a^{n-1} + a^{n-2}b+a^{n-3}b^{2} + \cdots+ab^{n-2}+b^{n-1})$$

In this case $a = 1 , b = r$

So $(1^{n}-r^{n}) = (1-r)(1^{n-1} + 1^{n-2}r+1^{n-3}r^{2} + \cdots+r^{n-2}+r^{n-1})$

Hence $\lim_{n\to\infty}[\frac{(1-r^{n})}{(1-r)}] = \lim_{n\to\infty}[\frac{(1-r)(1^{n-1}+1^{n-2}r+1^{n-3}r^{2}+\cdots+r^{n-2}+r^{n-1})}{(1-r)}]$

$$= (1+r+r^{2}+...+r^{\infty-2} + r^{\infty-1})$$

$$= \infty$$

So $\sum_{n=1}^{+\infty} ar^{(n-1)} = \sum_{n=1}^{+\infty} (e^{(-x)})^{(n-1)}$

$$= \infty \quad , a = 1 \ , r = (e)^{(-x)} (>1) \ ,-\infty \le x \le 0$$

Thus $\int_{-\infty}^{0} \sum_{n=1}^{+\infty} (e^{(-x)})^{(n-1)} (e)^{(-x)} (x)^{(s-1)} dx$

$$= \int_{-\infty}^{0} (\infty)(e)^{(-x)} (x)^{(s-1)} dx$$

diverges to ∞ for $-\infty \leq x \leq 0$

Then $\zeta(s)\prod(s-1) = \int_{0}^{+\infty} \sum_{n=1}^{+\infty} (e^{(-x)})^{(n-1)} (e)^{(-x)} (x)^{(s-1)} dx$

$$+ \int_{-\infty}^{0} \sum_{n=1}^{+\infty} (e^{(-x)})^{(n-1)} (e)^{(-x)} (x)^{(s-1)} dx$$

diverges to ∞ for $-\infty \leq x \leq 0$

So extending $\zeta(s)\prod(s-1) = \int_{0}^{+\infty} \sum_{n=1}^{+\infty} (e^{(-x)})^{(n-1)} (e)^{(-x)} (x)^{(s-1)} dx$ to

$$\int_{0}^{+\infty} \sum_{n=1}^{+\infty} (e^{(-x)})^{(n-1)} (e)^{(-x)} (x)^{(s-1)} dx + \int_{-\infty}^{0} \sum_{n=1}^{+\infty} (e^{(-x)})^{(n-1)} (e)^{(-x)} (x)^{(s-1)} dx$$

will cause it to diverge to ∞.

2.5.2 Secondly, he might try to take integration along a closed curve C covered the domain $(+\infty , +\infty)$, which by famous **Cauchy's theorem** " if two different paths connect the same two points, and a function is holomorphic everywhere "in between" the two paths, then the two path integrals of the function will be the same." And briefly, "the path integral along a Jordan curve of a function, holomorphic in the interior of the curve, is zero."

$$\oint_{c} f(u)du = 0$$

if a and b are two points on Jordan curve (simple closed curve)c

then $\oint_{c} f(u)du = \int_{a}^{b} f(u)du + \int_{b}^{a} f(u)du$

$$= 0$$

And let us consider improper integral when b$\rightarrow +\infty$, a $= 0^+$

Then $\oint_c f(u)du \quad = \lim_{b\rightarrow+\infty} \int_{0^+}^{b} f(u)du + \lim_{b\rightarrow+\infty} \int_{b}^{0^+} f(u)du$

Or $\oint_c f(u)du \quad = \int_{0^+}^{\infty} f(u)du \; + \; \int_{\infty}^{0^+} f(u)du$

$= \; 0$

And for $f(u)du \quad = \frac{(x)^{(s-1)}}{(e^x-1)}dx$

Then $\int_{+\infty}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)}dx = \int_{0^+}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)}dx \; + \; \int_{+\infty}^{0^+} \frac{(x)^{(s-1)}}{(e^x-1)}dx$

$= \int_{0^+}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)}dx \; - \; \int_{0^+}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)}dx$

$= \; 0$

$= \int_{0^+}^{+\infty} \frac{(1)^{(s)}}{(1)} \frac{(x)^{(s-1)}}{(e^x-1)}dx \; - \; \int_{0^+}^{+\infty} \frac{(1)^{(s)}}{(1)} \frac{(x)^{(s-1)}}{(e^x-1)}dx$

$= (1)^{(s)} \int_{0^+}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)}dx \; - (1)^{(s)} \int_{0^+}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)}dx$

From **Euler's Formula** again

$(e)^{\pm i\pi} \; = \; -1$

$(-e)^{\pm i\pi} \; = \; 1$

$(\cos\pi \; \pm i\sin\pi) \; = \; -1$

$\cos\pi \; = \; -1$, $\sin\pi \; = \; 0$

Hence $\int_{+\infty}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)}dx \; = \; 0$

$= (1)^{(s)} \int_{0^+}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)}dx \; - \; (1)^{(s)} \int_{0^+}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)}dx$

$= \left(-e^{i\pi}\right)^{(s)} \int_{0^+}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)}dx - \left(-e^{-i\pi}\right)^{(s)} \int_{0^+}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)}dx$

$= [\left(-e^{i\pi}\right)^{(s)} - \left(-e^{-i\pi}\right)^{(s)}] \int_{0^+}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)}dx \qquad \ldots(6)$

$$= [(-\cos \pi s - i\sin \pi s) - (-\cos \pi s + i\sin \pi s)] \int_{0^+}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)} dx$$

$$= -2i\sin \pi s \int_{0^+}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)} dx \qquad \ldots (7)$$

$$= (0) \int_{0^+}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)} dx$$

Or $\quad \int_{+\infty}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)} dx = -2i\sin \pi s \int_{0^+}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)} dx$

$$0 = -2i\sin \pi s\, \zeta(s) \prod(s-1)$$

$$0 = (0)\zeta(s) \prod(s-1)$$

Multiply by i both sides

$$i \int_{+\infty}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)} dx = -2(i)^2 \sin \pi s\, \zeta(s) \prod(s-1)$$

$$= 2\sin \pi s\, \zeta(s) \prod(s-1)$$

Or $2\sin \pi s\, \zeta(s) \prod(s-1) = i \int_{+\infty}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)} dx = 0 \qquad \ldots (8)$

$$(0)\, \zeta(s) \prod(s-1) = 0$$

That means the value of the equation $2\sin \pi s\, \zeta(s)\prod(s-1) = i \int_{+\infty}^{+\infty} \frac{(x)^{(s-1)}}{(e^x-1)} dx$ must always equal zero.

Now look at the many valued function again

$$(x)^{(s-1)} = (e)^{(s-1)\mathrm{Log}(x)}$$

The logarithm of x is determined to be real when x is positive number within the domain $(+\infty, +\infty)$.

2.6 Can we really get trivial zeroes $(-2, -4, -6, \ldots)$ from Riemann Zeta Function $\zeta(s) = 2^s(\pi)^{(s-1)}\sin(\pi \frac{s}{2})\Gamma(1-s)\zeta(1-s)$?

To answer this question, we have to study two functional equations and their relationship.

1. $(\pi)^{\left(-\frac{s}{2}\right)}\Gamma(\frac{s}{2})\zeta(\mathrm{s}) = (\pi)^{-\left(\frac{1-s}{2}\right)}\Gamma(\frac{1-s}{2})\zeta(1-\mathrm{s})$

2. $(\pi)^{-(s)}\Gamma(s)\zeta(\mathrm{s}) = (\pi)^{-(1-s)}\Gamma(1-s)\zeta(1-\mathrm{s})$

Firstly, you should pay attention to an interesting fact which is hidden in those equations.

2.6.1 Let us start from changing of $\prod(\mathrm{s}-1)$ of equation ...(1) to $\prod(\frac{s}{2}-1)$.

From $\prod(\mathrm{s}-1) = \int_{0^+}^{+\infty}(\mathrm{e})^{(-\mathrm{u})}(\mathrm{u})^{(\mathrm{s}-1)}\,\mathrm{du}$ ...(1)

Or $\Gamma(\mathrm{s}) = \int_{0^+}^{+\infty}(\mathrm{e})^{(-\mathrm{u})}(\mathrm{u})^{(\mathrm{s}-1)}\,\mathrm{du}$

Which converges when $\Re(\mathbf{s}) > 0$, $+\infty \geq \mathrm{u} \geq 0$

Thus $\prod\left(\frac{s}{2}-1\right) = \int_{0^+}^{+\infty}(\mathrm{e})^{(-\mathrm{u})}(\mathrm{u})^{\left(\frac{s}{2}-1\right)}\,\mathrm{du}$

Multiply by $\left(\frac{1}{n^s}\right)(\pi)^{\left(-\frac{s}{2}\right)}$ both sides and let $\mathrm{u} = \mathrm{nn}\pi\mathrm{x}$ (as Riemann tried to)

$$\left(\frac{1}{n^s}\right)(\pi)^{\left(-\frac{s}{2}\right)}\prod\left(\frac{s}{2}-1\right) = \int_{0^+}^{+\infty}\frac{1}{(\pi)^{\left(\frac{s}{2}\right)}}\left(\frac{1}{n^s}\right)(\mathrm{e})^{(-\mathrm{u})}(\mathrm{u})^{\left(\frac{s}{2}-1\right)}\,\mathrm{du}$$

$$= \int_{0^+}^{+\infty}\frac{1}{(\pi)^{\left(\frac{s}{2}\right)}}\left(\frac{1}{(\mathrm{nn})^{\left(\frac{s}{2}\right)}}\right)(\mathrm{e})^{(-\mathrm{nn}\pi x)}(\mathrm{nn}\pi x)^{\left(\frac{s}{2}-1\right)}\,\mathrm{d}(\mathrm{nn}\pi x)$$

$$= \int_{0^+}^{+\infty}\frac{(\mathrm{nn}\pi x)^{\left(\frac{s}{2}-1\right)}}{(\mathrm{nn}\pi)^{\left(\frac{s}{2}\right)}}(\mathrm{e})^{(-\mathrm{nn}\pi x)}\mathrm{nn}\pi\,\mathrm{d}x$$

$$= \int_{0^+}^{+\infty}\frac{(\mathrm{nn}\pi x)^{\left(\frac{s}{2}-1\right)}}{(\mathrm{nn}\pi)^{\left(\frac{s}{2}-1\right)}}(\mathrm{e})^{(-\mathrm{nn}\pi x)}\,\mathrm{d}x$$

$$= \int_{0^+}^{+\infty}(\mathrm{e})^{(-\mathrm{nn}\pi x)}(\mathrm{x})^{\left(\frac{s}{2}-1\right)}\,\mathrm{d}x$$

Take infinite summation both sides

$$\sum_{n=1}^{+\infty}\left(\frac{1}{n^s}\right)(\pi)^{\left(-\frac{s}{2}\right)}\prod\left(\frac{s}{2}-1\right) = \sum_{n=1}^{+\infty}\int_{0^+}^{+\infty}(\mathrm{e})^{(-\mathrm{nn}\pi x)}(\mathrm{x})^{\left(\frac{s}{2}-1\right)}\,\mathrm{d}x$$

$$= \int_{0^+}^{+\infty}\sum_{n=1}^{+\infty}(\mathrm{e})^{(-\mathrm{nn}\pi x)}(\mathrm{x})^{\left(\frac{s}{2}-1\right)}\,\mathrm{d}x$$

But **Riemann** denoted $\sum_{n=1}^{+\infty} (\mathrm{e})^{(-\mathrm{nn}\pi x)} = \psi(\mathrm{x})$

Then $\sum_{n=1}^{+\infty} (\frac{1}{n^s})(\pi)^{(-\frac{s}{2})} \prod\left(\frac{s}{2}-1\right) = \int_{0^+}^{+\infty} \psi(\mathrm{x})(x)^{(\frac{s}{2}-1)}\,\mathrm{dx}$

Or $(\pi)^{(-\frac{s}{2})}\prod(\frac{s}{2}-1)\zeta(\mathrm{s}) = \int_{0^+}^{+\infty} \psi(\mathrm{x})(x)^{(\frac{s}{2}-1)}\,\mathrm{dx}$ ...(9)

Let's consider the value of ψ(x).

$$\psi(\mathrm{x}) = \sum_{n=1}^{+\infty} (\mathrm{e})^{(-\mathrm{nn}\pi x)} = (\mathrm{e})^{(-\pi x)} + (\mathrm{e})^{(-4\pi x)} + \ldots + (\mathrm{e})^{(-\mathrm{nn}\pi x)}$$

$$= \left(\mathrm{e}^{-\pi x}\right)^{(1)^2} + \left(\mathrm{e}^{-\pi x}\right)^{(2)^2} + \ldots + \left(\mathrm{e}^{-\pi x}\right)^{(n)^2}$$

Let $\mathrm{k} = \sqrt{nn}$

So $\psi(\mathrm{x}) = \sum_{n=1}^{+\infty} (\mathrm{e})^{(-\mathrm{nn}\pi x)}$

$$= \sum_{k=1}^{+\infty} \left(\mathrm{e}^{-\pi x}\right)^{(k)}$$

$$= \left(\mathrm{e}^{-\pi x}\right)^{(1)} + \left(\mathrm{e}^{-\pi x}\right)^{(2)} + \ldots + \left(\mathrm{e}^{-\pi x}\right)^{(k)} \quad \text{... (a)}$$

(a)× $\left(\mathrm{e}^{-\pi x}\right)$; $\left(\mathrm{e}^{-\pi x}\right)\sum_{k=1}^{+\infty} \left(\mathrm{e}^{-\pi x}\right)^{(k)}$

$$= \left(\mathrm{e}^{-\pi x}\right)^{(2)} + \left(\mathrm{e}^{-\pi x}\right)^{(3)} + \ldots + \left(\mathrm{e}^{-\pi x}\right)^{(k+1)} \quad \text{... (b)}$$

(a)−(b) ; $[1-\left(\mathrm{e}^{-\pi x}\right)] \sum_{k=1}^{+\infty} \left(\mathrm{e}^{-\pi x}\right)^{(k)}$

$$= \left(\mathrm{e}^{-\pi x}\right)^{(1)} - \left(\mathrm{e}^{-\pi x}\right)^{(\mathrm{k}+1)}$$

$$\sum_{k=1}^{+\infty} \left(\mathrm{e}^{-\pi x}\right)^{(k)} = \frac{\left(\mathrm{e}^{-\pi x}\right)^{(1)} - \left(\mathrm{e}^{-\pi x}\right)^{(\mathrm{k}+1)}}{[1-\left(\mathrm{e}^{-\pi x}\right)]}$$

$$= \frac{1}{(\mathrm{e}^{\pi x}-1)}$$

Or $\psi(\mathrm{x}) = \sum_{n=1}^{+\infty} (\mathrm{e})^{(-\mathrm{nn}\pi x)} = \sum_{k=1}^{+\infty} \left(\mathrm{e}^{-\pi x}\right)^{(k)} = \frac{1}{(\mathrm{e}^{\pi x}-1)}$ ...(9.1)

So $\prod(\frac{s}{2}-1)(\pi)^{(-\frac{s}{2})}\zeta(\mathrm{s}) = \int_{0^+}^{+\infty} \psi(\mathrm{x})(x)^{(\frac{s}{2}-1)}\,\mathrm{dx}$

$$= \int_{0^+}^{+\infty} (\frac{1}{(e^{\pi x}-1)})(x)^{(\frac{s}{2}-1)}\,dx$$

But from $\frac{d[\mathrm{Ln}[(e)^{(\pi x)}-1]]}{d[(e)^{(\pi x)}-1]} = \frac{1}{(e^{\pi x}-1)}$

And $d[(x)^{(\frac{s}{2})}] = (\frac{s}{2})(x)^{(\frac{s}{2}-1)}dx$

Then $\prod(\frac{s}{2}-1)(\pi)^{(-\frac{s}{2})}\zeta(s) = \int_{0^+}^{+\infty} (\frac{d[\mathrm{Ln}[(e)^{(\pi x)}-1]]}{d[(e)^{(\pi x)}-1]})(\frac{2}{s})\, d[(x)^{(\frac{s}{2})}]$

$$= (\frac{\mathrm{Ln}[(e)^{(\pi x)}-1]}{(e)^{(\pi x)}-1})(\frac{2}{s})(x)^{(\frac{s}{2})}]_{0^+}^{+\infty}$$

By L'Hospital's Rule

$(\frac{\mathrm{Ln}[(e)^{(\pi x)}-1]}{(e)^{(\pi x)}-1})(\frac{2}{s})(x)^{(\frac{s}{2})}]_{0^+}^{+\infty} = \infty$ (diverges to ∞) ...(9.2)

Or $\Gamma(\frac{s}{2})(\pi)^{(-\frac{s}{2})}\zeta(s) = \int_{0^+}^{+\infty} \psi(x)(x)^{(\frac{s}{2}-1)}\,dx$

$= \infty$ (diverges to ∞)

2.6.2 From $\Gamma(s) = \int_{0^+}^{+\infty} (e)^{(-u)}\,(u)^{(s-1)}\,du$

$$\left(\frac{1}{n^{(1-s)}}\right)(\pi)^{-\left(\frac{1-s}{2}\right)}\Gamma\left(\frac{1-s}{2}\right) = dx$$

$$= \int_{0^+}^{+\infty} (e)^{(-nn\pi x)}\,(x)^{(\frac{1-s}{2}-1)}\,dx)$$

$$\sum_{n=1}^{+\infty} \left(\frac{1}{n^{(1-s)}}\right)(\pi)^{-\left(\frac{1-s}{2}\right)}\Gamma\left(\frac{1-s}{2}\right) = \sum_{n=1}^{+\infty} \int_{0^+}^{+\infty} (e)^{(-nn\pi x)}(x)^{(\frac{1-s}{2}-1)}\,dx$$

$$= \int_{0^+}^{+\infty} \sum_{n=1}^{+\infty} (e)^{(-nn\pi x)}(x)^{(\frac{1-s}{2}-1)}\,dx$$

Riemann denoted $\sum_{n=1}^{+\infty} (e)^{(-nn\pi x)} = \psi(x)$

And from ... (9.1) $\psi(x) = \frac{1}{(e)^{(\pi x)}-1}$

But from $\frac{d[\mathrm{Ln}[(e)^{(\pi x)}-1]]}{d[(e)^{(\pi x)}-1]} = \frac{1}{(e)^{(\pi x)}-1}$

And $d[(x)^{(\frac{1-s}{2})}] = (\frac{1-s}{2})(x)^{(\frac{1-s}{2}-1)}dx$

so $\Gamma\left(\frac{1-s}{2}\right)(\pi)^{-(\frac{1-s}{2})}\zeta(1-s) = \int_{0^+}^{+\infty} \psi(x)(x)^{(\frac{1-s}{2}-1)}\,dx$

$$= \int_{0^+}^{+\infty} \left(\frac{1}{(e)^{(\pi x)}-1}\right)(x)^{\left(\frac{1-s}{2}-1\right)}\,dx$$

$$= \int_{0^+}^{+\infty} \frac{d[\mathrm{Ln}[(e)^{(\pi x)}-1]]}{d[(e)^{(\pi x)}-1]} \left(\frac{2}{1-s}\right) d[(x)^{\left(\frac{1-s}{2}\right)}]$$

$$= \left(\frac{\mathrm{Ln}[(e)^{(\pi x)}-1]}{(e)^{(\pi x)}-1}\right)\left(\frac{2}{1-s}\right)(x)^{\left(\frac{1-s}{2}\right)}\Big]_{0^+}^{+\infty}$$

$$= \infty \quad \text{(diverges to } \infty)$$

Consider if $\int_{0^+}^{+\infty} \psi(x)(x)^{\left(\frac{1-s}{2}-1\right)}\,dx \neq \infty$,then

$$(\pi)^{-\left(\frac{1-s}{2}\right)}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s) \neq (\pi)^{\left(-\frac{s}{2}\right)}\Gamma\left(\frac{s}{2}\right)\zeta(s) \quad \text{except when } s = \frac{1}{2}$$

But exactly $\int_{0^+}^{+\infty} \psi(x)(x)^{\left(\frac{1-s}{2}-1\right)}\,dx = \infty$ (diverges to ∞) , so

$$(\pi)^{-\left(\frac{1-s}{2}\right)}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s) = (\pi)^{\left(-\frac{s}{2}\right)}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

$$= \infty$$

2.6.3 From $\Gamma(s) = \int_{0^+}^{+\infty} (e)^{(-u)}\,(u)^{(s-1)}\,du$

Let $u = n\pi x$

$$\left(\frac{1}{n^{(1-s)}}\right)(\pi)^{-(1-s)}\Gamma(1-s) = \int_{0^+}^{+\infty} \frac{(e)^{(-n\pi x)}}{(n)^{(1-s)}(\pi)^{(1-s)}}(n\pi x)^{(1-s-1)}\, n\pi\, dx$$

$$= \int_{0^+}^{+\infty} (e)^{(-n\pi x)}\,(x)^{(1-s-1)}\,dx)$$

$$\sum_{n=1}^{+\infty}\left(\frac{1}{n^{(1-s)}}\right)(\pi)^{-(1-s)}\Gamma(1-s) = \sum_{n=1}^{+\infty}\int_{0^+}^{+\infty} (e)^{(-n\pi x)}(x)^{(1-s-1)}\,dx$$

$$= \int_{0^+}^{+\infty}\sum_{n=1}^{+\infty} (e)^{(-n\pi x)}(x)^{(1-s-1)}\,dx$$

denote $\sum_{n=1}^{+\infty} (e)^{(-n\pi x)} = \phi(x)$

so $(\pi)^{-(1-s)}\Gamma(1-s)\zeta(1-s) = \int_{0^+}^{+\infty} \phi(x)(x)^{(1-s-1)}\,dx$

Let’s consider the value of $\phi(x)$.

$$\phi(x) = \sum_{n=1}^{+\infty} (e)^{(-n\pi x)} = (e)^{(-\pi x)} + (e)^{(-2\pi x)} + \ldots + (e)^{(-n\pi x)}$$

$$= \left(\mathrm{e}^{-\pi x}\right)^{(1)} + \left(\mathrm{e}^{-\pi x}\right)^{(2)} + ... + \left(\mathrm{e}^{-\pi x}\right)^{(n)} \quad ...\ (a)$$

(a)× $\left(\mathrm{e}^{-\pi x}\right)$; $\left(\mathrm{e}^{-\pi x}\right)\sum_{n=1}^{+\infty} (\mathrm{e})^{(-\mathrm{n}\pi x)}$

$$= \left(\mathrm{e}^{-\pi x}\right)^{(2)} + \left(\mathrm{e}^{-\pi x}\right)^{(3)} + ... + \left(\mathrm{e}^{-\pi x}\right)^{(n+1)} \quad ...\ (b)$$

(a)−(b) ; $[1-\left(\mathrm{e}^{-\pi x}\right)]\sum_{n=1}^{+\infty} (\mathrm{e})^{(-\mathrm{n}\pi x)}$

$$= \left(\mathrm{e}^{-\pi x}\right)^{(1)} - \left(\mathrm{e}^{-\pi x}\right)^{(n+1)}$$

$$\sum_{n=1}^{+\infty} (\mathrm{e})^{(-\mathrm{n}\pi x)} = \frac{\left(\mathrm{e}^{-\pi x}\right)}{[1-\left(\mathrm{e}^{-\pi x}\right)]}$$

Or $\phi(\mathrm{x}) = \frac{1}{(\mathrm{e})^{(\pi x)}-1}$

But from $\frac{\mathrm{d}[\mathrm{Ln}[(\mathrm{e})^{(\pi x)}-1]]}{\mathrm{d}[(\mathrm{e})^{(\pi x)}-1]} = \frac{1}{(\mathrm{e})^{(\pi x)}-1}$

And $\mathrm{d}[(x)^{(1-s)}] = (1-s)(x)^{(1-s-1)}\,\mathrm{dx}$

So $(\pi)^{-(1-s)}\Gamma(1-s)\zeta(1-\mathrm{s}) = \int_{0^+}^{+\infty} \phi(\mathrm{x})(x)^{(1-\mathrm{s}-1)}\,\mathrm{dx}$

$$= \int_{0^+}^{+\infty} \left(\frac{1}{(\mathrm{e})^{(\pi x)}-1}\right)(x)^{(1-\mathrm{s}-1)}\,\mathrm{dx}$$

$$= \int_{0^+}^{+\infty} \frac{\mathrm{d}[\mathrm{Ln}[(\mathrm{e})^{(\pi x)}-1]]}{\mathrm{d}[(\mathrm{e})^{(\pi x)}-1]}\left(\frac{1}{1-s}\right)\mathrm{d}(x)^{(1-s)}]$$

$$= \left(\frac{\mathrm{Ln}[(\mathrm{e})^{(\pi x)}-1]}{(\mathrm{e})^{(\pi x)}-1}\right)\left(\frac{1}{1-\mathrm{s}}\right)(x)^{(1-s)}]_{0^+}^{+\infty}$$

$$= \infty \quad \text{(diverges to } \infty)$$

2.6.4 From $\Gamma(\mathrm{s}) = \int_{0^+}^{+\infty} (\mathrm{e})^{(-\mathrm{u})}\,(\mathrm{u})^{(\mathrm{s}-1)}\,\mathrm{du}$

Let $\mathrm{u} = \mathrm{n}\pi x$

$$\left(\frac{1}{n^{(s)}}\right)(\pi)^{-(s)}\Gamma(\mathrm{s}) = \int_{0^+}^{+\infty} \frac{(\mathrm{e})^{(-n\pi x)}}{(n)^{(s)}(\pi)^{(s)}}(n\pi x)^{(s-1)}\mathrm{n}\pi\,\mathrm{d}x$$

$$= \int_{0^+}^{+\infty} (\mathrm{e})^{(-\mathrm{n}\pi x)}\,(x)^{(s-1)}\,\mathrm{d}x$$

$$\sum_{n=1}^{+\infty} \left(\frac{1}{n^{(s)}}\right)(\pi)^{-(s)}\Gamma(\mathrm{s}) = \sum_{n=1}^{+\infty} \int_{0^+}^{+\infty} (\mathrm{e})^{(-\mathrm{n}\pi x)}(x)^{(s-1)}\, \mathrm{d}x$$

$$= \int_{0^+}^{+\infty} \sum_{n=1}^{+\infty} (\mathrm{e})^{(-\mathrm{n}\pi x)}(x)^{(s-1)}\, \mathrm{d}x$$

denote $\sum_{n=1}^{+\infty} (\mathrm{e})^{(-\mathrm{n}\pi x)} = \phi(\mathrm{x})$

And from $\phi(\mathrm{x}) = \frac{1}{(\mathrm{e})^{(\pi x)}-1}$

But from $\frac{\mathrm{d}[\mathrm{Ln}[(\mathrm{e})^{(\pi x)}-1]]}{\mathrm{d}[(\mathrm{e})^{(\pi x)}-1]} = \frac{1}{(\mathrm{e})^{(\pi x)}-1}$

And $\mathrm{d}[(x)^{(s)}] = (s)(x)^{(s-1)}\,\mathrm{dx}$

so $(\pi)^{-(s)}\Gamma(s)\zeta(\mathrm{s}) = \int_{0^+}^{+\infty} \phi(\mathrm{x})(x)^{(s-1)}\,\mathrm{dx}$

$$= \int_{0^+}^{+\infty} \left(\frac{1}{(\mathrm{e})^{(\pi x)}-1}\right)(x)^{(s-1)}\,\mathrm{dx}$$

$$= \int_{0^+}^{+\infty} \frac{\mathrm{d}[\mathrm{Ln}[(\mathrm{e})^{(\pi x)}-1]]}{\mathrm{d}[(\mathrm{e})^{(\pi x)}-1]}\left(\frac{1}{s}\right)\mathrm{d}[(x)^{(s)}]$$

$$= \left(\frac{\mathrm{Ln}[(\mathrm{e})^{(\pi x)}-1]}{(\mathrm{e})^{(\pi x)}-1}\right)\left(\frac{1}{\mathrm{s}}\right)(x)^{(s)}\Big]_{0^+}^{+\infty}$$

By L'Hospital's Rule

$\left(\frac{\mathrm{Ln}[(\mathrm{e})^{(\pi x)}-1]}{(\mathrm{e})^{(\pi x)}-1}\right)\left(\frac{1}{\mathrm{s}}\right)(x)^{(s)}\Big]_{0^+}^{+\infty} = \infty$ (diverges to ∞)

Consider if $\int_{0^+}^{+\infty} \phi(\mathrm{x})(x)^{(s-1)}\,\mathrm{dx} \neq \infty$, then

$(\pi)^{-(1-s)}\Gamma(1-s)\zeta(1-\mathrm{s}) \neq (\pi)^{-(s)}\Gamma(s)\zeta(\mathrm{s})$ except when $\mathrm{s} = \frac{1}{2}$

But exactly $\int_{0^+}^{+\infty} \phi(\mathrm{x})(x)^{(s-1)}\,\mathrm{dx} = \infty$, so

$(\pi)^{-(1-s)}\Gamma(1-s)\zeta(1-\mathrm{s}) = (\pi)^{-(s)}\Gamma(s)\zeta(\mathrm{s}) = \infty$

Next from $(\pi)^{-(1-s)}\Gamma(1-s)\zeta(1-\mathrm{s}) = \int_{0^+}^{+\infty} \phi(\mathrm{x})(x)^{(1-s-1)}\,\mathrm{dx}$.

If we try to extend $\int_{0^+}^{+\infty} \phi(\mathrm{x})(x)^{(1-s-1)}\,\mathrm{dx}$ to $\int_{+\infty}^{+\infty} \phi(\mathrm{x})(x)^{(1-s-1)}\,\mathrm{dx}$ by taking integration along a closed curve C covered the domain $(+\infty, +\infty)$, then, by famous **Cauchy's theorem**, we will get

$$\int_{+\infty}^{+\infty} \phi(\mathrm{x})(x)^{(1-s-1)}\,\mathrm{dx} = \int_{0^+}^{+\infty} \phi(\mathrm{x})(x)^{(1-s-1)}\,\mathrm{dx} + \int_{+\infty}^{0^+} \phi(\mathrm{x})(x)^{(1-s-1)}\,\mathrm{dx}$$

$$0 = \int_{0^+}^{+\infty} \phi(\mathrm{x})(x)^{(1-s-1)}\,\mathrm{dx} - \int_{0^+}^{+\infty} \phi(\mathrm{x})(x)^{(1-s-1)}\,\mathrm{dx}$$

$$= [\frac{(1)^{(1-s)}}{(1)} - \frac{(1)^{(1-s)}}{(1)}]\int_{0^+}^{+\infty} \phi(\mathrm{x})(x)^{(1-s-1)}\,\mathrm{dx}$$

From **Euler's Formula**

$$(e)^{\pm i\pi} = -1$$

$$(\cos\pi \pm i\sin\pi) = -1$$

$$\cos\pi = -1, \quad \sin\pi = 0$$

$$[\frac{(1)^{(1-s)}}{(1)} - \frac{(1)^{(1-s)}}{(1)}]\int_{0^+}^{+\infty} \phi(\mathrm{x})(x)^{(1-s-1)}\,\mathrm{dx}$$

$$= [\left(-e^{i\pi}\right)^{(1-s)} - \left(-e^{-i\pi}\right)^{(1-s)}] \int_{0^+}^{+\infty} \frac{(\mathrm{x})^{(s-1)}}{(\mathrm{e^x}-1)}\,\mathrm{dx}$$

$$= [(-\cos\pi(1-s) - \mathrm{i}\sin\pi(1-s) - (-\cos\pi(1-s) + \mathrm{i}\sin\pi(1-s)] \int_{0^+}^{+\infty} \frac{(\mathrm{x})^{(1-s-1)}}{(\mathrm{e^x}-1)}\,\mathrm{dx}$$

$$= -2i\sin\pi(1-s) \int_{0^+}^{+\infty} \frac{(\mathrm{x})^{(1-s-1)}}{(\mathrm{e^x}-1)}\,\mathrm{dx}$$

So $\int_{+\infty}^{+\infty} \phi(\mathrm{x})(x)^{(1-s-1)}\,\mathrm{dx} = 0$

$$= -2i\sin\pi(1-s)\,\pi^{-(1-s)}\Gamma(1-s)\zeta(1-\mathrm{s})$$

But $\pi^{-(1-s)}\Gamma(1-s)\zeta(1-\mathrm{s}) = \infty$

So $\sin\pi(1-s) = (\sin\pi\cos\pi s - \cos\pi\sin\pi s)$

$$= \sin\pi s$$

$$= 0$$

$\sin\pi s$ must equal zero to cause $\int_{+\infty}^{+\infty} \phi(\mathrm{x})(x)^{(1-s-1)}\,\mathrm{dx} = 0$

$$= -2i\sin\pi(1-s)\,\pi^{-(1-s)}\Gamma(1-s)\zeta(1-\mathrm{s})$$

$$= -2i\sin\pi(1-s)(\infty)$$

$$= -2i\sin\pi\mathrm{s}\,(\infty)$$

And $\sin\pi s = 2\sin\frac{\pi s}{2}\cos\frac{\pi s}{2}$

$= 0$ (in this case)

so $\int_{+\infty}^{+\infty} \phi(x)(x)^{(1-s-1)}\,dx = 0$

$$= -2i2\sin\frac{\pi s}{2}\cos\frac{\pi s}{2}\ \pi^{-(1-s)}\Gamma(1-s)\zeta(1-s) \ldots (9.3)$$

$$0 = -2i(0)\ \pi^{-(1-s)}\Gamma(1-s)\zeta(1-s)$$

And from

$$(\pi)^{-\left(\frac{1-s}{2}\right)}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s) = \int_{0^+}^{+\infty}\phi(x)(x)^{\left(\frac{1-s}{2}-1\right)}\,dx$$

$$= \infty$$

Next, extend $\int_{0^+}^{+\infty}\phi(x)(x)^{\left(\frac{1-s}{2}-1\right)}\,dx$ to $\int_{+\infty}^{+\infty}\phi(x)(x)^{\left(\frac{1-s}{2}-1\right)}\,dx$ by taking integration along a closed curve C covered the domain (+∞ , +∞), then by famous **Cauchy's theorem** we get

$$\int_{+\infty}^{+\infty}\phi(x)(x)^{\left(\frac{1-s}{2}-1\right)}\,dx = \int_{0^+}^{+\infty}\phi(x)(x)^{\left(\frac{1-s}{2}-1\right)}\,dx - \int_{0^+}^{+\infty}\phi(x)(x)^{\left(\frac{1-s}{2}-1\right)}\,dx$$

$$= 0$$

$$0 = \left[\frac{(1)^{\left(\frac{1-s}{2}\right)}}{(1)} - \frac{(1)^{\left(\frac{1-s}{2}\right)}}{(1)}\right]\int_{0^+}^{+\infty}\phi(x)(x)^{\left(\frac{1-s}{2}-1\right)}\,dx$$

$$= \left[\frac{(-e^{i\pi})^{\left(\frac{1-s}{2}\right)}}{(1)} - \frac{(-e^{-i\pi})^{\left(\frac{1-s}{2}\right)}}{(1)}\right]\int_{0^+}^{+\infty}\phi(x)(x)^{\left(\frac{1-s}{2}-1\right)}\,dx$$

$$= \left[\frac{\left(-\cos\pi\left(\frac{1-s}{2}\right)-i\sin\pi\left(\frac{1-s}{2}\right)\right)}{(1)} - \frac{\left(-\cos\pi\left(\frac{1-s}{2}\right)+i\sin\pi\left(\frac{1-s}{2}\right)\right)}{(1)}\right]\int_{0^+}^{+\infty}\phi(x)(x)^{\left(\frac{1-s}{2}-1\right)}\,dx$$

$$0 = (0)\int_{0^+}^{+\infty}\phi(x)(x)^{\left(\frac{1-s}{2}-1\right)}\,dx$$

$$0 = (0)\ \pi^{-\left(\frac{1-s}{2}\right)}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$$

From $\sin\pi\left(\frac{1-s}{2}\right) = \left(\sin\frac{\pi}{2}\cos\frac{\pi s}{2} - \cos\frac{\pi}{2}\sin\frac{\pi s}{2}\right)$

$$= \cos\frac{\pi s}{2}$$

Then $$0 = -2i\sin\pi(\tfrac{1-s}{2})\,\pi^{-(\frac{1-s}{2})}\Gamma\left(\tfrac{1-s}{2}\right)\zeta(1-\mathrm{s})$$

$$= -2i\cos\frac{\pi s}{2}\pi^{-\left(\frac{1-s}{2}\right)}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-\mathrm{s}) \qquad \text{.... (9.4)}$$

$$0 = (0)\,\pi^{-\left(\frac{1-s}{2}\right)}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-\mathrm{s})$$

so $$\cos\frac{\pi s}{2} = 0$$

Thus from ....(9.3) and ....(9.4)

$$-2i2\sin\frac{\pi s}{2}\cos\frac{\pi s}{2}\pi^{-(1-s)}\Gamma(1-s)\zeta(1-\mathrm{s}) = 0$$

$$= -2i\cos\frac{\pi s}{2}\pi^{-\left(\frac{1-s}{2}\right)}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-\mathrm{s})$$

cancel term $-2i\cos\frac{\pi s}{2}$ (which = 0) both sides **(remember that the aim of our process is just only to follow or prove all of Riemann's process of deriving equation to see whether it is true or not, so we have to go on although the way to derive the equation look so strange !)**

$$2\sin\frac{\pi s}{2}\pi^{-(1-s)}\Gamma(1-s)\zeta(1-\mathrm{s}) = \pi^{-\left(\frac{1-s}{2}\right)}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-\mathrm{s})$$

$$= \infty$$

And from $(\pi)^{-(s)}\Gamma(s)\zeta(\mathrm{s}) = \int_{0^+}^{+\infty}\phi(\mathrm{x})(x)^{(s-1)}\,\mathrm{dx}$

$$= \infty$$

so $$(\pi)^{-(s)}\Gamma(s)\zeta(\mathrm{s}) = 2\sin\frac{\pi s}{2}\pi^{-(1-s)}\Gamma(1-s)\zeta(1-\mathrm{s}) \qquad \text{... (9.5)}$$

$$= \infty \text{ (always diverges to } \infty\text{)}$$

$$(\infty) = 2\sin\frac{\pi s}{2}(\infty)$$

Because $\pi^{-(1-s)}\Gamma(1-s)\zeta(1-\mathrm{s})$ and $(\pi)^{-(s)}\Gamma(s)\zeta(\mathrm{s})$ are always equal to ∞, and because $\sin\frac{\pi s}{2}\cos\frac{\pi s}{2} = 0$ while $\cos\frac{\pi s}{2}$ always= 0, so $\sin\frac{\pi s}{2}$ must always equal 1.

Because $\Gamma(s)$ alone $\neq$ 0 and $\Gamma(s) = \infty$ only for some values of s and $(\pi)^{-(s)}$alone $\neq$ 0 and $(\pi)^{-(s)}$never $= \infty$, so ζ(s) itself must always equal ∞ to cause $(\pi)^{-(s)}\Gamma(s)\zeta(s)$ always equal ∞ (diverge to ∞).

Finally $\quad \zeta(s) = 2\sin\frac{\pi s}{2}\pi^{(s-1)}\Gamma(1-s)\zeta(1-s)$ ... (9.6)

$$= 2(1)(\infty)$$

$$= \infty \quad \text{(always diverges to } \infty \text{)}$$

and $\sin\frac{\pi s}{2}$ must always equal 1.

If you need the exact $\zeta(s)= 2^{s}\sin\frac{\pi s}{2}\pi^{(s-1)}\Gamma(1-s)\zeta(1-s)$ instead of $2\sin\frac{\pi s}{2}\pi^{(s-1)}\Gamma(1-s)\zeta(1-s)$, you can get it by multiplying equation

$$(\pi)^{-(1-s)}\Gamma(1-s)\zeta(s) = \int_{0^+}^{+\infty} \phi(x)(x)^{(1-s-1)}\,dx$$

$$= \infty$$

by $(2)^{-(1-s)}$, the above equation then becomes

$$(2\pi)^{-(1-s)}\Gamma(1-s)\zeta(s) = \int_{0^+}^{+\infty} \frac{\phi(x)(x)^{(1-s-1)}}{(2)^{(1-s)}}\,dx$$

$$= \infty$$

and then extend R.H.S. by famous **Cauchy's theorem** to

$$\int_{+\infty}^{+\infty} \frac{\phi(x)(x)^{(1-s-1)}}{(2)^{(1-s)}}\,dx = \int_{0^+}^{+\infty} \frac{\phi(x)(x)^{(1-s-1)}}{(2)^{(1-s)}}\,dx + \int_{+\infty}^{0^+} \frac{\phi(x)(x)^{(1-s-1)}}{(2)^{(1-s)}}\,dx$$

$$= \int_{0^+}^{+\infty} \frac{\phi(x)(x)^{(1-s-1)}}{(2)^{(1-s)}}\,dx - \int_{0^+}^{+\infty} \frac{\phi(x)(x)^{(1-s-1)}}{(2)^{(1-s)}}\,dx$$

$$= \left[\frac{(1)^{(1-s)}}{(1)} - \frac{(1)^{(1-s)}}{(1)}\right]\int_{0^+}^{+\infty} \frac{\phi(x)(x)^{(1-s-1)}}{(2)^{(1-s)}}\,dx$$

$$= [(-\cos\pi(1-s) - i\sin\pi(1-s) - (-\cos\pi(1-s) + i\sin\pi(1-s)]\int_{0^+}^{+\infty} \frac{(x)^{(1-s-1)}}{(2)^{(1-s)}}\,dx$$

$$= -2i\sin\pi(1-s)(2\pi)^{-(1-s)}\Gamma(1-s)\zeta(1-s)$$

$$0 = -2i(0)(2\pi)^{-(1-s)}\Gamma(1-s)\zeta(1-s)$$

But $\quad \sin\pi(1-s) = (\sin\pi\cos\pi s - \cos\pi\sin\pi s)$

$$= \sin\pi s$$

And $\quad \sin\pi s \quad = 2\sin\frac{\pi s}{2}\cos\frac{\pi s}{2}$

so $\int_{+\infty}^{+\infty} \frac{\phi(\mathrm{x})(x)^{(1-s-1)}}{(2)^{(1-s)}}\mathrm{dx} = -2i\sin\pi s(2\pi)^{-(1-s)}\Gamma(1-s)\zeta(1-\mathrm{s})$

$$= -2i2\sin\frac{\pi s}{2}\cos\frac{\pi s}{2}(2\pi)^{-(1-s)}\Gamma(1-s)\zeta(1-\mathrm{s}) \quad ...(9.7)$$

$$0 = -2i(0)(2\pi)^{-(1-s)}\Gamma(1-s)\zeta(1-\mathrm{s})$$

Then follow the previous process, and finally you will get

$$\zeta(\mathrm{s}) = 2^s\sin\frac{\pi s}{2}\pi^{(s-1)}\Gamma(1-s)\zeta(1-\mathrm{s}) \quad ...(9.8)$$

$= 2^s(1)(\infty)$

$= \infty$ (always diverges to ∞) and $\sin\frac{\pi s}{2}$ must always equal 1.

It looks as if the functional equation $\zeta(\mathrm{s}) = 2^s\sin\frac{\pi s}{2}\pi^{(s-1)}\Gamma(1-s)\zeta(1-\mathrm{s})$ will be equal to zero if and only if the value of $\sin\frac{\pi s}{2} = 0$(or values of s (of $\sin\frac{\pi s}{2}$) are equal to $-2, -4, -6, \ldots$), which are the trivial zeroes of $\zeta(\mathrm{s})$ as many people think. That is not true, actually from process above, it is shown that $\zeta(\mathrm{s}) = 2^s\sin\frac{\pi s}{2}\pi^{(s-1)}\Gamma(1-s)\zeta(1-\mathrm{s}) = \infty$ (always diverges to ∞) and $\sin\frac{\pi s}{2}$ must equal 1 only. There is no trivial zero of Riemann zeta function $\zeta(\mathrm{s}) = 2^s\sin\frac{\pi s}{2}\pi^{(s-1)}\Gamma(1-s)\zeta(1-\mathrm{s}) = \infty$ at all!

I would like to specify (confirm) that the value of $\zeta(\mathrm{s})$ from equation $\zeta(\mathrm{s}) = \sum_{n=1}^{\infty}\left(\frac{1}{n^s}\right)$, which is up to the value of s and converges only when **$\Re$(s)>1**, is <u>not</u> the same as the value of $\zeta(\mathrm{s})$ from functional equation

$\zeta(\mathrm{s}) = 2^s\sin\frac{\pi s}{2}\pi^{(s-1)}\Gamma(1-s)\zeta(1-\mathrm{s})$ which is always equal to ∞ (diverges to ∞).

3. Integral of the remaining complex quantities

Next Riemann tried to find the integral of the remaining complex quantities in negative sense around the domain. He mentioned that the integrand had discontinuities where x was equal to the whole multiple of $\pm 2\pi i$, if the real part of s was negative (integer). And the integral was thus equal to the sum of the integrals taken in negative sense around these values. The integral around the value n$2\pi i$ was $= (-2\pi n i)^{(s-1)}(-2\pi i)$, then

Riemann denoted

$$2\sin \pi s\, \zeta(s) \prod(s-1) = (2\pi)^{(s)} \sum(n)^{(s-1)} [(-i)^{(s-1)} + (i)^{(s-1)}]$$

Let us prove together,

Last time when Riemann talked about positive sense around a domain, he worked with values of x on $(+\infty, +\infty)$. This time he talked about negative sense around that domain and worked with x which were imaginary numbers $= \pm n2\pi i$.

From ...(5) $\quad 2\sin \pi s\, \zeta(s)\prod(s-1) = i\int_{+\infty}^{+\infty} \frac{(x)^{(s-1)}}{(e^{x}-1)}\,dx = 0$

$$0 = i\int_{0^+}^{+\infty} \frac{(x)^{(s-1)}}{(e^{x}-1)}\,dx + i\int_{+\infty}^{0^+} \frac{(x)^{(s-1)}}{(e^{x}-1)}\,dx$$

$$= i\int_{0^+}^{+\infty} \frac{(x)^{(s-1)}}{(e^{x}-1)}\,dx - i\int_{0^+}^{+\infty} \frac{(x)^{(s-1)}}{(e^{x}-1)}\,dx$$

$$= i\int_{0^+}^{+\infty} \frac{(x)^{(s-1)}}{(1-e^{(-x)})(e^{x})}\,dx - i\int_{0^+}^{+\infty} \frac{(x)^{(s-1)}}{(1-e^{(-x)})(e^{x})}\,dx$$

$$= i\int_{0^+}^{+\infty} \frac{(x)^{(s-1)}(e)^{(-x)}}{(1-e^{(-x)})}\,dx - i\int_{0^+}^{+\infty} \frac{(x)^{(s-1)}(e)^{(-x)}}{(1-e^{(-x)})}\,dx$$

For $x = \pm\, x_n = \pm\, n2\pi i$

$$0 = i\int_{0^+}^{+\infty} (x_n)^{(s-1)}\left[\frac{(e^{-(x_n)})}{(1-e^{-(x_n)})}\right] dx_n - i\int_{0^+}^{+\infty} (-x_n)^{(s-1)}\left[\frac{(e^{-(-x_n)})}{(1-e^{-(-x_n)})}\right] d(-x_n)$$

From **Riemann Sum**

$\int_{0}^{+\infty} f(x)\,dx = \sum_{n=1}^{+\infty} f(s_n)\,\Delta x_n \qquad$ for $x_{n+1} \geq s_n \geq x_n$

$= \sum_{n=1}^{+\infty} f(x_n)\,\Delta x_n \qquad$ if $x_n = n2\pi i$ = right-hand end point on $[(x_n) - (x_{n+1})]$ of the interval $[0, +\infty)$.

$\int_0^{+\infty} f(-\mathrm{x})\,\mathrm{dx} = \sum_{n=1}^{+\infty} f(-\mathrm{s}_n)\,\Delta x_n$ for $(-x_{n-1}) \leq (-\mathrm{s}_n) \leq (-x_n)$

$= \sum_{n=1}^{+\infty} f(-x_n)\,\Delta(-x_n)$ if $(-x_n) = (-n2\pi i) =$ right-hand end point on $[(-x_{n-1}) - (-x_n)]$ of the interval $[0, +\infty)$

Thus by **Riemann Sum**

$$i\int_{0^+}^{+\infty} (x_n)^{(\mathrm{s}-1)}[\frac{(e^{-(x_n)})}{(1-e^{-(x_n)})}]\,\mathrm{d}x_n - i\int_{0^+}^{+\infty} (-x_n)^{(\mathrm{s}-1)}[\frac{(e^{-(-x_n)})}{(1-e^{-(-x_n)})}]\,\mathrm{d}(-x_n) = 0$$

$$= i\int_{0^+}^{+\infty} (x_n)^{(\mathrm{s}-1)} \sum_{n=1}^{+\infty} \left(e^{-(x_n)}\right)^{(n-1)} e^{-(x_n)}\mathrm{d}x_n$$

$$-i\int_{0^+}^{+\infty} (-x_n)^{(\mathrm{s}-1)} \sum_{n=1}^{+\infty} \left(e^{-(-x_n)}\right)^{(n-1)} e^{-(-x_n)}\mathrm{d}(-x_n)$$

$$= i\int_{0^+}^{+\infty} (x_n)^{(\mathrm{s}-1)} \sum_{n=1}^{+\infty} \left(e^{-(x_n)}\right)^{(n)} \mathrm{d}x_n$$

$$-i\int_{0^+}^{+\infty} (-x_n)^{(\mathrm{s}-1)} \sum_{n=1}^{+\infty} \left(e^{-(-x_n)}\right)^{(n)} \mathrm{d}(-x_n)$$

$$= i\int_{0^+}^{+\infty} (n2\pi i)^{(\mathrm{s}-1)} \sum_{n=1}^{+\infty} \left(e^{-(2\pi i)}\right)^{(nn)} \mathrm{d}x_n$$

$$-i\int_{0^+}^{+\infty} (-n2\pi i)^{(\mathrm{s}-1)} \sum_{n=1}^{+\infty} \left(e^{-(-2\pi i)}\right)^{(nn)} \mathrm{d}(-x_n)$$

$$= i\sum_{n=1}^{+\infty} (n2\pi i)^{(\mathrm{s}-1)}\,[\cos 2\pi - i\sin 2\pi]^{(nn)}[2\pi i]$$

$$-i\sum_{n=1}^{+\infty} (-n2\pi i)^{(\mathrm{s}-1)}\,[\cos 2\pi + i\sin 2\pi]^{(nn)}[-2\pi i]$$

$$= i\sum_{n=1}^{+\infty} (n2\pi i)^{(\mathrm{s}-1)}\,[1]^{(nn)}[2\pi i]$$

$$-i\sum_{n=1}^{+\infty} (-n2\pi i)^{(\mathrm{s}-1)}\,[1]^{(nn)}[-2\pi i]$$

$$= i\sum_{n=1}^{+\infty} (\mathrm{n}2\pi i)^{(\mathrm{s}-1)}\,[2\pi i] - i\sum_{n=1}^{+\infty} (-\mathrm{n}2\pi i)^{(\mathrm{s}-1)}\,[-2\pi i]$$

$$= i\sum_{n=1}^{+\infty} (i)^{(\mathrm{s})}(2\pi)^{(\mathrm{s})}\,(\mathrm{n})^{(\mathrm{s}-1)} - i\sum_{n=1}^{+\infty} (-i)^{(\mathrm{s})}(2\pi)^{(\mathrm{s})}\,(\mathrm{n})^{(\mathrm{s}-1)}$$

$$0 = i(i)^{(\mathrm{s})}(2\pi)^{(\mathrm{s})} \sum_{n=1}^{+\infty} (\mathrm{n})^{(\mathrm{s}-1)} - i(-i)^{(\mathrm{s})}(2\pi)^{(\mathrm{s})} \sum_{n=1}^{+\infty} (\mathrm{n})^{(\mathrm{s}-1)}$$

$$= i\frac{i}{i}(i)^{(\mathrm{s})}(2\pi)^{(\mathrm{s})} \sum_{n=1}^{+\infty} (\mathrm{n})^{(\mathrm{s}-1)} + (-i)\frac{(-i)}{(-i)}(-i)^{(\mathrm{s})}(2\pi)^{(\mathrm{s})} \sum_{n=1}^{+\infty} (\mathrm{n})^{(\mathrm{s}-1)}$$

$= -1\ (i)^{(s-1)}(2\pi)^{(s)} \sum_{n=1}^{+\infty}(n)^{(s-1)} - (-i)^{(s-1)}(2\pi)^{(s)} \sum_{n=1}^{+\infty}(n)^{(s-1)}$

Multiply by -1 both sides

$= (i)^{(s-1)}(2\pi)^{(s)} \sum_{n=1}^{+\infty}(n)^{(s-1)} + (-i)^{(s-1)}(2\pi)^{(s)} \sum_{n=1}^{+\infty}(n)^{(s-1)}$

$= (2\pi)^{(s)} \sum_{n=1}^{+\infty}(n)^{(s-1)} [(-i)^{(s-1)} + (i)^{(s-1)}]$

The result is exactly the same as that of Riemann

$2\sin \pi s\ \zeta(s) \prod(s-1) = (2\pi)^{(s)} \sum(n)^{(s-1)} [(-i)^{(s-1)} + (i)^{(s-1)}]$

$= 0$

4. Finding nontrivial zeroes on critical line ($s = \frac{1}{2} + ti$)

From ... (9.2) $\prod\left(\frac{s}{2}-1\right)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(s) = \int_{0^+}^{+\infty} \psi(x)(x)^{\left(\frac{s}{2}-1\right)} dx = \infty$,

independent from the values of s.

Actually we can not go on anymore with this functional equation $\prod\left(\frac{s}{2}-1\right)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(s) = \int_{0^+}^{+\infty} \psi(x)(x)^{\left(\frac{s}{2}-1\right)} dx = \infty$. And so we have nothing to do further with the equation $\prod\left(\frac{s}{2}\right)(s-1)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(s) = \xi(t)$ denoted by Riemann. If someone tries to continue studying this Riemann's Hypothesis, he or she has to unavoidably solve the mysterious and doubtful equations below

4.1. $\prod\left(\frac{s}{2}-1\right)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(s) = \frac{1}{s(s-1)} + \int_{0^+}^{+\infty} \psi(x)[(x)^{\left(\frac{s}{2}-1\right)} + (x)^{-\left(\frac{1+s}{2}\right)}]dx$

4.2. $\xi(t) = \prod\left(\frac{s}{2}\right)(s-1)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(s)$

Let's see what's we can do with these two equations.

4.1. $\prod\left(\frac{s}{2}-1\right)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(s) = \frac{1}{s(s-1)} + \int_{0^+}^{+\infty} \psi(x)[(x)^{\left(\frac{s}{2}-1\right)} + (x)^{-\left(\frac{1+s}{2}\right)}]dx$

This equation is true. Let us prove together.

From $\prod\left(\frac{s}{2}-1\right)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(s) = \int_{0^+}^{+\infty} \psi(x)(x)^{\left(\frac{s}{2}-1\right)} dx$ ...(9)

$$= \int_{1}^{+\infty} \psi(x)(x)^{\left(\frac{s}{2}-1\right)}\,dx + \int_{0^+}^{1} \psi(x)(x)^{\left(\frac{s}{2}-1\right)}\,dx$$

From $(2\psi(x)+1) = (x)^{\left(-\frac{1}{2}\right)}\left(2\psi\left(\frac{1}{x}\right)+1\right)$ **(Jacobi, Fund. S.184)**

$$\prod\left(\tfrac{s}{2}-1\right)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(s) = \int_{1}^{+\infty} \psi(x)(x)^{\left(\frac{s}{2}-1\right)}\,dx + \int_{0^+}^{1} \psi\left(\frac{1}{x}\right)(x)^{\left(\frac{s-3}{2}\right)}\,dx$$

$$+\frac{1}{2}\int_{0^+}^{1}\left[(x)^{\left(\frac{s-3}{2}\right)} - (x)^{\left(\frac{s}{2}-1\right)}\right]dx$$

$$= \int_{1}^{+\infty} \psi(x)(x)^{\left(\frac{s}{2}-1\right)}\,dx + \int_{0^+}^{1} \psi\left(\frac{1}{x}\right)(x)^{\left(\frac{s-3}{2}\right)}\,dx$$

$$+\frac{1}{2}\left[\frac{(x)^{\left(\frac{s-1}{2}\right)}}{\left(\frac{s-1}{2}\right)}\right]_{0^+}^{1} - \frac{1}{2}\left[\frac{(x)^{\left(\frac{s}{2}\right)}}{\left(\frac{s}{2}\right)}\right]_{0^+}^{1}$$

$$= \int_{1}^{+\infty} \psi(x)(x)^{\left(\frac{s}{2}-1\right)}\,dx + \int_{0^+}^{1} \psi\left(\frac{1}{x}\right)(x)^{\left(\frac{s-3}{2}\right)}\,dx$$

$$+\frac{1}{2}\left[\frac{(1-0)}{\left(\frac{s-1}{2}\right)}\right] - \frac{1}{2}\left[\frac{(1-0)}{\left(\frac{s}{2}\right)}\right]$$

$$= \int_{1}^{+\infty} \psi(x)(x)^{\left(\frac{s}{2}-1\right)}\,dx + \int_{0^+}^{1} \psi\left(\frac{1}{x}\right)(x)^{\left(\frac{s-3}{2}\right)}\,dx$$

$$+ \frac{1}{(s)(s-1)}$$

So we get

$$\prod\left(\tfrac{s}{2}-1\right)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(s)$$

$$= \frac{1}{(s)(s-1)} + \int_{1}^{+\infty} \psi(x)(x)^{\left(\frac{s}{2}-1\right)}\,dx + \int_{0^+}^{1} \psi\left(\frac{1}{x}\right)(x)^{\left(\frac{s-3}{2}\right)}\,dx \qquad \text{... (10)}$$

Let's consider $\int_{0^+}^{1} \psi\left(\frac{1}{x}\right)(x)^{\left(\frac{s-3}{2}\right)}\,dx$

Suppose u= $\frac{1}{x}$ then $du = (-1)(x)^{-2}\,dx$, $dx = (-1)(u)^{-2}\,du$

Then $\int_{0^+}^{1} \psi\left(\frac{1}{x}\right)(x)^{\left(\frac{s-3}{2}\right)}\,dx = \int_{+\infty}^{1} \psi(u)(u)^{-\left(\frac{s-3}{2}\right)}(-1)(u)^{-2}du$

$$= \int_{1}^{+\infty} \psi(u)(u)^{-\left(\frac{1+s}{2}\right)}\,du$$

But $\int_{1}^{+\infty} \psi(u)(u)^{-\left(\frac{1+s}{2}\right)}\,du = \int_{1}^{+\infty} \psi(x)(x)^{-\left(\frac{1+s}{2}\right)}\,dx$

So $\int_{0^+}^{1} \psi\left(\frac{1}{x}\right)(x)^{\left(\frac{s-3}{2}\right)}\,dx = \int_{1}^{+\infty} \psi(x)(x)^{-\left(\frac{1+s}{2}\right)}\,dx$

And then $\prod\left(\frac{s}{2}-1\right)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(\mathrm{s})$

$$= \frac{1}{(s)(s-1)} + \int_{1}^{+\infty} \psi(\mathrm{x})(x)^{\left(\frac{s}{2}-1\right)}\,\mathrm{dx} + \int_{0^+}^{1} \psi\left(\frac{1}{x}\right)(x)^{\left(\frac{s-3}{2}\right)}\,\mathrm{dx}$$

$$= \frac{1}{(s)(s-1)} + \int_{1}^{+\infty} \psi(\mathrm{x})(x)^{\left(\frac{s}{2}-1\right)}\,\mathrm{dx} + \int_{1}^{+\infty} \psi(\mathrm{x})(\mathrm{x})^{-\left(\frac{1+s}{2}\right)}\,\mathrm{dx}$$

The same as found in the original Riemann's papers (1859)

$\prod\left(\frac{s}{2}-1\right)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(\mathrm{s})$

$$= \int_{1}^{+\infty} \psi(\mathrm{x})(x)^{\left(\frac{s}{2}-1\right)}\,\mathrm{dx} + \int_{0^+}^{1} \psi\left(\frac{1}{x}\right)(x)^{\left(\frac{s-3}{2}\right)}\,\mathrm{dx} + \frac{1}{2}\int_{0^+}^{1}\left[(x)^{\left(\frac{s-3}{2}\right)} - (x)^{\left(\frac{s}{2}-1\right)}\right]\mathrm{dx}$$

$$= \frac{1}{(s)(s-1)} + \int_{1}^{+\infty} \psi(\mathrm{x})\left[(x)^{\left(\frac{s}{2}-1\right)} + (x)^{-\left(\frac{1+s}{2}\right)}\right]\,\mathrm{dx} \qquad \ldots (11)$$

4.2. $\xi(\mathrm{t}) = \prod\left(\frac{s}{2}\right)(s-1)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(\mathrm{s})$

This equation may be not true because it looks as if there is a missing term. Let's prove .

From $\prod\left(\frac{s}{2}-1\right)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(\mathrm{s}) = \int_{0^+}^{+\infty} \psi(\mathrm{x})(x)^{\left(\frac{s}{2}-1\right)}\,\mathrm{dx}$ ... (9)

$$= \infty$$

Let us consider this equation of Riemann . **what we want here is only to prove from how or from where his new functional equation was derived. If it came from wrong sources (or former equations) or from wrong methods (of deriving equations), then it was a wrong equation and further using of it would be inappropriate.**

Now, from the equation,

$\prod\left(\frac{s}{2}-1\right)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(\mathrm{s})$

$$= \frac{1}{(s)(s-1)} + \int_{1}^{+\infty} \psi(\mathrm{x})\left[(x)^{\left(\frac{s}{2}-1\right)} + (x)^{-\left(\frac{1+s}{2}\right)}\right]\,\mathrm{dx} \qquad \ldots (11)$$

Multiply equation ...(11) by $\left(\frac{s}{2}\right)(s-1)$ both sides and set $\mathrm{s} = \frac{1}{2} + i\mathrm{t}$

(as Riemann did)

$$\prod\left(\frac{s}{2}-1\right)\left(\frac{s}{2}\right)(s-1)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(s)$$

$$=\frac{\left(\frac{s}{2}\right)(s-1)}{(s)(s-1)}+\left(\frac{s}{2}\right)(s-1)\int_1^{+\infty}\psi(\mathrm{x})\left[(x)^{\left(\frac{s}{2}-1\right)}+(x)^{-\left(\frac{1+s}{2}\right)}\right]\mathrm{dx}$$

$$=\frac{1}{2}+\frac{\left(\frac{1}{2}+it\right)\left(\frac{1}{2}+it-1\right)}{2}\int_1^{+\infty}\psi(\mathrm{x})\left[(x)^{\left(\frac{1}{4}+\frac{it}{2}-1\right)}+(x)^{-\left(\frac{1+\frac{1}{2}+it}{2}\right)}\right]\mathrm{dx}$$

$$=\frac{1}{2}-\frac{\left(tt+\frac{1}{4}\right)}{2}\int_1^{+\infty}\psi(\mathrm{x})\left[(x)^{\left(-\frac{3}{4}\right)}(x)^{\left(\frac{it}{2}\right)}+(x)^{\left(-\frac{3}{4}\right)}(x)^{\left(-\frac{it}{2}\right)}\right]\mathrm{dx}$$

$$=\frac{1}{2}-\frac{\left(tt+\frac{1}{4}\right)}{2}\int_1^{+\infty}\psi(\mathrm{x})(x)^{\left(-\frac{3}{4}\right)}\left[(e)^{\left(\frac{it}{2}\log\mathrm{x}\right)}+(e)^{\left(-\frac{it}{2}\log\mathrm{x}\right)}\right]\mathrm{dx}$$

$$=\frac{1}{2}-\frac{\left(tt+\frac{1}{4}\right)}{2}\int_1^{+\infty}\psi(\mathrm{x})(x)^{\left(-\frac{3}{4}\right)}\left[\left(\cos\left(\frac{1}{2}\mathrm{t}\log\mathrm{x}\right)+i\sin\left(\frac{1}{2}\mathrm{t}\log\mathrm{x}\right)\right)\right.$$

$$\left.+\left(\cos\left(\frac{1}{2}\mathrm{t}\log\mathrm{x}\right)-i\sin\left(\frac{1}{2}\mathrm{t}\log\mathrm{x}\right)\right)\right]\mathrm{dx}$$

$$=\frac{1}{2}-\frac{\left(tt+\frac{1}{4}\right)}{2}\int_1^{+\infty}\psi(\mathrm{x})(x)^{\left(-\frac{3}{4}\right)}\left(2\cos\left(\frac{1}{2}\mathrm{t}\log\mathrm{x}\right)\right)\mathrm{dx}$$

$$=\frac{1}{2}-\left(tt+\frac{1}{4}\right)\int_1^{+\infty}\psi(\mathrm{x})(x)^{\left(-\frac{3}{4}\right)}\cos\left(\frac{1}{2}\mathrm{t}\log\mathrm{x}\right)\mathrm{dx}$$

$=\xi(\mathrm{t})$ (the right hand side looks like that of Riemann, doesn't it? But the left hand side does not.)

You can see, there are two doubtful equations of Riemann here.

1. The left hand side of the above equation $\prod\left(\frac{s}{2}-1\right)\left(\frac{s}{2}\right)(s-1)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(s)=\xi(\mathrm{t})$ is different from the equation of Riemann $\prod\left(\frac{s}{2}\right)(s-1)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(s)=\xi(\mathrm{t})$. Has he made a mistake to write $\prod\left(\frac{s}{2}\right)(s-1)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(s)$ instead of $\prod\left(\frac{s}{2}-1\right)\left(\frac{s}{2}\right)(s-1)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(s)$? The answer is no! Let's prove from the relation

$$\Gamma\left(1+\frac{s}{2}\right)=\left(\frac{s}{2}\right)\Gamma\left(\frac{s}{2}\right)\quad\text{and}\quad\prod\left(\frac{s}{2}-1\right)=\Gamma\left(\frac{s}{2}\right)$$

Then $\prod\left(\frac{s}{2}\right)=\left(\frac{s}{2}\right)\prod\left(\frac{s}{2}-1\right)$. So in this case Riemann was right.

2. From the equation

$$\prod\left(\frac{s}{2}-1\right)\left(\frac{s}{2}\right)(s-1)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(\mathrm{s})$$

$$=\frac{1}{2}-\left(tt+\frac{1}{4}\right)\int_{1}^{+\infty}\psi(\mathrm{x})(x)^{\left(-\frac{3}{4}\right)}\cos\left(\frac{1}{2}\mathrm{tlogx}\right)\mathrm{dx}$$

$$=\infty$$

Hence the number of roots of the equation

$$\xi(\mathrm{t})=\frac{1}{2}-\left(tt+\frac{1}{4}\right)\int_{1}^{+\infty}\psi(\mathrm{x})(x)^{\left(-\frac{3}{4}\right)}\cos\left(\frac{1}{2}\mathrm{tlogx}\right)\mathrm{dx}$$

$$=\infty$$

derived by Riemann do not exist. It's impossible (not true)to show that the number of roots of $\xi(\mathrm{t})=0$ whose imaginary parts of t lie between $\frac{1}{2}i$ and $-\frac{1}{2}i$ and whose real parts lie between 0 and T is approximately $=\left(\frac{\mathrm{T}}{2\pi}\log\frac{\mathrm{T}}{2\pi}-\frac{\mathrm{T}}{2\pi}\right)$.

Next, let us consider the integral $\int d\log\xi(\mathrm{t})=\int d\log(\infty)$. It's impossible (not true) to show that the integral $\int d\log\xi(\mathrm{t})$, taken in a positive sense around the region consisting of the values of t whose imaginary parts lie between $\frac{1}{2}i$ and $-\frac{1}{2}i$ and whose real parts lie between 0 and T, is equal to $\left(\mathrm{T}\log\frac{\mathrm{T}}{2\pi}-\mathrm{T}\right)i$.

It is not right to denote that all $\propto$ from the complex numbers which are called the non trivial zeroes $\left(\frac{1}{2}+i\propto\right)$ of $\zeta(\mathrm{s})=$ roots of the equation

$$\xi(t)=\frac{1}{2}-\left(tt+\frac{1}{4}\right)\int_{1}^{+\infty}\psi(\mathrm{x})(x)^{\left(-\frac{3}{4}\right)}\cos\left(\frac{1}{2}\mathrm{tlogx}\right)\mathrm{dx}$$
$$=\infty\text{ (always)}$$

One can not express $\log\xi(\mathrm{t})$ as $\sum\log\left(1-\frac{tt}{\propto\propto}\right)+\log\xi(0)$, the reasons is that ξ(t) always = ∞.

5. Determination of the number of prime numbers that are

smaller than x

Next, Riemann tried to determine the number of prime numbers that are smaller than x with the assistance of all the methods he had derived before.

From the **identity by Riemann**

$$\log \zeta(s) = -\sum \log(1-(p)^{-s})$$

$$= \sum p^{-s} + \tfrac{1}{2}\sum p^{-2s} + \tfrac{1}{3}\sum p^{-3s} + \cdots \qquad ...(12)$$

Let's prove by using Maclaurin Series

$$-\frac{d}{dx}(\log(1-x)) = \frac{1}{(1-x)}$$

$$= \text{Geometric Series} \quad 1+X+X^2+X^3+\ldots \quad \text{for } x<1$$

By integration

$$-\log(1-x) = X + \tfrac{1}{2}X^2 + \tfrac{1}{3}X^3 + \tfrac{1}{4}X^4 + \ldots$$

Thus for $x = (p)^{-s} < 1$

$$-\log(1-(p)^{-s}) = (p)^{-s} + \tfrac{1}{2}(p)^{-2s} + \tfrac{1}{3}(p)^{-3s} + \ldots$$

For p = prime numbers 2, 3, 5, ...

$$-\log(1-(2)^{-s}) = (2)^{-s} + \tfrac{1}{2}(2)^{-2s} + \tfrac{1}{3}(2)^{-3s} + \ldots$$

$$-\log(1-(3)^{-s}) = (3)^{-s} + \tfrac{1}{2}(3)^{-2s} + \tfrac{1}{3}(3)^{-3s} + \ldots$$

$$-\log(1-(5)^{-s}) = (5)^{-s} + \tfrac{1}{2}(5)^{-2s} + \tfrac{1}{3}(5)^{-3s} + \ldots$$

Then $-[\log(1-(2)^{-s}) + \log(1-(3)^{-s}) + \log(1-(5)^{-s}) + \ldots]$

$$= (2)^{-s} + \tfrac{1}{2}(2)^{-2s} + \tfrac{1}{3}(2)^{-3s} + \ldots$$

$$+(3)^{-s} + \tfrac{1}{2}(3)^{-2s} + \tfrac{1}{3}(3)^{-3s} + \ldots$$

$$+(5)^{-s} + \tfrac{1}{2}(5)^{-2s} + \tfrac{1}{3}(5)^{-3s} + \ldots$$

$$+\ldots$$

Or $-\sum \log(1-(p)^{-s}) = \sum p^{-s} + \frac{1}{2}\sum p^{-2s} + \frac{1}{3}\sum p^{-3s} + \cdots$

For p = prime numbers = 2, 3, 5,...

n = all whole numbers = 1, 2, 3,...,∞

Riemann denoted that

$$\zeta(s) = \prod_{p\ prime} (1-\frac{1}{p^s})^{-1} = \sum_{n=1}^{+\infty} (\frac{1}{n^s}) \quad , \quad \mathfrak{R}(\mathbf{s}) > 1$$

$$\log \zeta(s) = \log \prod_{p\ prime} (1-\frac{1}{p^s})^{-1}$$

$$= \log\,[(1-(2)^{-s})^{-1}.\,(1-(3)^{-s})^{-1}.\,(1-(5)^{-s})^{-1}\ldots.]$$

$$= \log(1-(2)^{-s})^{-1} + \log(1-(3)^{-s})^{-1} + \log(1-(5)^{-s})^{-1} + \ldots$$

$$= -[\,\log(1-(2)^{-s}) + \log(1-(3)^{-s}) + \log(1-(5)^{-s}) + \cdots]$$

$$= -\sum \log\left(1-(p)^{(-s)}\right)$$

So $\log \zeta(s) = -\sum \log\left(1-(p)^{(-s)}\right) = \sum p^{-s} + \frac{1}{2}\sum p^{-2s} + \frac{1}{3}\sum p^{-3s} + \cdots$

One can replace $(p^{-s})^n$ by $s\int_{p^n}^{\infty}(x)^{-(s+1)}$dx.

Let's prove together

$$s\int_{p^n}^{\infty} (x)^{-(s+1)}\,dx = \frac{(\,s\,(x)^{(-s)}\,)}{(-s)}\Big]_{p^n}^{\infty}$$

$$= -\frac{1}{(x)^{(s)}}\Big]_{p^n}^{\infty}$$

$$= -(\frac{1}{\infty} - \frac{1}{(p^n)^s})$$

$$= (p)^{(-s)n} \qquad \ldots.\ (13)$$

Hope that my paper is clear enough to point out the mistakes or give disproof of the original **Riemann's Hypothesis** and explain the following sentences.

1. “All zeroes of the function ξ(t) are real”. This is not true because

$$\xi(t) = \prod\left(\frac{s}{2}-1\right)\left(\frac{s}{2}\right)(s-1)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(s), \text{ for } s = \frac{1}{2} + it$$

$$= \frac{1}{2} - \left(tt + \frac{1}{4}\right)\int_1^{+\infty} \psi(x)(x)^{\left(-\frac{3}{4}\right)} \cos\left(\frac{1}{2}\text{tlogx}\right) dx$$

is always equal to ∞. So there are no roots (all zeroes) of equation

$$\xi(t) = \prod\left(\frac{s}{2}-1\right)\left(\frac{s}{2}\right)(s-1)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(s), \text{ for } s = \frac{1}{2} + it$$

$$= \frac{1}{2} - \left(tt + \frac{1}{4}\right)\int_1^{+\infty} \psi(x)(x)^{\left(-\frac{3}{4}\right)} \cos\left(\frac{1}{2}\text{tlogx}\right) dx$$

$$= \infty$$

2. “The function (functional equation) ζ(s) has zeroes at the negative even integers −2, −4, … and one refers to them as the trivial zeroes”. This is not true , actually there are no trivial zeroes of ζ(s) because ζ(s) always = ∞ as proof above.

3. “The nontrivial zeroes of ζ(s) have real part equal to $\frac{1}{2}$ or the nontrivial zeroes are complex numbers $= \frac{1}{2} + i \propto$ where ∝ are zeroes of ξ(t)". This is not true because $\xi(t) = \prod\left(\frac{s}{2}-1\right)\left(\frac{s}{2}\right)(s-1)(\pi)^{\left(-\frac{s}{2}\right)}\zeta(s)$, for $s = \frac{1}{2} + it$ or $\xi(t) = \frac{1}{2} - \left(tt + \frac{1}{4}\right)\int_1^{+\infty} \psi(x)(x)^{\left(-\frac{3}{4}\right)} \cos\left(\frac{1}{2}\text{tlogx}\right) dx$ is always equal to ∞, for any values of s (or t). So ∝ = zero of equation ξ(t)= ∞ can not be found by this equation and so the nontrivial zeroes of ζ(s) or $\frac{1}{2} + i \propto$ can not be found by this way too.

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