Tricritical quantum point and inflationary cosmology

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Abstract

The holographic protection due to inflationary cosmology is a consequence of a quantum tricritical point. In this scenario a closed spacetime solution transitions into an inflationary de Sitter spacetime. Saturation of the holographic entropy bound is prevented by the phase change in the topology of the early universe.

\footnote{This essay received an \honorable mention" in the 2012 Essay Competition of the Gravity Research Foundation.} \vfill\eject

1 Introduction

The role of entropy bounds, holography and the accelerated universe constitutes a considerable body of work. Fischler and Susskind proposed ways the holographic principle (HP) does not conflict with inflation and the accelerated cosmology [1]. The holographic principle is satisfied for entropy/energy ratios for bounded systems with event horizons [2]. The closed Friedman-Lemaitre-Robertson-Walker (FLRW) spacetime violates the HP. A proposed cosmological model in [3] advances how the HP may accommodate inflation and dark energy, and derives the equation of state for vacuum energy and pressure [4]. It is thought the HP is more fundamental than inflation, where the HP constructs states on a surface one dimension smaller than the general spacetime, while inflation is spacetime dynamics based entirely on some initial conditions. A closed FLRW spacetime will saturate the entropy bound or HP once the universe reaches its turn around point to a big crunch[5].

The FLRW metric for a closed homogeneous and isotropic cosmology is

$$ds^2 = dt^2 - a^2(t)(d\chi^2 + sin^2\chi d\Omega^2)$$

for χ the azimuthal angle of the sphere $\mathbb{S}^3$ and Ω the steradian measure on any 2-sphere for a given χ. The particle horizon is

$$\chi_h \ = \ \int_{t_p}^{t} \frac{dt'}{a(t')}.$$

for t_p a fiducial start time. The entropy density $\sigma \ = \ (\rho+p)/T$ is constant and the entropy to area ratio is

$$\frac{S}{A} \ = \ \sigma\frac{2\chi_h \ - \ sin(2\chi_h)}{4a^2(\chi_h)sin^2(\chi_h)}$$

This saturates for $\chi_h = \pi/2$ at maximal expansion of the closed FLRW cosmology [5].

The inflationary model with a scalar inflaton field ϕ and potential $V(\phi)$ evolves as $\dot\rho = 3(\rho + p) = 3(1+w)\rho$, which gives the equation of state parameter $w = -1$ for $\dot\rho = 0$. The inflaton field ϕ evolves as

$$\nabla^2\phi \ - \ \dot\phi \ - \ 3H\dot\phi \ - \ \frac{\partial V(\phi)}{\partial\phi} \ = \ 0,$$

for the Hubble parameter $H \ = \ \dot a/a$. Isotropy $\nabla^2\phi \ \simeq \ 0$ simplifies this to $\dot\phi \ + \ 3H\dot\phi \ + \ \frac{\partial V(\phi)}{\partial\phi} \ = \ 0$. The scale factor obeys

$$\Big(\frac{\dot a}{a}\Big)^2 \ = \ H^2 \ = \ H_0^2\Big[\Omega_\phi\Big(\frac{a_0}{a}\Big)^{3(1+w)} \ + \ \Omega_m\Big(\frac{a_0}{a}\Big)^4 \ - \ \Omega_k\Big(\frac{a_0}{a}\Big)^2\Big],$$

where a_0 and H_0 are the initial scale factor and Hubble parameter. Ω_ϕ, Ω_m and Ω_k are the density ratios for the inflaton field, relativistic matter or radiation and the spatial curvature for $k = 1$. The equation of state parameter $-1 < w < -1/3$ are limits for inflation and k-dynamics due to spatial curvature. Inflation occurs as $w \ \rightarrow \ -1$ as Ω_ϕ grows larger than Ω_k which vanishes. The total density $\Omega \ = \ \Omega_\phi \ + \ \Omega_m \ = \ 1 \ + \ \Omega_k \ \simeq \ 1$ is sufficiently close to unity for $\Omega_k \ = \ \rho_{k0}/\rho_c \ = \ 3M_{pl}^2/8\pi(a_0H_0)^2 \ << \ 1$. The $k \ = \ 1$ FLRW for closed spacetime such that Ω_k, no matter how small, saturates the HP at the turn about in the cosmic evolution.

2 The three phases

The earliest phase is the pre-inflationary phase given by the FLRW equation

$$\Big(\frac{\dot a}{a}\Big)^2 = H^2 \ = \ \frac{8\pi G}{3M_p^2}\Big(\frac{a_0}{a}\Big)^4 \ - \ \frac{1}{a^2},$$

for a spherical space containing relativistic particles. The spatial dynamics of the space is $a(\chi_h) \ = \ Asin(\chi_h)$ for $A \ = \ \sqrt{8\pi Ga_0^4/3M_p^2}$. This solution for

$\chi_h \to \pi$ where $S/A \to \infty$, where the singularity at $\chi_h = 0$ is precluded by the Planck scale cut-off in a_0.

During this phase $\Omega_k >> \{\Omega_m,\ \Omega_\phi\}$ and $\Omega_k \lesssim 1$ the scale factor dynamics is

$$\dot{a} \simeq a_0 H_0 \sqrt{\Omega_k}$$

with

$$\chi_h \simeq \frac{1}{a_0 H_0 \sqrt{\Omega_k}} ln\left(\frac{t}{t_0}\right),$$

for $t_0 \sim t_p$. Assume the cutoff distance is the string length $a_0 \simeq \ell_s$ and the Hubble inverse time $H_0 \simeq 1/\ell_p$, for ℓ_p the Planck length. Thus $a_0 H_0 \sim g^{-1/4}$, for g the string coupling constant. The condition $\chi_h = \pi/2$ for maximum expansion corresponds to the time

$$t = t_0 exp(\chi_h a_0 H_0 \sqrt{\Omega_k}) = t_0 exp(\pi/2g^{1/4})$$

The string parameter $g^{1/4} \simeq .06$ computes $t = 2.3 \times 10^{11} t_p$, or $t \simeq 2 \times 10^{-32} sec$, which agrees with current models. The S/A ratio is

$$\frac{S}{A}\Big|_{\chi_h=\pi/2} = \sigma \frac{\pi}{4a_{max}^2},$$

for σ the constant entropy density. The entropy is given directly by entropy density $S_m = \pi\sigma$. The derivative of this function at $\chi_h = \pi/2$ is

$$\frac{d}{d\chi_h}\frac{S}{A}\Big|_{\pi/2} = \frac{\sigma}{a_{max}^2} = \frac{4}{\pi}\frac{S}{A}\Big|_{\pi/2},$$

giving the entropy bound at the maximal expansion $S = kA/4\ell_p^2$, the saturation point, where continued evolution surpasses the Bekenstein bound and S/A diverges.

The cosmology at $t \sim 10^{-35} sec$ gives a Bekenstein bound of $\sim 10^{16}$ bits of information, or $S \sim 10^{-7} J/K$. The Boltzmann factor $E/kT \sim 10^{16}$ equals the quantum phase $E\tau/\hbar$, for τ Euclidean time. For the temperature approximately 10^{-8} times the Hagedorn temperature $\tau = \hbar/kT \sim 10^{-34} sec$, which corresponds to the onset of inflation and a phase transition. At this point the energy density of the inflaton field became larger than the curvature; pre-inflation $\rho_k >> \rho_\phi$ and inflationary $\rho_\phi >> \rho_k \sim\to\sim 0$. The strict $\Omega_k = 0$ preserves the HP, and the universe converts from $\mathbb{S}^3$ to $\mathbb{R}^3$. The geometry

$$ds^2 = -dt^2 + a^2(t) d\Omega_3^2,$$

transitions to one of greater symmetry. The scale factor $a(t) = sin(t/t_0)$ is replaced with $a(t) = cosh(t\Lambda/3)$ for a de Sitter spacetime.

The HP protection is a quantum critical phase transition. The FLRW equation

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2 \;=\; \frac{8\pi G}{3}\rho \;-\; \frac{1}{a^2},$$

for a closed spherical spacetime predicts a large H for $a \geq 0$. The gravity action $S = \int d^4x\sqrt{-g}R$ is

$$S \;=\; \int d^4x\sqrt{g}\left(\frac{\partial g^{ij}}{\partial t}\mathcal{G}_{ijkl}\frac{\partial g^{kl}}{\partial t} \;-\; \frac{\delta W}{\delta g_{ij}}\mathcal{G}_{ijkl}\frac{\delta W}{\delta g_{kl}}\right)$$

with the superspace metric,

$$\mathcal{G}_{ijkl} \;=\; \frac{1}{2}g^{-1/2}(g_{ik}g_{jl} \;+\; g_{il}g_{jk} \;-\; g_{ij}g_{kl}).$$

The Ricci flow for the manifold is

$$\frac{\partial g^{ij}}{\partial t} \;=\; 2R^{ij} \;+\; \nabla^i N^j \;+\; \nabla^j N^i,$$

and $\partial W/\partial g^{ij}$ constructs a potential term. The metric for the FLRW gives the relevant superspace metric components

$$\mathcal{G}_{iiii} \;=\; -\mathcal{G}_{iijj} \;=\; \frac{1}{2}a.$$

The metric time derivative is $\partial g^{ii}/\partial t \;=\; \partial a^{-2}/\partial t = \;-2(\dot{a}/a^3) \;=\; 2R^{ii}$ and the kinetic energy is $K \;=\; -6(\dot{a}/a)^2a^{-3}$. The quadratic term $(\delta W(g_{ij})/\delta g_{ij})^2$ $=\; 4(1 \;-\; a^2/a_0^2)$ gives the potential

$$\frac{\delta W}{\delta g^{ij}}\mathcal{G}^{ijkl}\frac{\delta W}{\delta g^{kl}} \;=\; -6a^{-1}\left(1 \;-\; \frac{a^2}{a_0^2}\right).$$

The kinetic and potential energy give the Hamiltonian operator, and the irrelevant factor of 1/6 is ignored. The Hamiltonian acts on the wave functional $\Psi[g]$ as

$$\left(\frac{\dot{a}}{a}\right)^2 \;+\; a^2\left(1 \;-\; \frac{a^2}{a_0^2}\right) \;\rightarrow\; \left[\hat{\pi}_a^2 \;+\; a^2\left(1 \;-\; \frac{a^2}{a_0^2}\right)\right]\Psi[a] \;=\; 0,$$

for $\dot{a}/a \;\rightarrow \hat{\pi}_a \;=\; -i\partial_a$, which gives this Wheeler-Dewitt equation. The numerical solution for $\Psi[a]$ for $k \;=\; 1$ and $\Omega_k \;\lessapprox\; \Omega$ appears in figure 1. The inflationary phase resets the potential term with constant ρ, which defines a de Sitter space with a greater symmetry and where $\mathbb{S}^3 \;\rightarrow\; \mathbb{R}^3$.

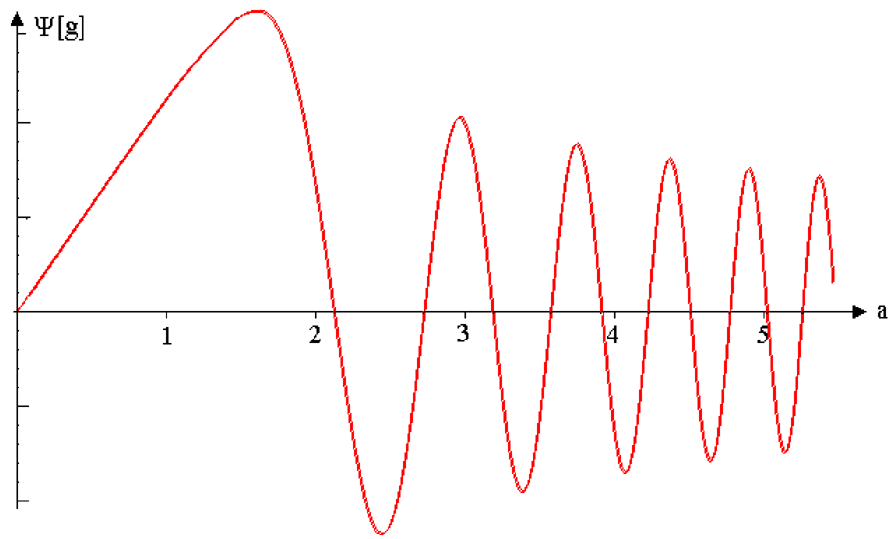

Figure 1. The solution to the Wheeler–DeWitt equation for the k = 1 homogenous, isotropic closed universe.

The transition from "k-dynamics," with $w = -1/3$ with equation of state $p = w\rho$, to the inflaton field, with $w = -1$ means the scale factor is replaced with $a(t) = cosh(Ht)/H$. The inflationary metric expands as $\sim e^{2\phi}g_{ij}$ with the inflaton field ϕ. The Ricci flow $\partial g_{ij}/\partial t = 2\dot\phi g_{ij}$ gives the dynamical equation $\ddot a/a = 4\pi G(\rho + 3p)$. The potential function, determined by W above is a polynomial form, typically $\frac{1}{2}m\phi^2$. The potential function for the evolution with a slowly varying ϕ $\partial_\phi V(\phi) \simeq H^2 = \Lambda/3$, is approximately constant $8\pi G\rho$. The "friction term" $3H\dot\phi$ due to dilution the scalar field with the wave equation in equation 4, so $\dot\phi < 0$. A slowly varying potential defines a de Sitter spacetime in the inflationary phase of the system.

The next phase of the system is reheating, where $\phi \simeq 0$. This is the period of particle creation and the thermalization of the universe. This is the thermal "bang" in the big bang, after inflation that lasts $t \simeq 10^{-35}$ seconds, with cosmic expansion under the influence of relativistic particle or radiation with $w = 1/3$. This phase is a bubble nucleation of a pocket universe in the de Sitter inflating space. The cosmic reheating and particle production governs inflaton decay

$$\ddot\phi + 3H\dot\phi + \partial_\phi V(\phi) = -\Gamma\dot\phi,$$

for Γ the rate of particle production. The dilution of radiation matter in the universe eventually pushes the universe into a $w = -1$ equation of state after the radiation and matter dominated periods.

The three phases are from a Lifshitz tricritical quantum critical point, such as seen in condensed matter physics [6]. The first corresponds to a dynamical ϕ, which is the pre-inflationary phase. The next is the high potential phase or $\rho(\phi) \sim 10^{72}GeV^4$ and the last phase with small density $\rho(\phi) \sim 10^{-48}GeV^4$, corresponding to the current cosmological constant. This Lifshitz triple point is illustrated below, where the ordinate and abscissa are energy (temperature) and pressure. The system evolves from the top left as the closed cosmology in the pre-inflationary phase, to the phase corresponding to inflation, and the potential collapses in the bottom right which produces particles in the reheating phase. The scale of the diagram and the path drawn are heuristics.

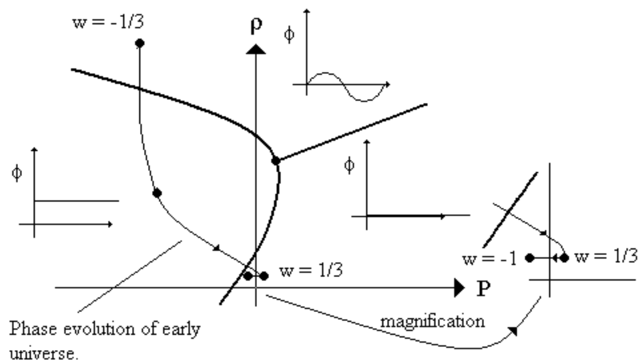

Figure 2. The tricritical point and the early evolution of the cosmological phase.

3 Quantum tricritical phases

The HP violation is avoided by a quantum critical point and a phase transition to the inflationary phase, which enters then into another phase, which is the reheating of the universe and the current cosmic state. The Lifshitz tricritical point connects the scalar field ϕ in a spatially modulated phase with the phases $\phi = const$ and $\phi \gtrsim 0$. The spatially modulated phase is the scalar or inflaton field in the closed spacetime configuration. The next phase is a large vacuum energy with $\phi \simeq const$, and inflation. The final transition to $\phi \gtrsim 0$ reheating phase is bubble nucleation of pocket universes. The tricritical point in connection to relativity is explored by Horava[7], with the reparameterizations $t \rightarrow b^z t$ and $x \rightarrow bx$, for z the critical exponent, which give conformal or Ricci flow $g_{ij} \rightarrow e^{2\phi} g_{ij}$, for $z = 1$.

The ground state wave functional $\Psi[\phi(x)] = exp(-W/2)$ has dynamics given by

$$S = \frac{1}{2}\int dtd^n x\Big(\dot{\phi}^2 - (\frac{1}{2}\Delta\phi)^2\Big).$$

The functional W for k-dynamics gives the critical exponent $x \rightarrow b^{[\phi]}x$ is then $[\phi] = 0$. This is the spatially modulated phase determined by W. The inflationary phase with a changed critical dimension and conformal flow $x_i \rightarrow x_i e^{\phi}$. The critical exponent $x_i \rightarrow x_i b^{[x]}$ defines the factor $b = e^{2\phi}$ scale factor with scale weight $[x] = 1$. The conversion of the closed sphere $\mathbb{S}^3$ to $\mathbb{R}^3$ in a simple model is a stereographic projection. Cartesian coordinates of $\mathbb{S}^3$, such at(x_1, x_2, x_3, z) for z some fictional embedding spatial coordinate, map as $x_i \rightarrow x_i/(1 - z)$.

We express the coordinate z according to a field ϕ as $1/(1 - z) = e^{\phi}$ so conformal dynamics is a manifestation of topological change $\mathbb{S}^3 \to \mathbb{R}^3$.

The renormalization group flow follows from a conformal rescaling, which has a rich structure stemming from Zamolodchikov [8] and the Cardy a-conjecture [9]. The Ricci flow $\partial g_{ij}/\partial t = 2\dot{\phi} g_{ij}$ computes the kinetic energy term

$$\frac{\partial g_{ij}}{\partial t}\mathcal{G}^{ijkl}\frac{\partial g_{kl}}{\partial t} = 4\left(\frac{\partial \phi}{\partial t}\right)^2 g_{ij}g_{kl}\mathcal{G}^{ijkl} = \frac{k}{2}\left(\frac{\partial \phi}{\partial t}\right)^2$$

for $k = 2g_{ij}g_{kl}\mathcal{G}^{ijkl}$ and ϕ rescaled to $\phi/2$. The potential energy term is

$$\frac{\delta W}{\delta g^{ij}} = \frac{\delta W}{\delta \phi}\frac{\partial \phi}{\partial g^{ij}} = \frac{1}{2}\frac{\delta W}{\delta \phi}g_{ij}$$

so

$$\frac{\delta W}{\delta g^{ij}}G^{ijkl}\frac{\delta W}{\delta g^{kl}} = \frac{1}{4}\left(\frac{\delta W}{\delta \phi}\right)^2 g_{ij}g_{kl}G^{ijkl} = \frac{k}{2}\left(\frac{\delta W}{\delta \phi}\right)^2.$$

The Lagrangian

$$\mathcal{L} = \frac{k}{2}\left(\frac{\partial \phi}{\partial t}\right)^2 - \frac{k}{2}\left(\frac{\delta W}{\delta \phi}\right)^2.$$

with the functional $W = W[\phi]$ of the form $\dot{\phi}^2 - (\frac{1}{2}\Delta\phi)^2$ for $\Delta\phi = P(\phi)$, a polynomial so

$$\mathcal{L} = \frac{k}{2}\left(\frac{\partial \phi}{\partial t}\right)^2 - k\sum_n \frac{a_n}{n}\phi^n.$$

A possible first choice of a potential is

$$\left(\frac{\delta W}{\delta \phi}\right)^2 = V(\phi) = \frac{a_2}{2}\phi^2 + \frac{a_4}{4}\phi^4$$

which is zero for $\phi = 0$ and $\phi = \sqrt{-2a_2/a_4}$ for $a_2 < 0$. The Thomas-Fermi approximation [10] [11] for phase transitions employs a cubic term

$$V(\phi) = \frac{a_2}{2}\phi^2 - \frac{a_3}{3}\phi^3 + \frac{a_4}{4}\phi^4$$

For $a_2 < a_3^2/4a_4$ there is a third minimum

$$\phi_0 = \frac{a_3}{2a_4}\left(1 + \sqrt{1 - 4a_2a_4/a_3^2}\right)$$

The potential for different values of a_3 appears in figure 3

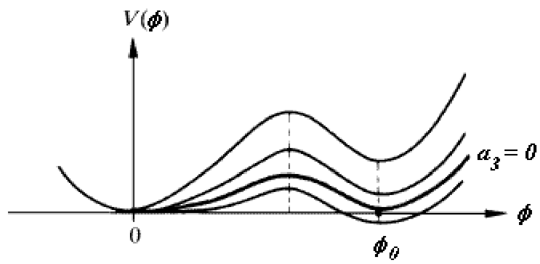

Figure 3. The Fermi-Thomas potential for the polynomial function of the field ϕ.

The field exhibits a phase transition jump to $\phi = 0$ from $\phi_0 = 2a_3/3a_4$. The coherence length of the ϕ-fluctuations for $a_3 = 0$ is computed from the nonzero expectation of the field $\langle\phi\rangle = \phi_0 = \sqrt{-a_2/a_4}$, which corresponds to a Higgs-like mass, and the coherence length $\xi = 1/\sqrt{-2a_2}$[12]. In superconductivity this connects the Meissner-Higgs mass term to the penetration depth of a magnetic field $\lambda \sim 1/\phi$. The ratio of the two length scales is the Ginsburg parameter $k = \lambda/\sqrt{2}\xi$, which is large for type I superconductors and small for type II. Including the cubic term the coherence length is

$$\xi = \frac{1}{\sqrt{a_2 + 3a_4\phi^2 - 2a_3\phi}},$$

which around $\phi = 0$ is $\xi = 1/\sqrt{a_2}$. For $a_3 = 0$ the field jumps to $\phi_0 = \sqrt{2a_3/3a_4}$ and the ϕ-fluctuation length jumps to

$$\xi_1 = \frac{3}{a_3}\sqrt{\frac{a_4}{2}}$$

corresponding to a first order phase transition.

The expected field $\phi_0 = \langle\phi\rangle = \sqrt{2a_3/3a_4}$ is the critical parameter for inflation and HP protection. The metric at this critical value $g_{ij} = exp(2\phi_0) \sim a(t')$ cuts off at the value of $\pi/2 < \theta < \pi$ where the entropy bound is saturated. The phase change initiates inflation or de Sitter dynamics with a large vacuum energy density and pressure.

Transition to the third phase with $\phi \gtrsim 0$ causes production of particles. The vacuum energy becomes very small, with inflationary expansion with a larger e-fold time $t_e = 1/H$. This is bubble nucleation [13] transpires on the de Sitter spacetime inflating on a time scale $t \simeq \ell_s$, where each bubble subsequently expands on a time scale $1/H$. This is the first multiverse scenario where the space $\mathbb{R}^3$ rapidly inflates and local regions tunnel into a lower energy configuration. These regions define pocket universes within the Guth-Linde-Vilenkin multiverse scenario [14][15].

4 Final Remarks

This potentially connects "braney" physics with the standard cosmological model. A cosmology is then the stringy content of a D3-brane. Some models have cosmologies generated by the collision between two branes [16]. However, it may be quite the opposite. It is possible that the elementary particles in our universe are truly point-like according to zero electric dipole moments, as experiments are beginning to suggest, with dipole moment $0.07 \pm 0.07 \times 10^{-26} e \cdot cm$[17], which is close to some SUSY predictions of an electric dipole. The strings for these particles connect two D3-branes. We might think of some field flux which causes open strings to connect two branes together, instead of both endpoints on a single brane. The branes may be dynamical and are moving apart, which stretches the string. The string may break, but rather than having free endpoints they are attached to a new braney object. This object has the topology of a sphere, which transforms its topology to $\mathbb{R}^3$. Consider a foliation of D3-branes in expanding bulk space which pulls branes apart. As teh brane foliation separates gaps are filled by the generation of new branes, which contain nascent inflationary cosmologies, which in turn generate pocket universes. The tricritical phase transitions are associated with both the generation of the brane and subsequent bubble nucleation on it.

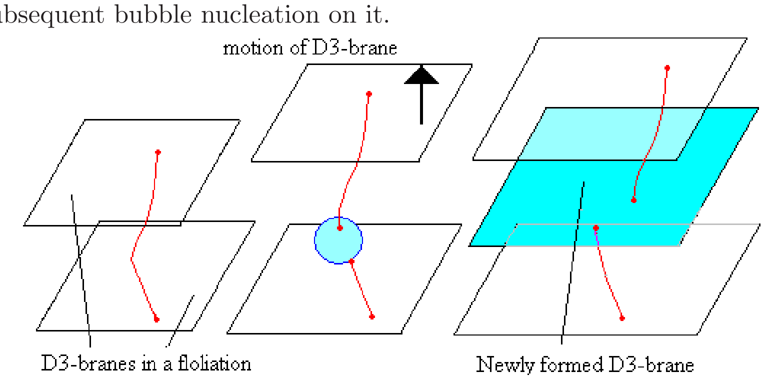

Figure 4. An M-theory or brane interpretation of the tricritical point.

Dp-branes have a large $N\hbar$ content and are classical. The emergent small $\mathbb{S}^3$ with the end points of the cut string has more quantum mechanical content. $\mathbb{S}^3 \rightarrow \mathbb{R}^3$ is a transition to classical mechanics for the large D3-brane. The tricritical phase transition from a varying field to constant field transforms $\mathbb{S}^3$ into a classical spacetime $\mathbb{R}^3$. This middle phase is the inflationary period of the universe. The final phase is bubble nucleation and the formation of pocket universes. This reheating phase produces the radiation, matter and small accelerated phases of the universe we observe.

The reheating phase of the universe breaks the Weyl transformation or conformal scaling of the inflationary period. The large inflationary Λ is reduced to near zero, which accompanies the breaking of conformal symmetry if $\Lambda = 0$ identically. The dilaton in this setting emerges in vacua with spontaneous broken conformal symmetry. However, physically it is not identically zero. It is possible high energy scale physics of inflation is dual to the low energy physics of the cosmological constant of the current universe in a T-duality or $k \rightarrow 1/k$. The high energy physics is $k \sim M_{pl}$, and the low energy physics is $k \rightarrow K^2/k \sim M_\Lambda = 10^{-12} GeV$. The implicit constant at work here is the intermediary momentum or mass $K = \sqrt{M_\Lambda M_{pl}} \sim 10^3 GeV$, or about the LHC energy scale. The dual when written according to proper units is $M_{pl} \rightarrow K^2/M_{pl} \sim M_\Lambda$. This is the value of the vacuum energy for the universe and the cosmological constant. This suggests some underlying UV/IR duality to renormalization group flow.

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