A Finite Reflection Formula For A Polynomial Approximation To The Riemann Zeta Function

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June 23, 2012

Abstract. The Riemann zeta function can be written as the Mellin transform of the unit interval map $w(x) = \lfloor x^{-1} \rfloor (x \lfloor x^{-1} \rfloor + x - 1)$ multiplied by $s\frac{s+1}{s-1}$. A finite-sum approximation to $\zeta(s)$ denoted by $\zeta_w(N; s)$ which has real roots at $s = -1$ and $s = 0$ is examined and an associated function $\chi(N; s)$ is found which solves the reflection formula $\zeta_w(N; 1 - s) = \chi(N; s)\zeta_w(N; s)$. A closed-form expression for the integral of $\zeta_w(N; s)$ over the interval $s = -1...0$ is given. The function $\chi(N; s)$ is singular at $s = 0$ and the residue at this point changes sign from negative to positive between the values of $N = 176$ and $N = 177$. Some rather elegant graphs of $\zeta_w(N; s)$ and the reflection functions $\chi(N; s)$ are also provided. The values $\zeta_w(N; 1 - n)$ for integer values of n are found to be related to the Bernoulli numbers.

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1. The Riemann Zeta Function as the Mellin Transform of a Unit Interval Map

The Riemann zeta function can be written as the Mellin transform of the unit interval map $w(x) = \lfloor x^{-1} \rfloor (x \lfloor x^{-1} \rfloor + x - 1)$ multiplied by $s\frac{s+1}{s-1}$. [2]

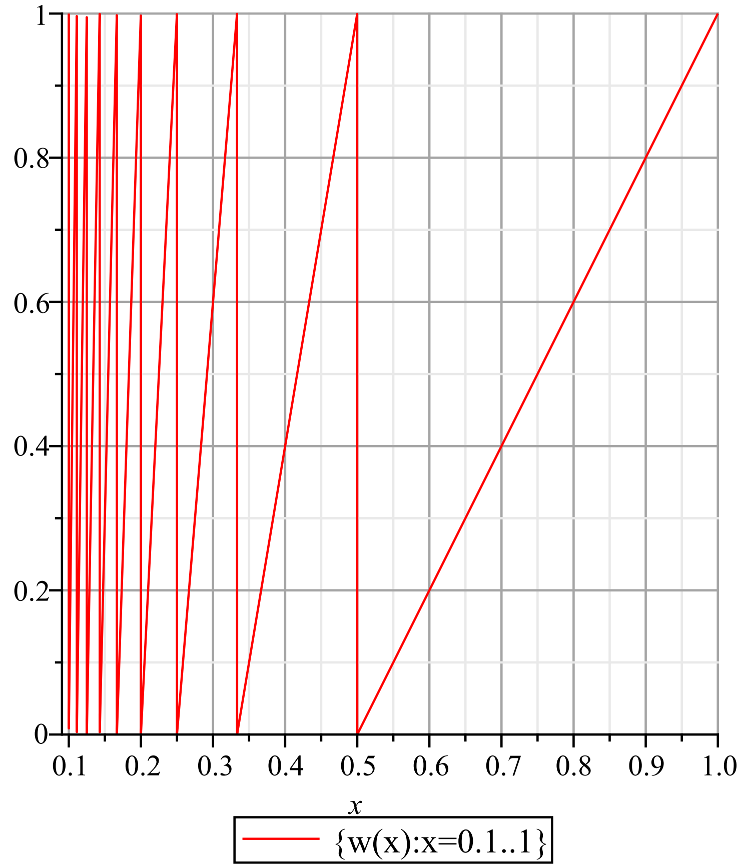

Figure 1. The Harmonic Sawtooth map

$$\begin{aligned}\zeta_w(s) &= \zeta(s) \forall -s \notin \mathbb{N}^* \\ &= s\frac{s+1}{s-1}\int_0^1 \lfloor x^{-1} \rfloor (x \lfloor x^{-1} \rfloor + x - 1) x^{s-1} \mathrm{d}x \\ &= s\frac{s+1}{s-1}\sum_{n=1}^{\infty} \int_{\frac{1}{n+1}}^{\frac{1}{n}} n(xn + x - 1) x^{s-1} \mathrm{d}x \\ &= \sum_{n=1}^{\infty} s\frac{s+1}{s-1}\left(-\frac{n^{1-s} - n(n+1)^{-s} - sn^{-s}}{s(s+1)}\right) \\ &= \sum_{n=1}^{\infty} \frac{n(n+1)^{-s} - n^{1-s} + sn^{-s}}{s-1} \\ &= \frac{1}{s-1}\sum_{n=1}^{\infty} n(n+1)^{-s} - n^{1-s} + sn^{-s}\end{aligned} \tag{1}$$

1.1. The Truncated Zeta Function.

The substition $\infty \to N$ is made in the infinite sum appearing the expression for $\zeta_w(s)$ to get a finite polynomial approximation

$$\begin{aligned}\zeta_w(N;s) &= \frac{1}{s-1}\sum_{n=1}^{N} n(n+1)^{-s} - n^{1-s} + s n^{-s}\\ &= \frac{1}{s-1}\left(s + (N+1)^{1-s} - 1 + s\sum_{n=2}^{N} n^{-s} - \sum_{n=2}^{N+1} n^{-s}\right)\\ &= \frac{N}{(s-1)(N+1)^s} - \frac{\cos(\pi s)\,\Psi(s-1,N+1)}{\Gamma(s)} + \zeta(s)\forall s\in\mathbb{N}^*\end{aligned} \tag{2}$$

with equality in the limit except at the negative integers

$$\lim_{N\to\infty}\zeta_w(N;s) = \zeta(s)\forall -s\notin\mathbb{N}^* \tag{3}$$

The functions $\zeta_w(N;s)$ have real zeros at $s=-1$ and $s=0$, that is

$$\lim_{s\to-1}\zeta_w(N;s) = \lim_{s\to 0}\zeta_w(N;s) = 0 \tag{4}$$

One possible idea is that the functions $\zeta_w(N;s)$ can be orthonormalized over the interval $s=-1...0$ via the Gram-Schmidt process[3] and that the result might possibly shed some light on the zeroes of $\zeta(s)$. Let the logarithmic integral be defined

$$\mathrm{Li}(x) = \int_0^{\ln(x)} \frac{e^y - 1}{y}\mathrm{d}y + \ln(\ln(x)) + \gamma \tag{5}$$

where $\gamma = 0.57721...$ is Euler's constant, then the normalization factors are given by the integral

$$\begin{aligned}\int_{-1}^{0}\zeta_w(N;s)\mathrm{d}s &= \int_{-1}^{0}\sum_{n=1}^{N}\frac{n(n+1)^{-s} - n^{1-s} + s n^{-s}}{s-1}\mathrm{d}s\\ &= 1 + \frac{N}{N+1}(\mathrm{Li}(N+1) - \mathrm{Li}((N+1)^2)) + \sum_{n=1}^{N-1}\frac{n}{\ln(n+1)}\end{aligned} \tag{6}$$

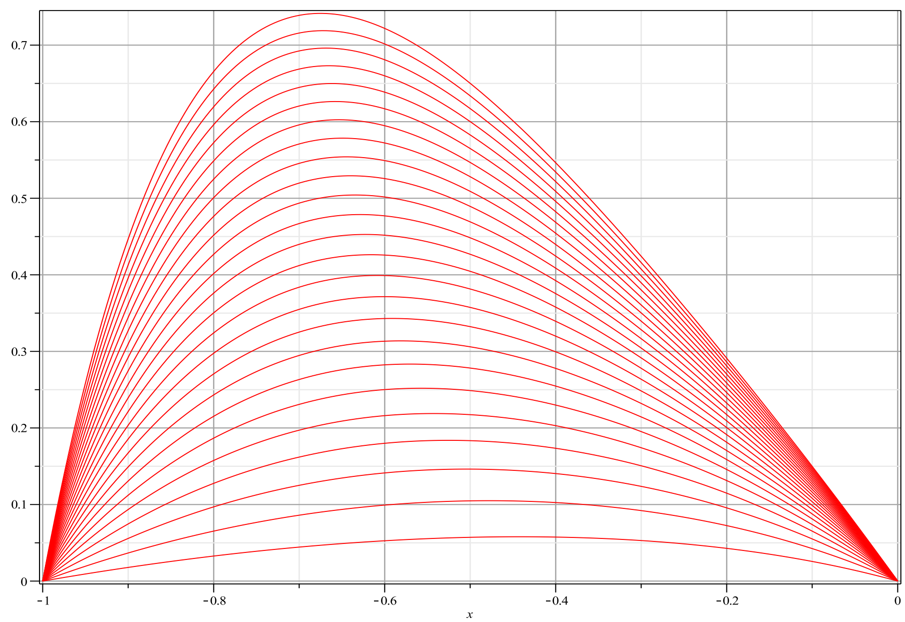

Figure 2. $\{\zeta_w(N;s)\colon s=-1...0, N=1...25\}$

The following table lists the values of $\zeta_w(N;1-n)$ for $n=2...12$.

$$\left[\begin{array}{c} 0 \\ -\frac{1}{6}N-\frac{1}{6}N^2 \\ -\frac{1}{4}N-\frac{1}{2}N^2-\frac{1}{4}N^3 \\ -\frac{7}{30}N-\frac{4}{5}N^2-\frac{13}{15}N^3-\frac{3}{10}N^4 \\ -\frac{1}{6}N-\frac{11}{12}N^2-\frac{5}{3}N^3-\frac{5}{4}N^4-\frac{1}{3}N^5 \\ -\frac{5}{42}N-\frac{6}{7}N^2-\frac{97}{42}N^3-\frac{20}{7}N^4-\frac{23}{14}N^5-\frac{5}{14}N^6 \\ -\frac{1}{8}N-\frac{19}{24}N^2-\frac{21}{8}N^3-\frac{14}{3}N^4-\frac{35}{8}N^5-\frac{49}{24}N^6-\frac{3}{8}N^7 \\ -\frac{13}{90}N-\frac{8}{9}N^2-\frac{26}{9}N^3-\frac{56}{9}N^4-\frac{371}{45}N^5-\frac{56}{9}N^6-\frac{22}{9}N^7-\frac{7}{18}N^8 \\ -\frac{1}{10}N-\frac{21}{20}N^2-\frac{18}{5}N^3-\frac{79}{10}N^4-\frac{63}{5}N^5-\frac{133}{10}N^6-\frac{42}{5}N^7-\frac{57}{20}N^8-\frac{2}{5}N^9 \\ -\frac{1}{66}N-\frac{10}{11}N^2-\frac{101}{22}N^3-\frac{120}{11}N^4-\frac{199}{11}N^5-\frac{252}{11}N^6-\frac{221}{11}N^7-\frac{120}{11}N^8-\frac{215}{66}N^9-\frac{9}{22}N^{10} \\ -\frac{1}{12}N-\frac{1}{2}N^2-\frac{55}{12}N^3-\frac{121}{8}N^4-\frac{55}{2}N^5-\frac{110}{3}N^6-\frac{77}{2}N^7-\frac{231}{8}N^8-\frac{55}{4}N^9-\frac{11}{3}N^{10}-\frac{5}{12}N^{11} \end{array}\right] \tag{7}$$

1.1.1. Integrating Over the Critical Strip.

There is a formula similiar to (6) which gives the integral of $\zeta_w(N;s)$ over the critical strip $0\leqslant \mathrm{Re}(s)\leqslant 1$.

$$\int_0^1 \zeta_w(N;c+is)\mathrm{d}c \;=1+\frac{N}{N+1}(\mathrm{Ei}_1(is\ln(N+1)-\ln(N+1))-\mathrm{Ei}_1(is\ln(N+1)))+\sum_{n=1}^{N-1}\frac{n(n+1)^{-is}}{(n+1)\ln(n+1)} \tag{8}$$

where $\mathrm{Ei}_1(t)$ is the exponential integral defined by

$$\mathrm{Ei}_1(t) = t\int_0^1 \int_0^1 e^{-txy}\mathrm{d}y\mathrm{d}x - \gamma - \ln(t) \tag{9}$$

The contribution from the Ei term vanishes as $s \to \infty$, that is

$$\lim_{s\to\infty} \frac{N}{N+1}(\mathrm{Ei}_1(is\ln(N+1) - \ln(N+1)) - \mathrm{Ei}_1(is\ln(N+1))) = 0 \tag{10}$$

1.1.2. The Reflection Formula.

There is a reflection equation for the finite-sum approximation $\zeta_w(N; s)$ which is similiar to the well-known formula $\zeta(1-s) = \chi(s)\zeta(s)$ with $\chi(s) = 2(2\pi)^{-s}\cos\left(\frac{\pi s}{2}\right)\Gamma(s)$. The solution to

$$\zeta_w(N; 1-s) = \chi(N; s)\zeta_w(N; s) \tag{11}$$

is given by the expression

$$\begin{aligned}\chi(N; s) &= \frac{\zeta_w(N; 1-s)}{\zeta_w(N; s)} \\ &= \frac{\sum_{n=1}^N -\frac{-n^s + (n+1)^{s-1}n + n^{s-1} - n^{s-1}s}{s}}{\sum_{n=1}^N \frac{-n^{1-s} + (n+1)^{-s}n + n^{-s}s}{s-1}} \\ &= -\frac{(s-1)\sum_{n=1}^N -n^s + (n+1)^{s-1}n + n^{s-1} - n^{s-1}s}{s\sum_{n=1}^N -n^{1-s} + (n+1)^{-s}n + n^{-s}s}\end{aligned}$$

which satisfies

$$\chi(N; 1-s) = \chi(N; s)^{-1} \tag{12}$$

The functions $\chi(N; s)$, indexed by N, have singularities at $s = 0$. Let

$$\begin{aligned} a(N) &= \sum_{n=1}^N n(\ln(n+1) - \ln(n)) \\ b(N) &= \sum_{n=1}^N \frac{\ln(n)n^2 - \ln(n+1)n^2 - \ln(n)}{n(n+1)} \\ c(N) &= \frac{1}{2}\sum_{n=1}^N n(\ln(n+1)^2 - \ln(n)^2) \end{aligned} \tag{13}$$

then the residue at the singular point $s = 0$ is given by the expression

$$\begin{aligned}\operatorname*{Res}_{s=0}(\chi(N; s)) &= -\operatorname*{Res}_{s=1}(\chi(N; s)^{-1}) \\ &= \frac{1 + \gamma + \Psi(n+2) - \frac{2}{N+1} + b(N) - \frac{N(\ln(\Gamma(N+1)) - c(N))}{(N - a(N))(N+1)}}{a(N) - N} \\ &= \frac{1 + \gamma + \Psi(n+2) - \frac{2}{N+1} + \sum_{n=1}^N \frac{\ln(n)n^2 - \ln(n+1)n^2 - \ln(n)}{n(n+1)} - \frac{N\left(\ln(\Gamma(N+1)) - \frac{1}{2}\sum_{n=1}^N n(\ln(n+1)^2 - \ln(n)^2)\right)}{(N - \sum_{n=1}^N n(\ln(n+1) - \ln(n)))(N+1)}}{\left(\sum_{n=1}^N n(\ln(n+1) - \ln(n))\right) - N}\end{aligned} \tag{14}$$

which has the limit

$$\lim_{N\to\infty} \operatorname*{Res}_{s=0}(\chi(N; s)) = 1 \tag{15}$$

We also have the residue of the reciprocal at $s=2$

$$\operatorname*{Res}_{s=2}(\chi(N;s)^{-1}) = \frac{\frac{2N}{(N+1)^2} - 2\Psi(1,N+1) + 2\zeta(2)}{\frac{(N+1)^2}{2} - \frac{N}{2} - \frac{1}{2} - \sum_{n=1}^{N} n(\ln(n+1) + \ln(n+1)n - \ln(n) - n\ln(n))} \tag{16}$$

which vanishes as N tends to infinity

$$\lim_{N\to\infty} \operatorname*{Res}_{s=2}(\chi(N;s)^{-1}) = 0 \tag{17}$$

As can be seen in the figures below, the residue at $s=0$ changes sign from negative to positive between the values of $N=176$ and $N=177$.

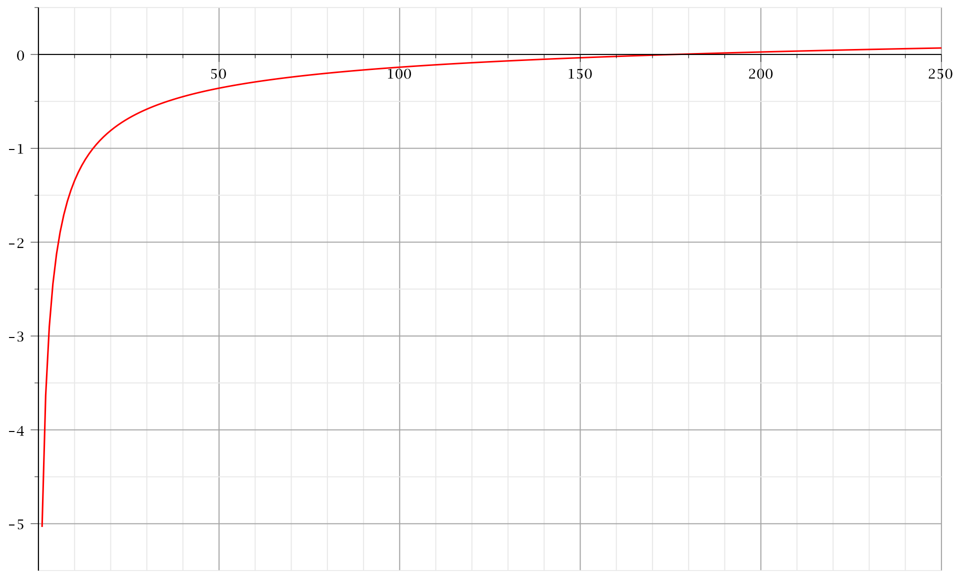

Figure 3. $\left\{\operatorname*{Res}_{s=0}(\chi(N;s)) \colon N = 1...250\right\}$

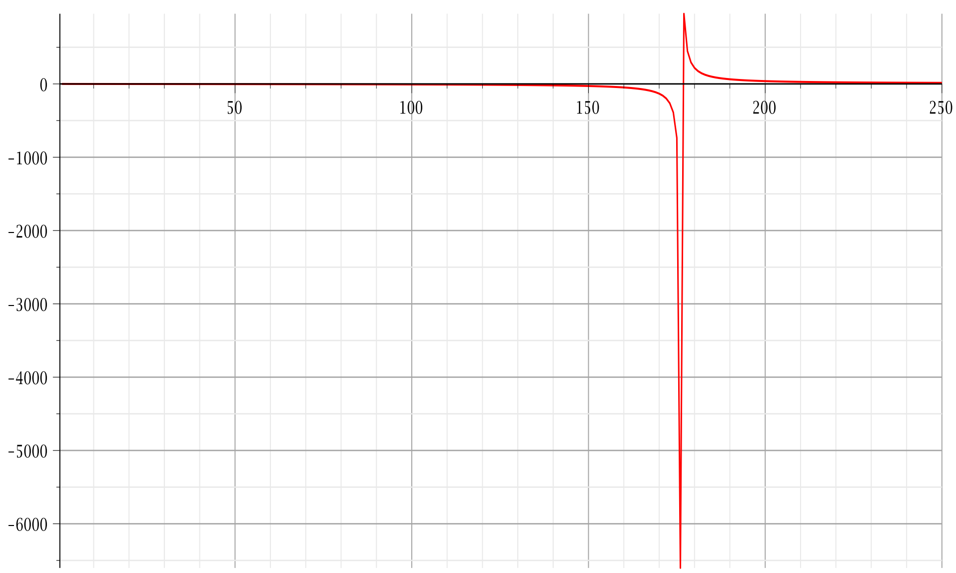

Figure 4. $\left\{\operatorname*{Res}_{s=0}(\chi(N;s))^{-1} \colon N = 1...250\right\}$

For any positive integer N, we have the limits

$$\begin{aligned} \lim_{s\to 0}\chi(N;s) &= \infty \\ \lim_{s\to 0}\frac{\mathrm{d}^n}{\mathrm{d}s^n}\chi(N;s) &= \infty \\ \lim_{s\to\frac{1}{2}}\chi(N;s) &= 1 \\ \lim_{s\to 1}\chi(N;s) &= 0 \\ \lim_{s\to 2}\chi(N;s) &= 0 \\ \lim_{s\to 1}\frac{\mathrm{d}}{\mathrm{d}s}\chi(N;s) &= 0 \end{aligned} \tag{18}$$

The line $\mathrm{Re}(s)=\frac{1}{2}$ has a constant modulus

$$\left|\chi\left(N;\frac{1}{2}+i s\right)\right| = 1 \tag{19}$$

There is also the complex conjugate symmetry

$$\chi(N;x+i y) = \overline{\chi(N;x-i y)} \tag{20}$$

If $s=n\in\mathbb{N}^*$ is a positive integer then $\chi(N;n)$ can be written as

$$\begin{aligned} \chi(N;n) &= \frac{\zeta_w(N;1-n)}{\zeta_w(N;n)} \\ &= \frac{\sum_{m=1}^{N} -\sum_{k=1}^{n-2}\frac{m^k}{n}\binom{n-1}{k-1}}{\frac{N}{(n-1)(N+1)^n} - \frac{\cos(\pi n)\Psi(n-1,N+1)}{\Gamma(n)} + \zeta(n)} \\ &= \frac{-\sum_{m=1}^{N}\frac{1}{n}((n-1)m^{n-1} + m^n - (m+1)^{n-1}m)}{\frac{N}{(n-1)(N+1)^n} - \frac{\cos(\pi n)\Psi(n-1,N+1)}{\Gamma(n)} + \zeta(n)} \end{aligned} \tag{21}$$

where $\binom{n-1}{k-1}$ is of course a binomial. The Bernoulli numbers[1] make an appearance since

$$\chi(N;2n)\zeta_w(N;2n) = B_{2n}(N+1)^2\frac{(2n+1)}{2} + \ldots \tag{22}$$

The denominator of $\chi(N;n)$ has the limits

$$\begin{aligned} \lim_{N\to\infty}\zeta_w(N;n) &= \zeta(n) \\ \lim_{n\to\infty}\zeta_w(N;n) &= 1 \end{aligned} \tag{23}$$

Another interesting formula gives the limit at $s=1$ of the quotient of successive functions

$$\begin{aligned} \lim_{s=1}\frac{\chi(N+1;s)}{\chi(N;s)} &= \frac{(N+2)N(N+1-a(N+1))}{(N+1)^2(N-a(N))} \\ &= \frac{(N+2)N\left(N+1-\sum_{n=1}^{N+1} n(\ln(n+1)-\ln(n))\right)}{(N+1)^2\left(N-\sum_{n=1}^{N} n(\ln(n+1)-\ln(n))\right)} \end{aligned} \tag{24}$$

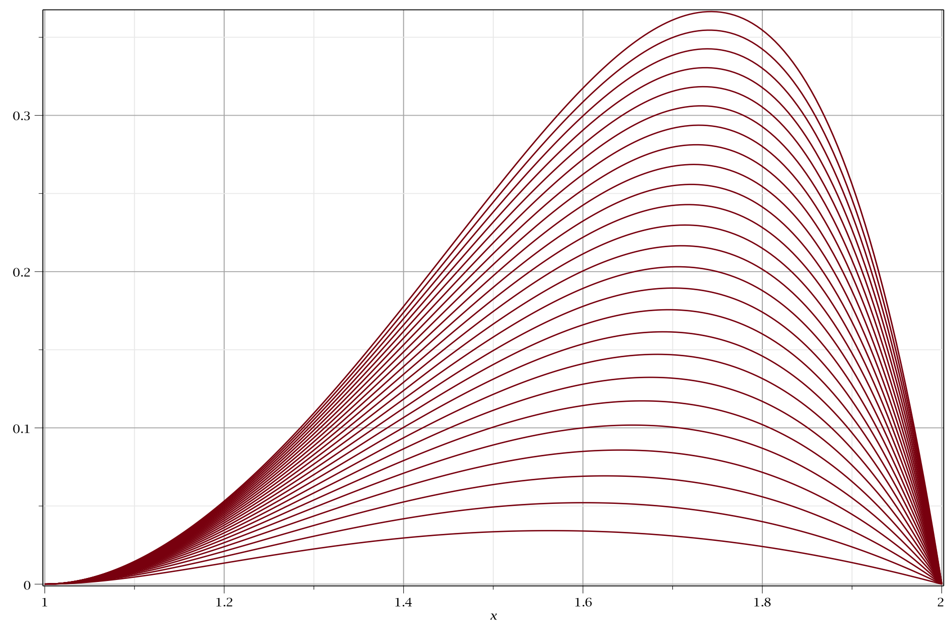

Figure 5. $\{\chi(N;s): s=1...2, N=1...25\}$

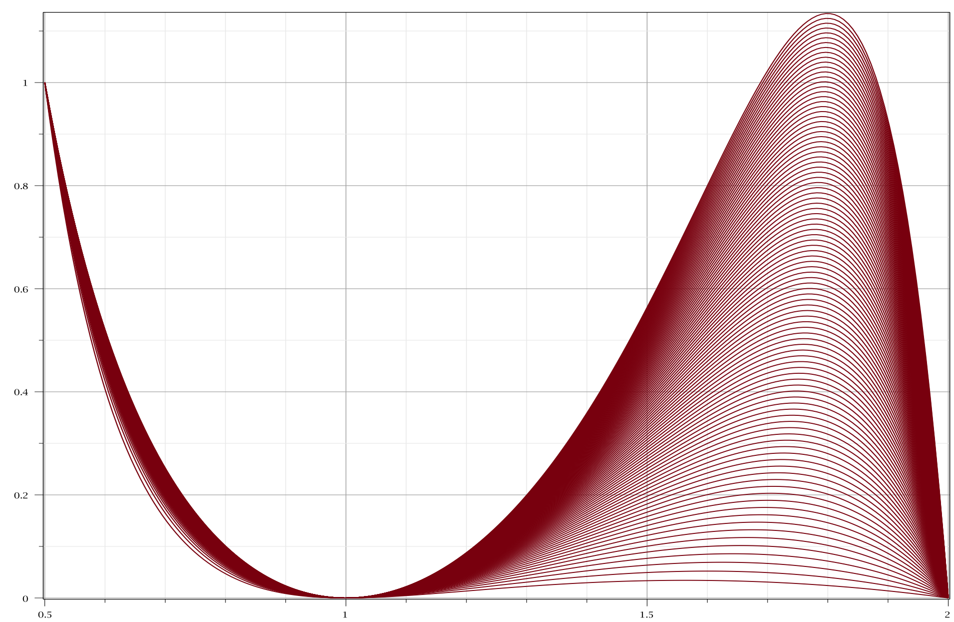

Figure 6. $\left\{\chi(N;s): s=\frac{1}{2}...2, N=1...100\right\}$

Let

$$\nu(s)=\chi(\infty;s)=\frac{\zeta(1-s)}{\zeta(s)} \tag{25}$$

Then the residue at the even negative integers is

$$\operatorname*{Res}_{s=-n}(\nu(s)) = \begin{cases} \frac{\zeta(1-n)}{\frac{\mathrm{d}}{\mathrm{d}s}\zeta(s)|_{s=-n}} & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \tag{26}$$

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