A Finite Reflection Formula For A Polynomial Approximation To The Riemann Zeta Function

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ABSTRACT. The Riemann zeta function can be written as the Mellin transform of the unit interval map $w(x) = \lfloor x^{-1} \rfloor (x \lfloor x^{-1} \rfloor + x - 1)$ multiplied by $s\frac{s+1}{s-1}$. A finite-sum approximation to $\zeta(s)$ denoted by $\zeta_w(N; s)$ is examined and an associated function $\chi(N; s)$ is found which solves the reflection formula $\zeta_w(N; 1 - s) = \chi(N; s)\zeta_w(N; s)$. The function $\chi(N; s)$ is singular at $s = 0$ and the residue at this point changes sign from negative to positive between the values of $N = 176$ and $N = 177$. Some rather elegant graphs of the reflection functions $\chi(N; s)$ are also provided. The values $\zeta_w(N; 1 - s)$ are found to be related to the Bernoulli numbers.

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1. The Riemann Zeta Function as the Mellin Transform of a Unit Interval Map

The Riemann zeta function can be written as the Mellin transform of the unit interval map $w(x) = \lfloor x^{-1} \rfloor (x \lfloor x^{-1} \rfloor + x - 1)$ multiplied by $s\frac{s+1}{s-1}$. [2]

$$
\begin{aligned}
\zeta_w(s) \ &= \zeta(s) \forall -s \notin \mathbb{N}^* \\
&= s\frac{s+1}{s-1} \int_0^1 \lfloor x^{-1} \rfloor (x \lfloor x^{-1} \rfloor + x - 1) x^{s-1} \mathrm{d}x \\
&= s\frac{s+1}{s-1} \sum_{n=1}^{\infty} \int_{\frac{1}{n+1}}^{\frac{1}{n}} n(xn + x - 1) x^{s-1} \mathrm{d}x \\
&= \sum_{n=1}^{\infty} s\frac{s+1}{s-1} \left(-\frac{n^{1-s} - n(n+1)^{-s} - sn^{-s}}{s(s+1)} \right) \\
&= \sum_{n=1}^{\infty} \frac{n(n+1)^{-s} - n^{1-s} + sn^{-s}}{s-1} \\
&= \frac{1}{s-1} \sum_{n=1}^{\infty} n(n+1)^{-s} - n^{1-s} + sn^{-s}
\end{aligned}
\tag{1}
$$

The substitution $\infty \to N$ is made in the infinite sum appearing the expression for $\zeta_w(s)$ to get a finite polynomial approximation

$$
\begin{aligned}
\zeta_w(N;s) &= \frac{1}{s-1}\sum_{n=1}^{N} n(n+1)^{-s} - n^{1-s} + s\,n^{-s} \\
&= \frac{1}{s-1}\left(s + (N+1)^{1-s} - 1 + s\sum_{n=2}^{N} n^{-s} - \sum_{n=2}^{N+1} n^{-s} \right) \\
&= \frac{N}{(s-1)(N+1)^s} - \frac{\cos(\pi s)\,\Psi(s-1,N+1)}{\Gamma(s)} + \zeta(s)\,\forall s \in \mathbb{N}^*
\end{aligned}
\tag{2}
$$

with equality in the limit except at the negative integers

$$
\lim_{N\to\infty} \zeta_w(N;s) = \zeta(s)\,\forall -s \notin \mathbb{N}^*
\tag{3}
$$

The following table lists the values of $\zeta_w(N;1-n)$ for $n=2\ldots12$.

$$
\begin{bmatrix}
0 \\
-\dfrac{1}{6}N - \dfrac{1}{6}N^2 \\
-\dfrac{1}{4}N - \dfrac{1}{2}N^2 - \dfrac{1}{4}N^3 \\
-\dfrac{7}{30}N - \dfrac{4}{5}N^2 - \dfrac{13}{15}N^3 - \dfrac{3}{10}N^4 \\
-\dfrac{1}{6}N - \dfrac{11}{12}N^2 - \dfrac{5}{3}N^3 - \dfrac{5}{4}N^4 - \dfrac{1}{3}N^5 \\
-\dfrac{5}{42}N - \dfrac{6}{7}N^2 - \dfrac{97}{42}N^3 - \dfrac{20}{7}N^4 - \dfrac{23}{14}N^5 - \dfrac{5}{14}N^6 \\
-\dfrac{1}{8}N - \dfrac{19}{24}N^2 - \dfrac{21}{8}N^3 - \dfrac{14}{3}N^4 - \dfrac{35}{8}N^5 - \dfrac{49}{24}N^6 - \dfrac{3}{8}N^7 \\
-\dfrac{13}{90}N - \dfrac{8}{9}N^2 - \dfrac{26}{9}N^3 - \dfrac{56}{9}N^4 - \dfrac{371}{45}N^5 - \dfrac{56}{9}N^6 - \dfrac{22}{9}N^7 - \dfrac{7}{18}N^8 \\
-\dfrac{1}{10}N - \dfrac{21}{20}N^2 - \dfrac{18}{5}N^3 - \dfrac{79}{10}N^4 - \dfrac{63}{5}N^5 - \dfrac{133}{10}N^6 - \dfrac{42}{5}N^7 - \dfrac{57}{20}N^8 - \dfrac{2}{5}N^9 \\
-\dfrac{1}{66}N - \dfrac{10}{11}N^2 - \dfrac{101}{22}N^3 - \dfrac{120}{11}N^4 - \dfrac{199}{11}N^5 - \dfrac{252}{11}N^6 - \dfrac{221}{11}N^7 - \dfrac{120}{11}N^8 - \dfrac{215}{66}N^9 - \dfrac{9}{22}N^{10} \\
-\dfrac{1}{12}N - \dfrac{1}{2}N^2 - \dfrac{55}{12}N^3 - \dfrac{121}{8}N^4 - \dfrac{55}{2}N^5 - \dfrac{110}{3}N^6 - \dfrac{77}{2}N^7 - \dfrac{231}{8}N^8 - \dfrac{55}{4}N^9 - \dfrac{11}{3}N^{10} - \dfrac{5}{12}N^{11}
\end{bmatrix}
\tag{4}
$$

1. The Reflection Formula for $\zeta_w(N; s)$.

There is a reflection equation for the finite-sum approximation $\zeta_w(N; s)$ which is similiar to the well-known formula $\zeta(1-s) = \chi(s)\zeta(s)$ with $\chi(s) = 2(2\pi)^{-s}\cos\left(\frac{\pi s}{2}\right)\Gamma(s)$. The solution to

$$\zeta_w(N; 1-s) = \chi(N; s)\zeta_w(N; s) \tag{5}$$

is given by the expression

$$
\begin{aligned}
\chi(N; s) \ &= \frac{\zeta_w(N; 1-s)}{\zeta_w(N; s)} \\
&= \frac{\sum_{n=1}^{N} -\frac{-n^s + (n+1)^{s-1}n + n^{s-1} - n^{s-1}s}{s}}{\sum_{n=1}^{N} \frac{-n^{1-s} + (n+1)^{-s}n + n^{-s}s}{s-1}} \\
&= -\frac{(s-1)\sum_{n=1}^{N} -n^s + (n+1)^{s-1}n + n^{s-1} - n^{s-1}s}{s\sum_{n=1}^{N} -n^{1-s} + (n+1)^{-s}n + n^{-s}s}
\end{aligned}
$$

which satisfies

$$\chi(N; 1-s) = \chi(N; s)^{-1} \tag{6}$$

The functions $\chi(N; s)$, indexed by N, have singularities at $s = 0$. Let

$$
\begin{aligned}
a(N) \ &= \sum_{n=1}^{N} n(\ln(n+1) - \ln(n)) \\
b(N) \ &= \sum_{n=1}^{N} \frac{\ln(n)n^2 - \ln(n+1)n^2 - \ln(n)}{n(n+1)} \\
c(N) \ &= \frac{1}{2}\sum_{n=1}^{N} n(\ln(n+1)^2 - \ln(n)^2)
\end{aligned}
\tag{7}
$$

then the residue at the singular point $s = 0$ is given by the expression

$$
\begin{aligned}
\operatorname*{Res}_{s=0}(\chi(N; s)) \ &= -\operatorname*{Res}_{s=1}(\chi(N; s)^{-1}) \\
&= \frac{1 + \gamma + \Psi(n+2) - \frac{2}{N+1} + b(N) - \frac{N(\ln(\Gamma(N+1)) - c(N))}{(N - a(N))(N+1)}}{a(N) - N} \\
&= \frac{1 + \gamma + \Psi(n+2) - \frac{2}{N+1} + \sum_{n=1}^{N} \frac{\ln(n)n^2 - \ln(n+1)n^2 - \ln(n)}{n(n+1)} - \frac{N\left(\ln(\Gamma(N+1)) - \frac{1}{2}\sum_{n=1}^{N} n(\ln(n+1)^2 - \ln(n)^2)\right)}{(N - \sum_{n=1}^{N} n(\ln(n+1) - \ln(n)))(N+1)}}{\left(\sum_{n=1}^{N} n(\ln(n+1) - \ln(n))\right) - N}
\end{aligned}
\tag{8}
$$

which has the limit

$$\lim_{N \to \infty} \operatorname*{Res}_{s=0}(\chi(N; s)) = 1 \tag{9}$$

We also have the residue of the reciprocal at $s = 2$

$$\operatorname*{Res}_{s=2}(\chi(N; s)^{-1}) \ = \frac{\frac{2N}{(N+1)^2} - 2\Psi(1, N+1) + 2\zeta(2)}{\frac{(N+1)^2}{2} - \frac{N}{2} - \frac{1}{2} - \sum_{n=1}^{N} n(\ln(n+1) + \ln(n+1)n - \ln(n) - n\ln(n))} \tag{10}$$

which vanishes as N tends to infinity

$$\lim_{N \to \infty} \operatorname*{Res}_{s=2}(\chi(N;s)^{-1}) = 0 \tag{11}$$

As can be seen in the figures below, the residue at $s=0$ changes sign from negative to positive between the values of $N=176$ and $N=177$.
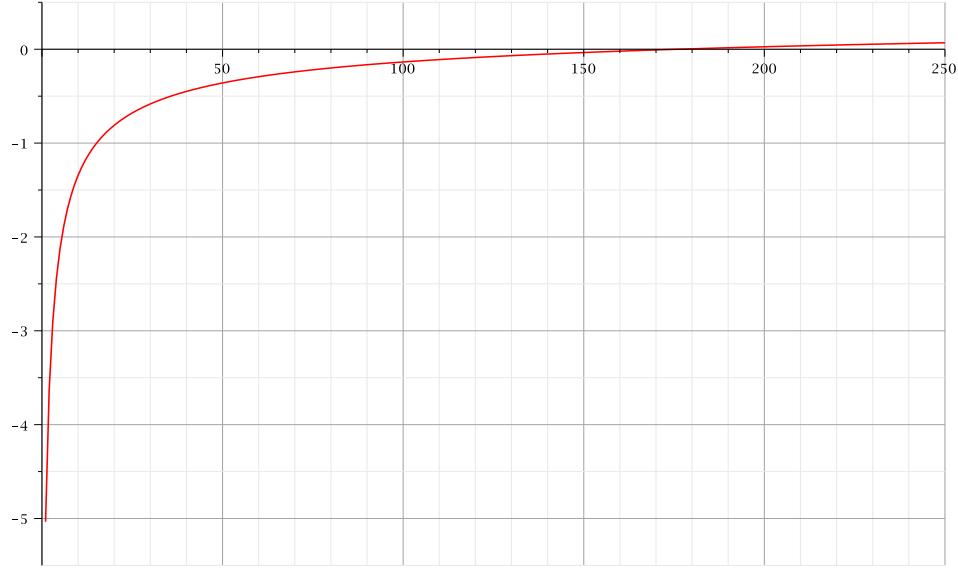
**Figure 1.** $\left\{ \operatorname*{Res}_{s=0}(\chi(N;s)) : N = 1...250 \right\}$
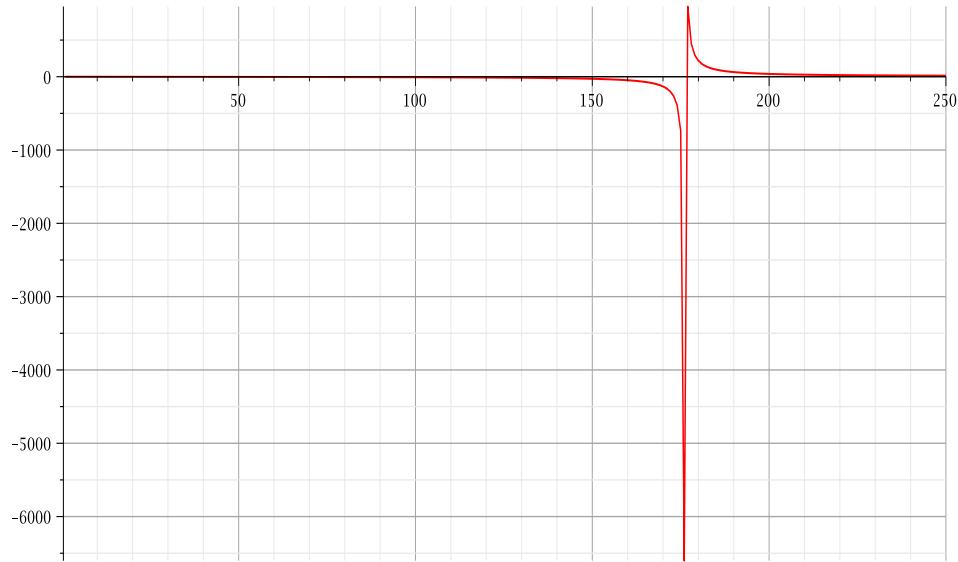
Figure 2. $\left\{ \operatorname*{Res}_{s=0}(\chi(N;s))^{-1} : N = 1...250 \right\}$

For any positive integer N, we have the limits

$$
\begin{aligned}
\lim_{s\to 0} \chi(N;s) &= \infty \\
\lim_{s\to 0} \frac{\mathrm{d}^n}{\mathrm{d}s^n}\chi(N;s) &= \infty \\
\lim_{s\to \frac{1}{2}} \chi(N;s) &= 1 \\
\lim_{s\to 1} \chi(N;s) &= 0 \\
\lim_{s\to 2} \chi(N;s) &= 0 \\
\lim_{s\to 1} \frac{\mathrm{d}}{\mathrm{d}s}\chi(N;s) &= 0
\end{aligned}
\tag{12}
$$

The line $\mathrm{Re}(s) = \frac{1}{2}$ has a constant modulus

$$
\left| \chi\left(N; \frac{1}{2}+is\right) \right| = 1
\tag{13}
$$

There is also the complex conjugate symmetry

$$
\chi(N; x+iy) = \overline{\chi(N; x-iy)}
\tag{14}
$$

If $s = n \in \mathbb{N}^*$ is a positive integer then $\chi(N; n)$ can be written as

$$
\begin{aligned}
\chi(N; n) &= \frac{\zeta_w(N; 1-n)}{\zeta_w(N; n)} \\
&= \frac{\frac{\sum_{m=1}^{N} -\sum_{k=1}^{n-2} \frac{m^k}{n}\binom{n-1}{k-1}}{N}}{\frac{1}{(n-1)(N+1)^n} - \frac{\cos(\pi n)\Psi(n-1, N+1)}{\Gamma(n)} + \zeta(n)} \\
&= \frac{\frac{-\sum_{m=1}^{N} \frac{1}{n}\left((n-1)m^{n-1}+m^n-(m+1)^{n-1}m\right)}{N}}{\frac{1}{(n-1)(N+1)^n} - \frac{\cos(\pi n)\Psi(n-1, N+1)}{\Gamma(n)} + \zeta(n)}
\end{aligned}
\tag{15}
$$

The Bernoulli numbers[1] make an appearance since

$$
\chi(N; 2n)\zeta_w(N; 2n) = B_{2n}(N+1)^2 \frac{(2n+1)}{2} + \dots
\tag{16}
$$

The denominator of $\chi(N; n)$ has the limits

$$
\begin{aligned}
\lim_{N\to\infty} \zeta_w(N; n) &= \zeta(n) \\
\lim_{n\to\infty} \zeta_w(N; n) &= 1
\end{aligned}
\tag{17}
$$

Another interesting formula gives the limit at $s = 1$ of the quotient of successive functions

$$
\begin{aligned}
\lim_{s\to 1} \frac{\chi(N+1; s)}{\chi(N; s)} &= \frac{(N+2)N(N+1-a(N+1))}{(N+1)^2(N-a(N))} \\
&= \frac{(N+2)N\left(N+1-\sum_{n=1}^{N+1} n(\ln(n+1)-\ln(n))\right)}{(N+1)^2\left(N-\sum_{n=1}^{N} n(\ln(n+1)-\ln(n))\right)}
\end{aligned}
\tag{18}
$$
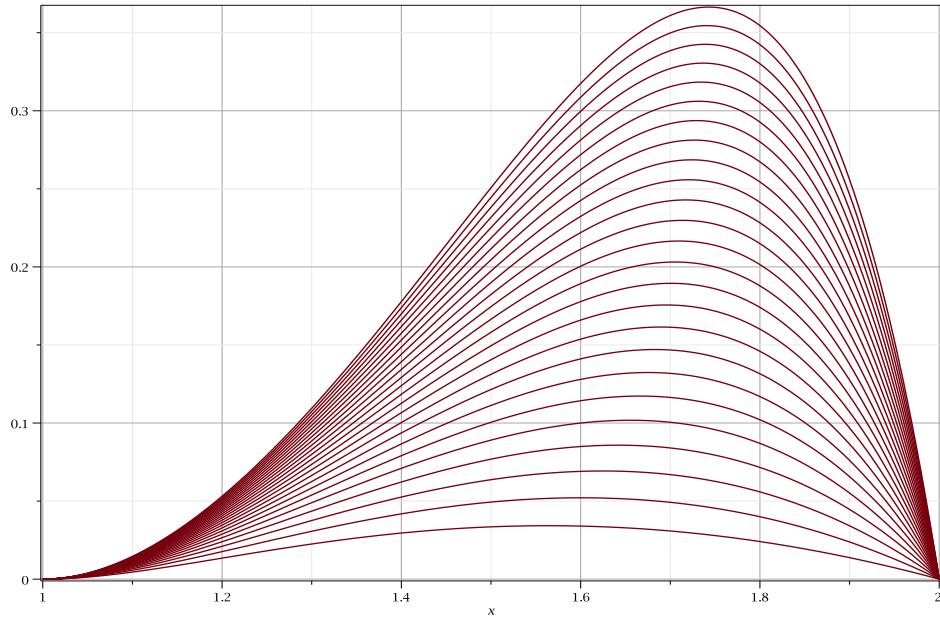
Figure 3. $\{\chi(N;s)\colon s=1...2, N=1...25\}$
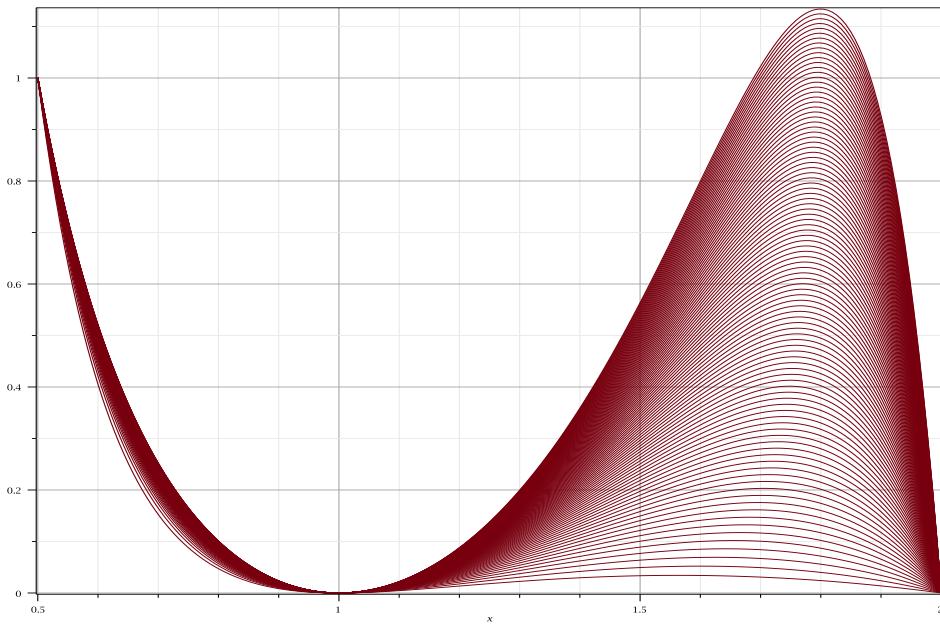
Figure 4. $\left\{\chi(N;s)\colon s=\frac{1}{2}...2, N=1...100\right\}$

Let

$$\nu(s) = \chi(\infty;s) = \frac{\zeta(1-s)}{\zeta(s)} \tag{19}$$

Then the residue at the even negative integers is

$$\operatorname{Res}_{s=-n}\left(\nu(s)\right)=\begin{cases} \dfrac{\zeta(1-n)}{\frac{\mathrm{d}}{\mathrm{d}s}\zeta(s)|_{s=-n}} & n\text{ even} \\ 0 & n\text{ odd} \end{cases} \tag{20}$$

Bibliography

[1] G Arfken. *Mathematical Methods for Physicists, 3rd ed.*, chapter 5.9, Bernoulli Numbers, Euler-Maclaurin Formula., pages 327–338. Academic Press, 1985.

[2] Stephen Crowley. Integral transforms of the harmonic sawtooth map, the riemann zeta function, and fractal strings. *viXra.org*, (1202.0079), 2012.