A Simple Proof of Bernoulli's Inequality

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Bernoulli's inequality states that for $r \geq 1$ and $x \geq -1$:

$$(1+x)^r \geq 1+rx$$

The inequality reverses for $r \leq 1$.

In this note an elementary proof of this inequality for rational r is described. The proof is only based on the fact that for any n non-negative numbers, geometric mean can not exceed arithmetic mean (see e.g. [1, Section 250,253])[1]. I think, part(s) of this proof may be appearing as exercise in some text books, but unfortunately as search engines are unable to find those, I have written this note.

Let us first consider the case when $r \leq 1$, i.e., let $r = \frac{p}{q}$ with $p \leq q$.

Geometric mean of q numbers ($(1+x)$ occurs p times):

$$1, 1, \ldots, 1, (1+x), (1+x), \ldots (1+x)$$

is $(1+x)^{p/q}$ and arithmetic mean of these numbers is: $1 + \frac{p}{q}x$. As geometric mean is less than arithmetic mean, we get

$$(1+x)^{p/q} \leq 1 + \frac{p}{q}x \tag{1}$$

Or substituting $r = \frac{p}{q}$,

$$(1+x)^r \leq 1+rx$$

This proves the inequality for $r \leq 1$ case.

Next consider the case when exponent is more than 1. As $x \geq -1$, the condition1 is equivalent to:

$$(1+x) \leq \left(1 + \frac{p}{q}x\right)^{q/p}$$

Let $y = \frac{p}{q}x$, as $x \geq -1$, $y \geq -\frac{p}{q} \geq -1$.

As $x = \frac{q}{p}y$, in terms of y we have:

$$\left(1 + \frac{q}{p}y\right) \quad \leq \quad (1+y)^{q/p} \tag{2}$$

Now $s = \frac{q}{p} \geq 1$ is an arbitrary rational number. Hence, for any rational number $s \geq 1$ and $y \geq -1$, we have

$$(1+y)^s \geq 1+sy$$

This proves the inequality for $s \geq 1$ case.

As these inequalities are true for all rational numbers $r \leq 1$ and $s \geq 1$, they are also true for all real numbers. This is because, any real number can be approximated by rational numbers to arbitrary precision (this formally follows from Cauchy construction of real numbers).

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[1] The proof of [1] is also given in Appendix for completeness

References

[1] H.S.Hall and S.R.Knight, Higher Algebra, 1887.

Appendix: Geometric-Arithmetic Mean Inequality

For completeness we give the proof that arithmetic mean is greater than geometric mean[1, Section 253].

Let us assume that sum $M = x_1 + x_2 + \ldots + x_n$ is given and we want to maximise the product $x_1 x_2 \ldots x_n$. We claim that the product will have the largest value, when all x_i's are equal. For, if $x_i \neq x_j$, then replacing both x_i and x_j by $\frac{x_i + x_j}{2}$ will leave the sum M unchanged. But[2], the product will increase.

Thus the product will be maximum when each $x_i = \frac{M}{n} = \frac{\sum x_i}{n}$, and the product $x_1 x_2 \ldots x_n$ in this case will be $\left(\frac{\sum x_i}{n}\right)^n$.

Hence in all cases:

$$(x_1 x_2 \ldots x_n)^{\frac{1}{n}} \leq \left(\frac{x_1 + x_2 + \ldots + x_n}{n}\right)$$

[2] as $a + b - 2\sqrt{ab} = \left(\sqrt{a} - \sqrt{b}\right)^2 > 0$, we have $\left(\frac{a+b}{2}\right)^2 > ab$