Quasi-Isogonal Cevians

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In this article we will introduce the quasi-isogonal Cevians and we'll emphasize on triangles in which the height and the median are quasi-isogonal Cevians.

For beginning we'll recall:

Definition 1

In a triangle ABC the Cevians AD, AE are called isogonal if these are symmetric in rapport to the angle A bisector.

Observation

In figure 1, are represented the isogonal Cevians AD, AE

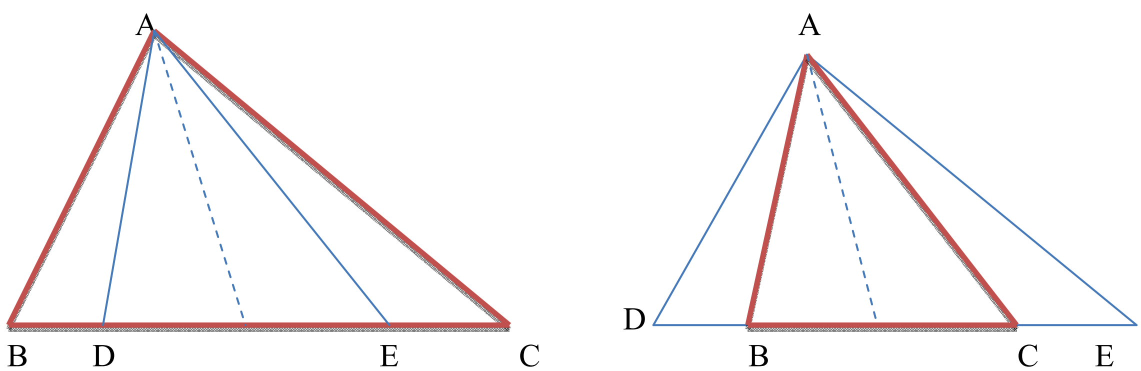

Fig. 1. Isogonal Cevians

Proposition 1.

In a triangle ABC, the height AD and the radius AO of the circumscribed circle are isogonal Cevians.

Definition 2.

We call the Cevians AD, AE in the triangle ABC quasi-isogonal if the point B is between the points D and E, the point E is between the points B and C, and $\sphericalangle DAB \equiv \sphericalangle EAC$.

Observation

In figure 2 we represented the quasi-isogonal Cevians AD, AE.

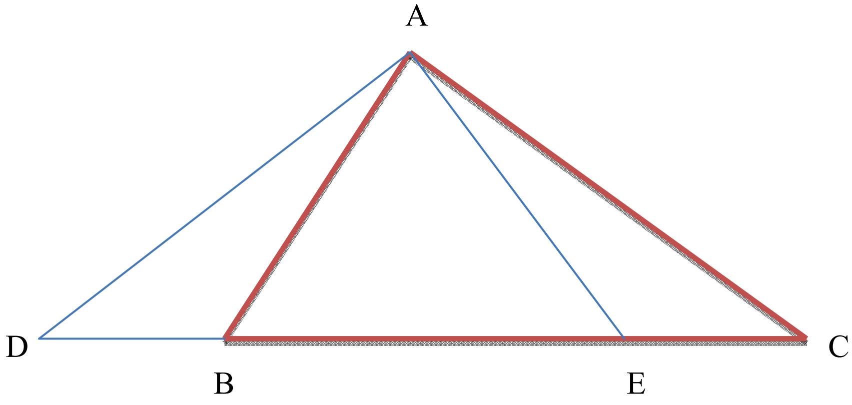

Fig. 2 quasi-isogonal Cevians

Proposition 2
There are triangles in which the height and the median are quasi-isogonal Cevians.

Proof
It is clear that if we look for triangles ABC for which the height and the median from the point A are quasi isogonal, then these must be obtuse-angled triangle. We'll consider such a case in which $m(\sphericalangle A) > 90°$ (see figure 3).

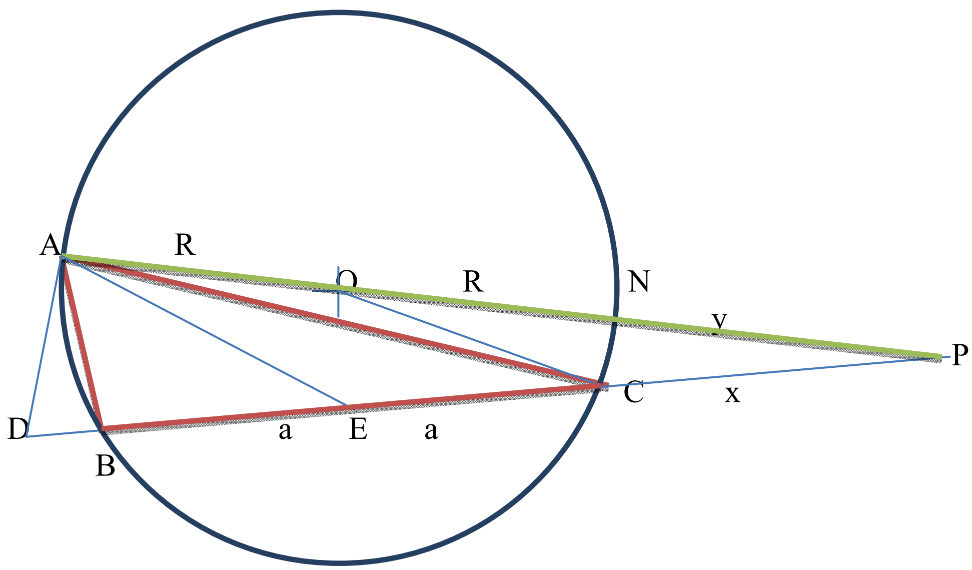

Fig. 3

Let O the center of the circumscribed triangle, we note with N the diametric point of A and with P the intersection of the line AO with BC.

We consider known the radius R of the circle and $BC = 2a$, $a < R$ and we try to construct the triangle ABC in which the height AD and the median AE are quasi isogonal Cevians; therefore $\sphericalangle DAB \equiv \sphericalangle EAC$. This triangle can be constructed if we find the lengths PC and PN in function of a and R. We note $PC = x$, $PN = y$.

We consider the power of the point P in function of the circle $\mathcal{C}(O,R)$. It results that

$$x\cdot(x+2a)=y\cdot(y+2R) \qquad (1)$$

From the Property 1 we have that $\sphericalangle DAB \equiv \sphericalangle OAC$. On the other side $\sphericalangle OAC \equiv \sphericalangle OCA$ and AD, AE are quasi isogonal, we obtain that $OC \parallel AE$.

The Thales' theorem implies that:

$$\frac{x}{a}=\frac{y+R}{R} \qquad (2)$$

Substituting x from (2) in (1) we obtain the equation:

$$\left(a^2-R^2\right)y^2-2R\left(R^2-2a^2\right)y+3a^2R^2=0 \qquad (3)$$

The discriminant of this equation is:

$$\Delta=4R^2\left(R^4-a^2R^2+a^4\right)$$

Evidently $\Delta>0$, therefore the equation has two real solutions.

Because the product of the solutions is $\frac{3a^2R^2}{a^2-R^2}$ and it is negative we obtain that one of solutions is strictly positive. For this positive value of y we find the value of x, consequently we can construct the point P, then the point N and at the intersection of the line PN we find A and therefore the triangle ABC is constructed.

For example, if we consider $R=\sqrt{2}$ and $a=1$, we obtain the triangle ABC in which $AB=\sqrt{2}$, $BC=2$ and $AC=1+\sqrt{3}$.

We leave to our readers to verify that the height and the median from the point A are quasi isogonal.