FERMAT'S LAST THEOREM-A ONE PAGE PROOF

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Abstract. This article presents the shortest possible proof of Fermat's Last Theorem of any that have ever been published. It might be the one that Fermat had hinted about in his copy of Diophanti's Arithmetic Book.

I Some philosophical aspects

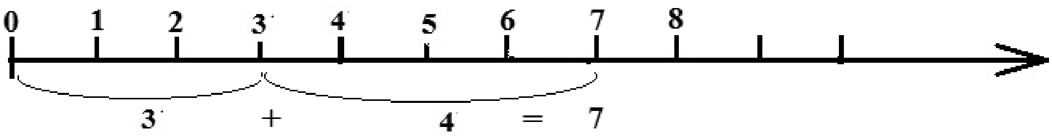
As is known, in accordance with Fermat's last theorem, the equation

$$a^n + b^n = c^n \tag{1}$$

does not have any whole number solution for $n > 2$.
Let us first relate to the first degree equation $n = 1$:

$$a^1 + b^1 = c^1$$

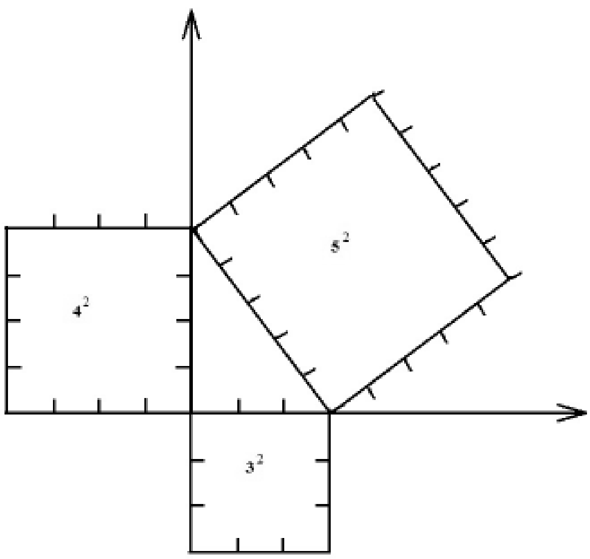
It is obvious that this equation is satisfied by any two whole numbers, as can see from its geometrical representation by an infinite line of natural numbers: the sum of two whole numbers always gives another whole number.



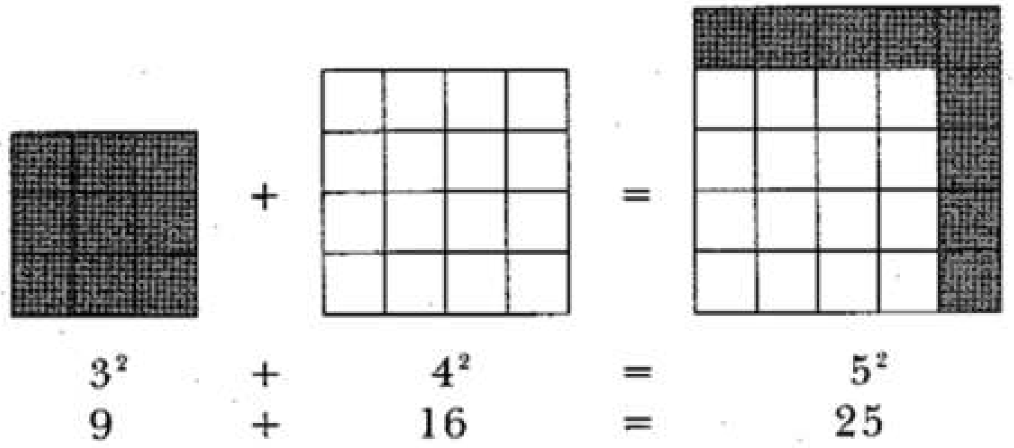
The second degree equation:

$$a^2 + b^2 = c^2 \tag{2}$$

is already more complicated, since it presents numbers on a plane, such as the case of Pythagoras' Triangle:



This equation is satisfied by only some whole numbers, as can be seen from the figure below (From the book "Fermat's Last Theorem" by Simon Singh).

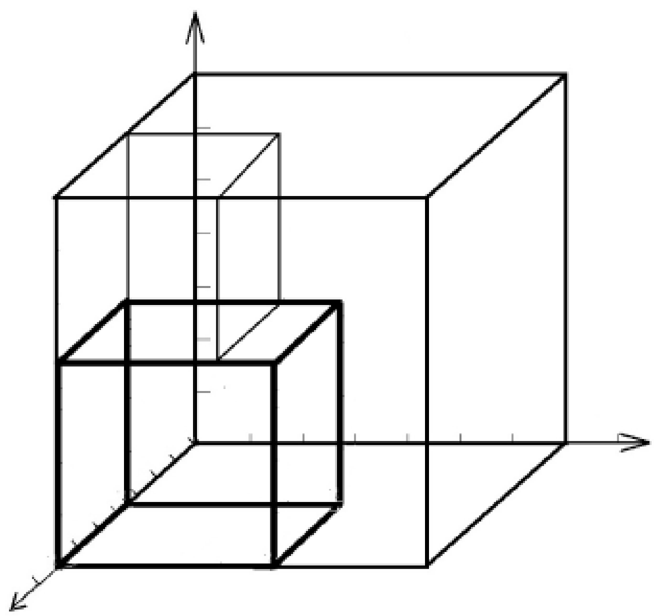


Finding whole number solutions to Pythagoras' equation can be thought of in terms of finding two squares which can be added together to form a third square. For example, a square made of 9 tiles can be added to a square of 16 tiles, and rearranged to form a third square made of 25 tiles.

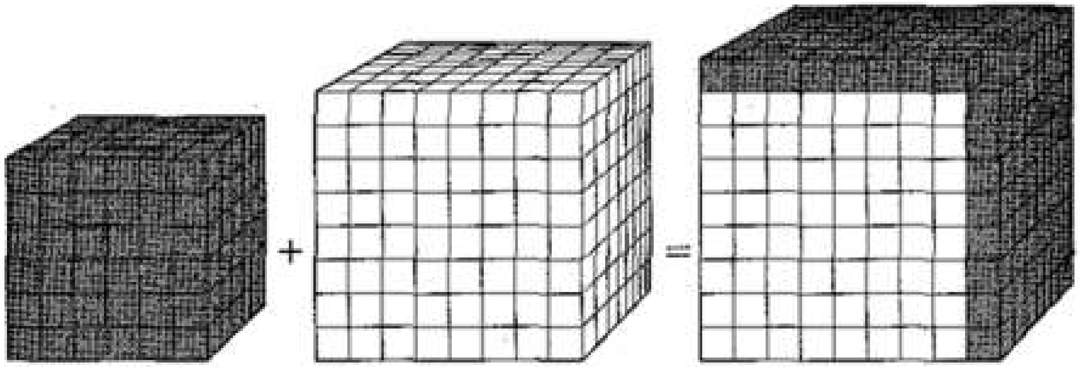
The third (and higher) degree equation:

$$a^3 + b^3 = c^3 \tag{3}$$

is much more complicated, since it presents numbers in a space:



This equation cannot be satisfied by any whole numbers, as can be seen from the figure below (From the book "Fermat's Last Theorem" by Simon Singh).



$6^3 \quad + \quad 8^3 \quad = \quad 9^3-1$

$216 \quad + \quad 512 \quad = \quad 729-1$

Is it possible to add the building blocks from one cube to another cube, to form a third, larger, cube? In this case a 6 × 6 × 6 cube added to an 8 × 8 × 8 cube does not have quite enough building blocks to form a 9 × 9 × 9 cube. There are 216 (6^3) building blocks in the first cube, and 512 (8^3) in the second. The total is 728 building blocks, which is 1 short of 9^3.

II One formula solution of a second degree equation

For the quadratic equation (2) we found a simple whole number solution.
Suppose that a is less than b and designate that $c = b + d$, where d is a whole number and understanding that $d < a$, we may write:

$$a^2 + b^2 = (b+d)^2 \tag{4}$$

or

$$a^2 + b^2 = b^2 + 2bd + d^2 .$$

After a reduction of b^2 and under the condition that a is given, we have for b a linear equation, where d plays the role of a parameter. We will solve this equation for b to obtain

$$b = \frac{a^2 - d^2}{2d} = \frac{\frac{a^2}{d} - d}{2} \tag{5}$$

Since b must be a whole number, d should be a divider of a^2, and also less than a, as mentioned above. Taking all this into consideration for each specific d, which fulfills the above conditions, we may find a corresponding b.
For example, suppose that $a = 8$, then the above equation:

$$b = \frac{\frac{8 \cdot 8}{d} - d}{2}$$

gives for d two possibilities $d_1 = 2$, $d_2 = 4$, which results in

$b_1 = \frac{\frac{8 \cdot 8}{2} - 2}{2} = 15$ and $c_1 = 15 + 2 = 17$, and the first solution is $8^2 + 15^2 = 17^2$. The second possibility results in $6^2 + 8^2 = 10^2$. Thus, by using the formula (5) one can find all the solutions of Fermat's equation for $n = 2$.

III The proof of the third degree equation

Next we analyzed in the same way the third-order equation, which gives

$$a^3 + b^3 = (b + d)^3 \text{ or}$$

$$a^3 + b^3 = b^3 + 3b^2 d + 3bd^2 + d^3.$$

After a reduction of b^3, we obtain for b a quadratic equation:

$$3d \cdot b^2 + 3d^2 \cdot b + d^3 - a^3 = 0 \tag{6}$$

whose only positive root is:

$$b = \frac{\sqrt{3}a\sqrt{4 - \left(\frac{d}{a}\right)^3}}{6} - \frac{d}{2} \tag{7}$$

Since $d < a$, this result cannot be a whole number (but an irrational number) for any value of d (note that for $d = a$, although b could be a whole number, but this case is not real because the sum of the cubes cannot be equal to the cube of the sums) and therefore $c = b + d$ cannot be a whole number.

Since a and b *can be any numbers, this is proof of Fermat's theorem for* $n = 3$.

IV The general proof

Now let us analyze Fermat's equation (1) in a general way. For this purpose we present the coefficient c as

$$c_{(n)} = \sqrt[n]{a^n + b^n} = b\sqrt[n]{1 + \left(\frac{a}{b}\right)^n} = bK_{(n)}\ , \qquad (8)$$

where $K_{(n)} = \sqrt[n]{1 + \left(\frac{a}{b}\right)^n} = \sqrt[n]{Q_{(n)}}$ and $Q_{(n)} = 1 + \left(\frac{a}{b}\right)^n$.

Since $\left(\frac{a}{b}\right)$ is a proper fraction, $Q_{(n)}$ being greater than 1, will decrease with an increasing n, so will $K_{(n)}$ and $c_{(n)}$:

$$\begin{aligned} &Q_{(1)} > Q_{(2)} > Q_{(3)} > \cdots Q_{(n-1)} > Q_{(n)} \\ &K_{(1)} > K_{(2)} > K_{(3)} > \cdots K_{(n-1)} > K_{(n)} \\ &c_{(1)} > c_{(2)} > c_{(3)} > \cdots c_{(n-1)} > c_{(n)} \end{aligned} \qquad (9)$$

Suppose that the second order equation of two numbers a and b has a whole number solution, which means that $Q_{(2)}$ is a full second power, and $K_{(2)}$ is rational (a necessary condition) and also, since $K_{(2)}$ is a fractional number, the product $bK_{(2)}$ should be a whole number (a sufficient condition). However, since for the next power ($n = 3$) the radicand $Q_{(3)}$ is obtained by raising only (a/b) to the next power and not the whole radicand, the radicand $Q_{(3)}$ cannot be a full third power (it should be larger then $Q_{(2)}$, but in accordance with (9), it is getting smaller) so that $K_{(3)}$ will be irrational and $c_{(3)}$ never will be a whole number. If $Q_{(2)}$ is not a full second power, $K_{(2)}$ is irrational and the coefficient $K_{(3)}$, which follows $K_{(2)}$, even more so, will also be irrational. It is obvious that the next radicands $Q_{(n)}$, since they are also getting smaller, cannot be full powers and all the coefficients $K_{(n)}$ are irrational, so that $c_{(n)} = bK_{(n)}$ cannot be whole numbers.

Since a and b *are any possible numbers, thereby Fermat's last theorem is proved.*

On the other hand, the possibility that the radicand $Q_{(n)}$ is the full power of some other number, say x, should also be checked, i.e.,

$$1 + (a/b)^n = (1 + x)^n .$$

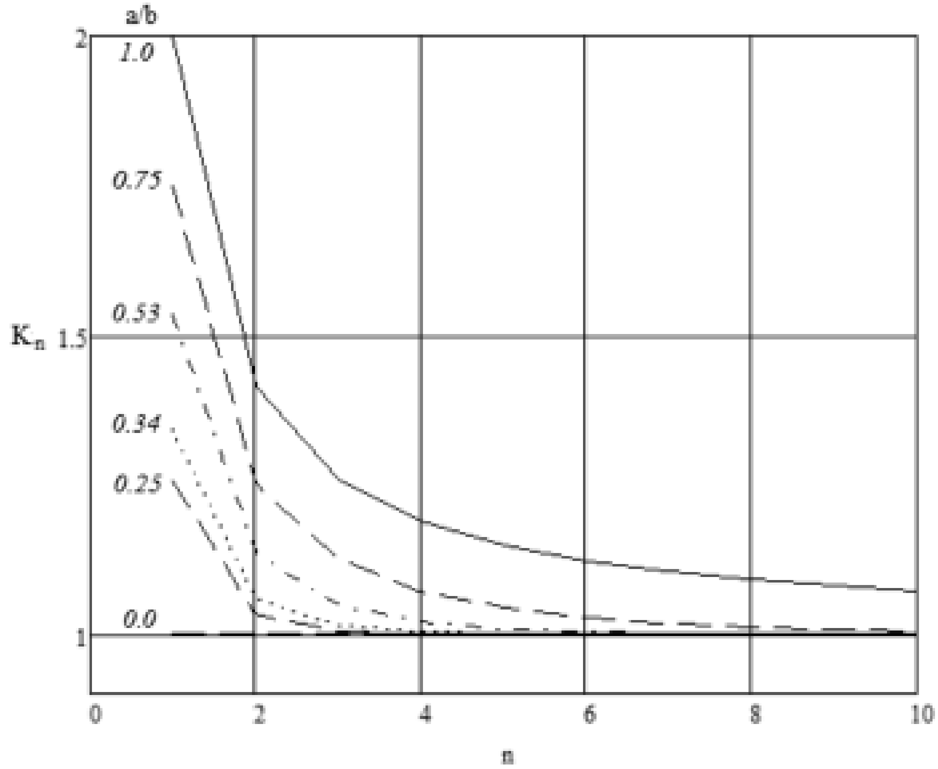
By solving this simple equation we have

$$x = \sqrt{1 + (a/b)^n} - 1 = K_{(n)} - 1,$$

which means that x is irrational if $K_{(n)}$ is irrational and vice versa. This brings us back to our previous decision that $K_{(n)}$ are irrational and $c_{(n)}$ cannot be whole numbers.

V Some additional clarifications

It is interesting to observe the curves of the coefficient $K_{(n)}$ versus n and a/b:



As can be seen, all the values of $K_{(n)}$, for all *n*-s ($1 < n < \infty$) and all ratios a/b ($0 < a/b < 1$), are concentrated in the narrow area of an envelope, between the curves *0.0* and *1.0*. This means that all the possibilities of Fermat's equations, for all *n-s* and all numbers a, b, are presented in this envelope. As can be seen these $K_{(n)}$ coefficients converge very fast (practically up to $n > 10$) to the unit and all the values of all the coefficients (for $n > 2$) are fractional and lay in between *1* and *~ 1.4*. This result gives an intuitive feeling that in such conditions $K_{(n)}$ cannot be rational numbers, but irrational.

For example, $K_{(2)} = \sqrt{1 + \left(\frac{3}{4}\right)^2} = \frac{5}{4}$ is rational, but $K_{(3)} = \sqrt[3]{1 + \left(\frac{3}{4}\right)^3} = \frac{54617618...}{48571213...}$ is already irrational.

Note that we proved Fermat's last theorem in a different way than usual, namely we proved it for a specific couple of numbers a, b for all the powers, in contrast to proving it for a specific power n and all the numbers a, b. So, here we see the reason for our success.

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