A Disconnect : Limitations of the Axiomatic Method in Physics

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Dedicated to Marie-Louise Nykamp

Abstract

This paper presents the phenomenon of *disconnect* in the axiomatic approach to theories of Physics, a phenomenon which appears due to the insistence on axioms which have a physical meaning. This insistence introduces a restriction which is *foreign* to the abstract nature of axiomatic systems as such. Consequently, it turns out to introduce as well the mentioned disconnect. The axiomatic approach in Physics has a longer tradition. It is there already in Newton's Principia. Recently for instance, a number of axiomatic approaches have been proposed in the literature related to Quantum Mechanics. Special Relativity, [2], had from its beginning in 1905 been built upon two axioms, namely, the Galilean Relativity and the Constancy of the Speed of Light in inertial reference frames. Hardly noticed in wider circles, the independence of these two axioms had quite early been subjected to scrutiny, [5,3], and that issue has on occasion been addressed ever since, see [8,4,24] and the literature cited there. Recently, [24], related to these two axioms in Special Relativity, the following phenomenon of wider importance in Physics was noted. As the example of axiomatization of Special Relativity shows it, it is possible to face a disconnect be-

tween a system of physically meaningful axioms, and on the other hand, one or another of the mathematical models used in the study of the axiomatized physical theory. The consequence is that, seemingly unknown so far, one faces in Physics the possibility that the axiomatic method has deeper, less obvious, and in fact not considered, or simply overlooked limitations. As there is no reason to believe that the system of the usual two axioms of Special Relativity is the only one subjected to such a disconnect, the various foundational ventures in modern Physics, related for instance to gravitation, quanta, or their bringing together in an overarching theory, may benefit from the study of the possible sources and reasons for such a disconnect. An attempt of such study is presented in this paper.

> "Of all things, good sense is the most fairly distributed : everyone thinks he is so well supplied with it that even those who are the hardest to satisfy in every other respect never desire more of it than they already have." :-) :-) :-)
>
> R Descartes
> Discourse de la Méthode

> "... creativity often consists of finding hidden assumptions. And removing those assumptions can open up a new set of possibilities ..."
>
> Henry R Sturman

> "History is written with the feet ..."
>
> Chinese Ex-Chairman Mao,
> of the Long March fame ...

Science is not done scientifically, since it is mostly done by non-scientists ...

Anonymous

Physics is too important to be left to physicists ...

Anonymous

Is the claim about the validity of the so called "physical intuition" but a present day version of medieval claims about the sacro-sanct validity of theological revelations ?

Anonymous

1. Brief Review of the Axiomatics of Special Relativity

As seen in [3-6,8,24] and the literature cited there, the usual two axioms of Special Relativity, namely, the Galilean Relativity and the Constancy of the Speed of Light in inertial reference frames, are *not* needed in their full power in order to obtain the Lorentz Transformations of inertial reference frames, transformations which, as well known, contain the essence of Special Relativity. Indeed, what are instead needed are only the following assumptions :

- the homogeneity and isotropy of space,
- the homogeneity of time,
- the Axiom of Reciprocity which means that, given two inertial reference frames S and S', and a speed $v \in \mathbb{R}$, the laws of Physics are the same whether S' moves related to S with speed v, or with velocity $-v$,

- the resulting upper limit of all physical speeds is the speed c of light in void.

This fact that the full power of the usual two physically meaningful axioms of Special relativity is *not* used in order to obtain the Lorentz Transformations was called a *disconnect* in [24]. And clearly, it raises the questions :

- Are the mentioned two usual and physically meaningful axioms of Special Relativity too strong for that theory of Physics ?

- And if not, then what are those parts of Special Relativity which have not yet been deduced from those two physically meaningful axioms, in view of the fact that the Lorentz Transformations do not need their full use ?

In short :

- Which may be a system of axioms - be it with all of its axioms physically meaningful, or not - for Special Relativity, such that it is both a sufficient and minimal axiomatic system ?

And why not, one may also ask the question :

- Is it possible that a sufficient and minimal system of axioms for Special Relativity may have to contain an axiom without manifest physical interpretation, thus not given by, and not even within the realms of "physical intuition" ?

In this regard, it may indeed be instructive in the pursuit of Physics in general to ponder for longer on the following two moments of major discontinuity in the history of Special Relativity.

Lorentz, Fitzgerald and Poincaré introduced, respectively, were aware of and supported the Lorentz Transformations, however, did not take the deeper conceptual step of seeing them as the indication for the need of a radical re-foundation of Newtonian Mechanics.
That deeper conceptual step was taken by Einstein who formulated the two axioms of Special Relativity, namely, the Galilean Relativity and

the Constancy of the Speed of Light in inertial reference frames. And as a consequence, dramatic new developments were obtained, among them the unification of mass and energy, expressed in the celebrated relation $E = mc^2$.

What was, however, missed in these two rather dramatic steps was what, with hindsight, may seem as the simple realization that a disconnect could possibly occur between, on the one hand, what appear to be eminently clear and physically meaningful axioms, and on the other hand, the resulting physical theory represented by its mathematical model, not to mention the particular theory of Physics as a whole, theory which is the object of axiomatization, such as for instance is the case with Special Relativity.

Such a disconnect is obviously but an expression of the *limitations* of the axiomatic method in Physics in general. And in particular, it questions the requirement that all axioms used in a given theory of Physics subjected to axiomatization have a clear and explicit physical meaning, and thus be given by, and hence be within the realms of "physical intuition".

And the fact is that, even nowadays, the idea of a deeply seated, yet as such rather simple, if not actually elementary limitation on axiomatic systems in Physics seems not to rise in the awareness of many ...
One reason for that may be the extraordinary success of the axiomatic method in Mathematics, ever since Euclid's Geometry.

Nevertheless, modern Mathematics, after more than two millennia of such a felicitous view of the axiomatic method had to wake up in the early 1930s to the two Incompleteness Theorems of Gödel, the first of which already showed that rather basic and simple theories of Mathematics, such as about the set $\mathbb{N}$ of natural integer numbers, can never be exhausted by any consistent system of axioms.

However, and as it is obvious here, as well as in [24], it is important to note that the disconnect which is the object of this paper is about a different issue than the Gödelian incompleteness. Indeed, the latter is a phenomenon which is of a purely abstract logical and formal nature.

On the other hand, the former is a consequence of the requirement of a completely different nature, namely, that all physical axioms in the formulation of a given theory of Physics have a physical meaning.

By the way, "having a physical meaning" is itself problematic, since it obviously can only mean "in the sense of the present knowledge and understanding of Physics". Thus it further imposes a limitation on the way the axiomatic method can be used in Physics.

2. A Brief Review of Axiomatization in Mathematics

Let us recall the way of axiomatization in Mathematics, which goes back at least as far as the Geometry of Euclid in ancient Egypt more than two millennia ago. In terms of modern Mathematical Logic, the process in its essence is as follows.

One starts with a setup of a *formal deductive system*. Namely, let A be an *alphabet* which can be given by any nonvoid finite or infinite set. Then a procedure is given according to which one constructs - by using the symbols in A - a set $\mathcal{F}$ of *well formed formulas*, or in short wff-s. Next, one chooses a set $\mathcal{R}$ of *deduction rules* which operate as follows

$$(2.1) \qquad \mathcal{F} \supseteq P \stackrel{\mathcal{R}}{\longmapsto} Q \subseteq \mathcal{F}$$

that is, from as set P of wff-s which are the *premises*, it leads to a set Q of wff-s which are the *consequences*.

And now come the *axioms* which can be any subset $\mathcal{A} \subseteq \mathcal{F}$.

Once the above is established, the respective *axiomatic theory* follows easily as being the *smallest* subset $\mathcal{T} \subseteq \mathcal{F}$ with the properties

$$(2.2) \qquad \mathcal{A} \subseteq \mathcal{T}$$

$$(2.3) \qquad \mathcal{T} \supseteq P \stackrel{\mathcal{R}}{\longmapsto} Q \subseteq \mathcal{T}$$

in which case the wff-s in $\mathcal{T}$ are called the *theorems* of the axiomatic system $\mathcal{A}$.

Of course, one should not forget that the set $\mathcal{T}$ of theorems depends essentially not only on the axioms in $\mathcal{A}$, but also on the deduction rules $\mathcal{R}$. Consequently, it is appropriate to write

(2.4) $\quad \mathcal{T}_{\mathcal{R}}(\mathcal{A})$

for the set $\mathcal{T}$ of theorems.

It is important to note that *no* disconnect of any kind can arise between a system of axioms $\mathcal{A}$ and its corresponding theory given by the theorems $\mathcal{T}_{\mathcal{R}}(\mathcal{A})$. Indeed, no any other condition, let alone of a physical nature, is required about the set of axioms $\mathcal{A}$, except that mentioned in (2.2) above.

Here are some of the relevant questions which can arise, however, regarding systems of axioms :

- are the axioms in $\mathcal{A}$ *independent* ?
- are the axioms in $\mathcal{A}$ *consistent* ?
- are the axioms in $\mathcal{A}$ *complete* ?

Independence means, of course, that for no axiom $P \in \mathcal{A}$, do we have $\mathcal{T}_{\mathcal{R}}(\mathcal{A}) = \mathcal{T}_{\mathcal{R}}(\mathcal{B})$, where $\mathcal{B} = \mathcal{A} \setminus \{P\}$. In other words, the axioms in $\mathcal{A}$ are minimal in order to obtain the theorems in $\mathcal{T}_{\mathcal{R}}(\mathcal{A})$. Of course, this condition can be formulated more simply and sharply by saying that for no axiom $P \in \mathcal{A}$, do we have $P \in \mathcal{T}_{\mathcal{R}}(\mathcal{B})$, where $\mathcal{B} = \mathcal{A} \setminus \{P\}$.

As for consistency, it means that there is no $P \in \mathcal{T}_{\mathcal{R}}(\mathcal{A})$, such that for its negation *non* P, we have *non* $P \in \mathcal{T}_{\mathcal{R}}(\mathcal{A})$.

Completeness, in one possible formulation, means that, given any axiom $P \in \mathcal{F} \setminus \mathcal{A}$ which is independent from $\mathcal{A}$, the axiom system $\mathcal{B} = \mathcal{A} \cup \{P\}$ is inconsistent.

3. Possible Reasons for Disconnect

Turning now to the axiomatization of various theories of Physics, the following essential *difference* easily appears, as illustrated in section 1, by the example of Special Relativity.

Namely, a *theory of Physics* may often be seen as a theory, like for instance, in Mathematics. Yet nevertheless, in reality it is a world of phenomena which is *not* a formal construct such as a set of theorems in (2.4) above. Indeed, a theory of Physics is rather a set - not necessarily limited and fixed - of various experiments, applications, and other effective direct physical, and not merely formal, involvements in which we humans or other physical entities participate.

Consequently, it may simply happen to be presumptuous to expect, let alone claim, that a perfect formulation by a finite number of assumed to be physically meaningful axioms would precisely describe the respective theory of physics.

In the sequel are listed certain facts in this regard which may as well help in the clarification of the disconnect mentioned in the case of Special Relativity, and why not, possibly more generally in Physics.

As a general remark, one should note the following.

When formulating any kind of axioms, and among them, physically meaningful ones, it is seldom - if at all - taken into account that not a small number of assumptions do quite automatically, and thus less than consciously and instead rather tacitly, are somewhat inevitably added to the respective axioms. Such assumptions do usually belong to what one may call the "cultural background" of those who happen to formulate the axioms, be they physicists or not, and such a background may contain elements of a scientific nature, as well as of a number of different kind, less or hardly at all related to science, let alone to Physics as such.

Here for the sake of a better focus we shall only consider assumptions of the kind physicists may make in a large variety of the situation they happen to be involved.

A typical such assumption is about Logic, more precisely, about the kind of logic one uses when building up a theory, or analyzing experiments, applications, and other effective direct physical involvements in which physicists or physical entities participate.
In this regard, rather as a norm, the usual binary valued Logic with the rule of double negation is used, the Logic set up more than two millennia ago by Aristotle.

Nowadays however, we humans are more sophisticated in this regard than the ancient Aristotelian view of Logic can ever allow.
For instance, about a century ago already, a Logic without the rule of double negation had been used in studies related to the Foundation of Set Theory.
More recently, in the 1980s, considerations in theoretical Computer Science led to two considerable enlargements of Logic, first by the inclusion of Self-Referential Logic, and then of Inconsistent Logic, [22].

Needless to say, the issue regrading Logic is not that physicists should now urgently all jump into such rather unfamiliar realms of Logic. No, the issue is simply to realize that, paraphrasing the dispute between Einstein and Bohr regarding whether God plays or not dice, one fact seems absolutely clear : God is not restricted to the good old Aristotelian binary valued Logic with the rule of double negation ...
And then, why should physicists assume - even if tacitly - that physical theories should ?

A second typical assumption is about the scalars used in Physics. These scalars, and in fact, many spaces, space-times or other backgrounds of physical theories are all constructed upon the usual field $\mathbb{R}$ of real numbers. This fact, quite unknown to nearly everybody in Physics, and quite amusingly in Mathematics as well, originates in ancient Egypt where Euclidean Geometry accepted the Archimedean Axiom, [10-23]. The severe limitations, as well as negative consequences, are of course inevitable, even if equally unknown. Among the limitations are the impossibility to have infinitely small and infinitely large quantities, while among the negative consequences are the so called “infinities in Physics” which, so far, have not found any kind of rigorous enough treatment.

Related to the mathematical framework used to express axioms and develop corresponding theories in Physics, a remarkable departure happened in the last few decades, and one far more impressive in its generality than the above related to setting aside the Archimedean Axiom. Namely, no less than the very general mathematics of Category Theory started to be used in Quantum Mechanics. And here it can be recalled that Category Theory is so general as to include as a mere particular case that of Set Theory which constitutes the background of nearly all of present day Mathematics.

The use of Category Theory in Physics has been pioneered and considerably developed by C Isham, B Coecke and S Abramsky, among others. And in this regard the following adds a further relevant aspect worth mentioning.
Initially, Category Theory was introduced by Eilenberg and MacLane in the early 1940s, due to certain motivations arising from such rather arcane branches of Mathematics like Algebraic Topology. And for a number of following decades, that theory was of interest to a rather exclusive group of mathematicians only. Recently however, a dramatic further extension of Category Theory, namely, the so called n-Category Theory, was initiated by theoretical physicists, and not by mathematicians ...

The above two assumptions - namely, related to Logic and scalars - were clearly of a rather general nature, thus not so much related to Physics alone, the kind of assumptions to which we turn now.

Here above all, it is about the inherent and inevitable limitation of each formal language upon which one develops an axiomatic system and the corresponding theory, as briefly indicated in section 2 above. Specifically, it has to be recognized as being in general no more than a mere assumption that a given axiomatic theory of a certain theory of Physics does indeed model that theory, that is, nothing more, and at the same time, nothing less. Indeed, as seen above, this is certainly not the case with the usual two axioms supposed to define Special Relativity. And then, why should it be the case with considerably more complex theories of Physics, such as those about gravity, quanta, or quanta and gravity considered together ?

Of course, the venture to set up axiomatic theories of Physics is so tempting and fascinating, giving among others a most rewarding opportunity for that famed "physical intuition" to show itself as operating in its fuller glory ...
As it happens, however, those intuitions end up under the realms of the above kind of limitations to which assumptions expressed in languages are subjected ...
And when it comes to limitations, well, only God may know about all of them, to return to yet another paraphrasing of the mentioned argument between Einstein and Bohr ...

Last but not least, it may perhaps be the time to draw some attention upon the following citation :

> Conditio Humana
>
> Evidently, the world is never given to us directly but only as it appears on our inner screen. This trivial fact, which in philosophical terminology is just the phenomenal character of the world, when taken seriously, has far reaching consequences. Everything we sensually or intellectually conceive of our world is shaped and conditioned in a categorial way by the mode of our existence as conscious individuals. Naive realism asserts that the world appears to us more or less "like it really is". Sometimes our categorial cognitive structure is compared to a pair of colored sunglasses, which can be taken off to allow a look at the real world. But also this optimistic belief underestimates the inexorable phenomenality of our existence, which must be the starting point of every reflection about the way we orient ourselves in our world. In particular, physics cannot lay its own foundations but has to be aware of the categorial prerequisites imposed by our cognitional system and our mode of existence. In this spirit we mention that a

measurement should not entirely be conceived as a physical process but also as an act of cognition. This also prevents a complete causal closure of physics. Of course, the physical process accompanying measurement has to be investigated and consistency with the possibility of cognition must be guaranteed. A strict physical reductionism, trying to reduce "everything" to physics, is unaware of the phenomenal character of the world and, hence, of its own foundations. Moreover, it runs into the naive methodological mistake to identify the model with what is modelled. The main structural features of the phenomenal mode of human existence have already been mentioned in passing. We briefly collect them here :

- The figure of oppositeness. In every act of cognition we experience ourselves as an observer, different and set apart from what we observe. This is sometimes referred to as the egocentricity of human existence. The epistemic cut between observer and observed is never absent.
- Temporality. Human existence is inescapably temporal in the sense of a future directed time with a privileged "now".
- Factuality. We live in a world of facts rather than a world of potentialities. Everything which appears to us, primarily touches us in the form of a fact. In particular, the "now" carries the imprint of prototypic factuality.

These basic existential features are deeply encoded in the structure of quantum theory. The naturalness and, in a way, a priori structure of quantum theory has been observed by many :

- The epistemic cut is present in the very special and fundamental role attributed to measurement in quantum theory. We saw that observables are located right on the epistemic cut. Standard reductive physicalism ignores the importance of the observer and the epistemic cut in favor of the outside world. In this sense, it is as one-sided and implausible as a solipsistic world view, which ignores the outside in favor of the inside world.

- Factuality is intimately related to quantum theoretical measurement, which basically amounts to a transition from potentiality to a measurement result of factual validity.

The categorial scheme of human existence is, of course, the product of a long development. The temporality of primitive animals is a total subjection to the undivided factuality of a simple "now". Memory and the possibility of preparing actions open up the horizon of temporality eventually resulting in a differentiation between past, present and future. Causality and personal freedom, which are often considered to be in contradictory relationship, actually rely on one another and are in fact offshoots of the same root of such a developed and differentiated temporality. This phylogenetic process is repeated in quick motion in the ontogenesis of every human individual. Related to the unfolding of temporality there is an emancipation from the close binding to primitive factuality. Free exploration of the of various possibilities comes into sight with the capacity for hypothetical and counterfactual thinking. Along with this emancipation goes a deepening of the epistemic cut. The precise form of human existence undergoes a process of varied cultural evolution and also shows large individual differences. Development goes on: Man is always rebellious against his categorical limitations. Philosophy, science and arts grant visions on timeless structures. Utopianism challenges factuality, while integrative world views by embedding man into a comprehensive universe try to alleviate the egocentricity of the epistemic cut.

Indeed, as pointed out, for instance in [9], and as mentioned in the citation above, a proper closure, or in other words, self-consistency of Physics, simply cannot be attained to any extent as long as the thought processes involved when doing Physics are kept as if being absolutely outside of Physics.

Well, until recent times, such an "epistemic cut" may have been motivated by the inordinate fear of self-referentiality imposed upon Western thinking ever since the ancient Greek Paradox of the Liar ...

Strangely enough, however, such a reaction to self-referentiality has not been the only one in human history. Indeed, as anthropologists tell us, three of the most important far older ideas than ancient Greece, ideas present already in pre-literate societies, have been change, self-reference and infinity ...
Also, in ancient Hebrew tradition, somewhat earlier than the Greek one, self-reference is far from being a horror to be avoided by all means, and instead, it is nothing less than the very name of God, as mentioned in Exodus 3:14.
But then, as mentioned above, there is by now a couple of decades old perfectly rigorous and valid mathematical approach to self-referentiality, an approach brought about by no less a modern and practical theory than that of theoretical Computer Science ...

And then, perhaps, it is high time that the so totally and relentlessly pursued avoidance of the "epistemic cut" in Physics may at last be abandoned, and thus together with it, its far more serious possible consequences in Physics, than the disconnect discussed in this paper ...

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