Apparent Measure and Relative Dimension

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A short version of this paper was published in 2008
Journal Européen des Systèmes Automatisés.
Fractional order systems, 42, p 733-746

In this paper, we introduce a concept of "apparent" measure in $\mathbb{R}^n$ and we define a concept of relative dimension (of real order) with it, which depends on the geometry of the object to measure and on the distance which separates it from an observer, at the end we discuss the relative dimension of the Cantor set. This measure enables us to provide a geometric interpretation of the Riemann-Liouville's integral of order $\alpha \in]0, 1]$.

Key Words: Riemann-Liouville fractional operators, measure, fractals.

1 Introduction

The theoretical and practical interest of fractional operator appears naturally in several fields of science and engineering, including fluid flow, rheology, modelling, signal processing, diffusion problem and others, ([8],[15],[11],[10]), in spite of the lack of a geometrical interpretation of this operator [1]. In this paper, we give a geometrical interpretation of the Riemann-Liouville's integral of order $\alpha \in]0, 1]$, ([12],[10],[4]), as a surface which can be evaluated and compared to the classical surface of an integral function. We introduce a concept of "apparent" measure, with respect to a point x_0, that changes value according to the distance of the object to be measured from x_0 (This kind of measure can be used in 3D animation). We introduce and prove some properties of this measure. The concept of relative dimension follows from this apparent measure. For example we discuss the "small irregular" Cantor set having Lebesgue measure zero and Hausdorff measure one. Thanks to the "apparent" measure we interpret the Riemann-Liouville's integral and we modified it.

2 Measure of real order

2.1 Measure in $\mathbb{R}$

Let $\mathcal{I}_{\mathbb{R}}$ be the set of all bounded open intervals of $\mathbb{R}$, of the form $I =]a, b[$, where $a \leq b$. In the case where $a = b$ we have $\emptyset \in \mathcal{I}_{\mathbb{R}}$. We define the length $\mathcal{L} : I_{\mathbb{R}} \longrightarrow \overline{\mathbb{R}}_+$ by $\mathcal{L}(]a, b[) = b - a$, and we have

Definition 1 *For any $x_0 \in \mathbb{R}$ we call the length of order $\alpha \in \mathbb{R}_+^*$, the map*

$\mu_{x_0}^\alpha : \mathcal{I}_{\mathbb{R}} \longrightarrow \overline{\mathbb{R}}_+$, *defined by*

$$\mu_{x_0}^\alpha(]a, b[) = \begin{cases} ||x_0 - a|^\alpha - |x_0 - b|^\alpha| & \text{if} \quad x_0 \notin]a, b[\\ |x_0 - a|^\alpha + |x_0 - b|^\alpha & \text{if} \quad x_0 \in]a, b[. \end{cases} \tag{1}$$

Remarks. 1) If $x_0 \notin]a, b[$, the length

$$\mathcal{L}(]a, b[) = b - a = b - x_0 + x_0 - b = \Big| \ |x_0 - a| - |x_0 - b| \ \Big| = \mu_{x_0}^1(]a, b[)$$

and if $x_0 \in]a, b[$, we have

$$\mathcal{L}(]a, b[) = (x_0 - a) + (b - x_0) = \mu_{x_0}^1(]a, b[).$$

This length is independent of the point x_0, it is intrinsic. On the other hand the length of real order $\mu_{x_0}^\alpha$ for $\alpha \in \mathbb{R}_+^*$ depends on the point x_0, for example, for $\alpha \in]0, 1[$, the further the point x_0 is from the interval $]a, b[$, the smaller the length of order α of this interval appears. Thanks to this dependence it can be called "apparent" length. It mimics the impression an observer gets for the measure of an objects: it diminishes when it gets further. The extension of the concept is done in section.

2) Obviously we have $\mu_{x_0}^\alpha(\emptyset) = 0$, and the function $\mu_{x_0}^\alpha$ can be considered as a density measure indeed:

a) If $x_0 \geq b$, we denote

$$\mu_{x_0}^\alpha(]a, b[) = (x_0 - a)^\alpha - (x_0 - b)^\alpha = \int_a^b \alpha(x_0 - t)^{\alpha - 1} dt = \int_a^b d\mu_{x_0}^\alpha(t). \tag{2}$$

with

$$d\mu_{x_0}^\alpha(t) = \alpha|x_0 - t|^{\alpha - 1} dt \tag{3}$$

b) If $x_0 \leq a$, we denote

$$\mu_{x_0}^\alpha(]a, b[) = (b - x_0)^\alpha - (a - x_0)^\alpha = \int_a^b \alpha(t - x_0)^{\alpha - 1} dt = \int_a^b d\mu_{x_0}^\alpha(t). \tag{4}$$

with

$$d\mu_{x_0}^{\alpha}(t) = \alpha|x_0 - t|^{\alpha-1}dt \tag{5}$$

c) If $x_0 \in [a, b]$, we have

$$\mu_{x_0}^{\alpha}(]a, b[) = (x_0 - a)^{\alpha} + (b - x_0)^{\alpha} = \int_a^{x_0} \alpha(x_0 - t)^{\alpha-1}dt + \int_{x_0}^b \alpha(t - x_0)^{\alpha-1}dt$$
$$= \int_a^b d\mu_{x_0}^{\alpha}(t). \tag{6}$$

$\mu_{x_0}^{\alpha}$ is then a measure of density $f(t) = \alpha|t - x_0|^{\alpha-1}$, it can be extended to any Borel set ("apparent" length of a Borel set).

2.2 Properties of $\mu_{x_0}^{\alpha}$

Let $]a, b[$ be a non empty interval of $\mathbb{R}$, thanks to a simple study, it can be said that the application $x \longmapsto g(x_0) = \mu_{x_0}^{\alpha}(]a, b[)$ is increasing in $]-\infty, \frac{a+b}{2}[$ and decreasing in $]\frac{a+b}{2}, +\infty[$ for $\alpha \in]0, 1[$ and conversely for $\alpha \in]1, +\infty[$, whereas it is constant for $\alpha = 1$. The measure of order α of the interval $]a, b[$ is maximum when the point x_0 is at the center of the interval $]a, b[$ and $\alpha \in]0, 1[$, and it becomes minimal at this point for $\alpha \in]1, +\infty[$. We state now, without proof, some simple properties of this measure.

Properties 1 *Let $]a, b[\subset \mathbb{R}$, with $a < b$, $x_0 \in \mathbb{R}$, we have*

1) (Scaling) For all $x_0 \in \mathbb{R}$, and all $\lambda \in \mathbb{R}_+^*, \alpha \in \mathbb{R}_+^*$,

$$\mu_{\lambda x_0}^{\alpha}(]\lambda a, \lambda b[) = \lambda^{\alpha}\mu_{x_0}^{\alpha}(]a, b[). \tag{7}$$

2) (Translation invariance) For all $c \in \mathbb{R}$, $\alpha \in \mathbb{R}_+^*$, we have

$$\mu_{x_0+c}^{\alpha}(]a + c, b + c[) = \mu_{x_0}^{\alpha}(]a, b[). \tag{8}$$

3) For all $\alpha \in \mathbb{R}_+^*$, we have

$$\mu_a^{\alpha}(]a, b[) = \mu_b^{\alpha}(]a, b[) = (b - a)^{\alpha}. \tag{9}$$

4) (Continuity) For all $x_0 \in \mathbb{R}$, $\mu_{x_0}^{\alpha}(]a, b[)$ tends towards $\mathcal{L}(]a, b[)$ when α tends to 1.

5) If $\alpha \in]0, 1[$, we have

$$\mu_a^{\alpha}(]a, b[) = \mu_b^{\alpha}(]a, b[) \geq \mathcal{L}(]a, b[) \quad \text{if} \quad b - a \leq 1 \tag{10}$$
$$\mu_a^{\alpha}(]a, b[) = \mu_b^{\alpha}(]a, b[) < \mathcal{L}(]a, b[) \quad \text{if} \quad b - a > 1. \tag{11}$$

6) For all real numbers a', b', such that $a' > a$, $b' < b$ and $x_0 \in \mathbb{R}\backslash]a, b[$, we have

$$\mu_{x_0}^{\alpha}(]a, b[) > \mu_{x_0}^{\alpha}(]a', b'[), \qquad \forall \alpha \in \mathbb{R}_+^*. \tag{12}$$

Remark. Equation (7) is equivalent to

$$\text{for all} \quad \rho > 0 \qquad , \qquad d\mu_x^\alpha(\rho t) = \rho^\alpha d\mu_{\frac{x}{\rho}}^\alpha(t). \tag{13}$$

2.3 Measure of real order in $\mathbb{R}^n$

We start with the definition of the measure of a simple set. A brick P is defined in $\mathbb{R}^n$ by n inequality of the form

$$a_i \leq x_i \leq b_i \qquad , \qquad (i = 1,\ 2, \dots,\ n) \tag{14}$$

where the a_i and b_i indicate given real numbers for all $i = 1,\ 2, \dots,\ n$.

Definition 2 *Let P be a parallelepiped in $\mathbb{R}^n$, and let $x_0 \in \mathbb{R}^n$, $\alpha \in \mathbb{R}_+^*$. We call measure of order α of P with respect to x_0 the quantity*

$$\mu_{x_0}^\alpha(P) = \prod_{i=1}^n \Big| |x_{0,i} - a_i|^\alpha - |x_{0,i} - b_i|^\alpha \Big| \qquad \textit{if} \quad x_0 \notin P \tag{15}$$

$$(\textit{resp.} \quad \mu_{x_0}^\alpha(P) = \prod_{i=1}^n (x_{0,i} - a_i)^\alpha + (b_i - x_{0,i})^\alpha \qquad \textit{if} \quad x_0 \in P). \tag{16}$$

Remark. We can obtain an equivalent of Fubini's theorem for the product of measure of real order. We let the readers formulate it. We can also build up in a traditional way product measure on $\mathbb{R}^n$.

2.4 Measure of a bounded set of $\mathbb{R}^n$

Let A be a subset of $\mathbb{R}^n$, and let 1_A be the characteristic function of A and $x_0 \in \mathbb{R}^n$, $\alpha \in \mathbb{R}_+^*$. We can define in a traditional way the measure of a bounded subset of $\mathbb{R}^n$.

Definition 3 *let $A \subset \mathbb{R}^n$, we will say that A is α-measurable if its characteristic function is $\mu_{x_0}^\alpha$-integrable. The number*

$$\mu_{x_0}^\alpha(A) = \int_{\mathbb{R}^n} 1_A(t) d\mu_{x_0}^\alpha(t) \tag{17}$$

is called the measure of order α of A compared to x_0.

Remark. For n=2 this measure is called apparent area with respect to x_0, and for n=3 it is called apparent volume with respect to x_0. The various properties of this measure follow easily from the the properties of the integral. We can also obtain a concept of apparent mass for a material system by proceeding in the same way as for the Lebesgue measure, and the same properties can be found. Remind finally that the measure of a set depends on the dimension of the space in which it is considered.

As an example the measure of order α of a segment, regarded as a subset of $\mathbb{R}$ is equal to its length of order α, but if it is regarded as plunged in $\mathbb{R}^2$, it's measure of order α is equal to zero.

In the continuation we will define a more precise measure of a real order without the use of integral. First of all we remind the definition of diameter of a set in a metric space.

Definition 4 *Let (E, d) be a metric space. If $A \subset E$, $A \neq \phi$, we call diameter of A and it is noted $diam(A)$, the element of $[0, +\infty[$ defined by*

$$diam(A) = \sup_{(x,y)\in A^2} d(x, y) \tag{18}$$

with $diam(\emptyset) = 0$

Definition 5 *Let $x_0 \in \mathbb{R}^n$, $B \subset \mathbb{R}^n$ and let P_i be a brick of $\mathbb{R}^n$ for all $i \in \mathbb{N}^*$. We define*

$$\begin{aligned} \mu_{x_0}^{\alpha,\delta}(B) &= \inf\Big\{\sum_{i=1}^{\infty}\mu_{x_0}^{\alpha}(P_i), \quad B \subset \bigcup_{i=1}^{\infty} P_i,\ diam(P_i) \le \delta\Big\} \\ &= \inf\Big\{\sum_{i=1}^{\infty}\prod_{j=1}^{n}\mu_{x_{0,j}}^{\alpha}([a_{i,j}, b_{i,j}]), \quad B \subset \bigcup_{i=1}^{\infty} P_i,\ diam(P_i) \le \delta\Big\}, \end{aligned} \tag{19}$$

with $\alpha \in]0, 1]$. The measure of order α of B with respect to x_0 is given by

$$\mu_{x_0}^{\alpha}(B) = \lim_{\delta \longrightarrow 0}\mu_{x_0}^{\alpha,\delta}(B) = \sup_{\delta>0}\mu_{x_0}^{\alpha,\delta}(B). \tag{20}$$

Remarks. 1) The choice to let δ tend towards 0 enables us to have a concept of more precise measure where the selected recovery follows the local geometry of the object.

2) The point x_0 represents the position from which the object is measured, this measure being therefore dependent on this position (Fig.1).

3) The last definition coincides with definition 2 and 3 for $\alpha \in]0, 1[$.

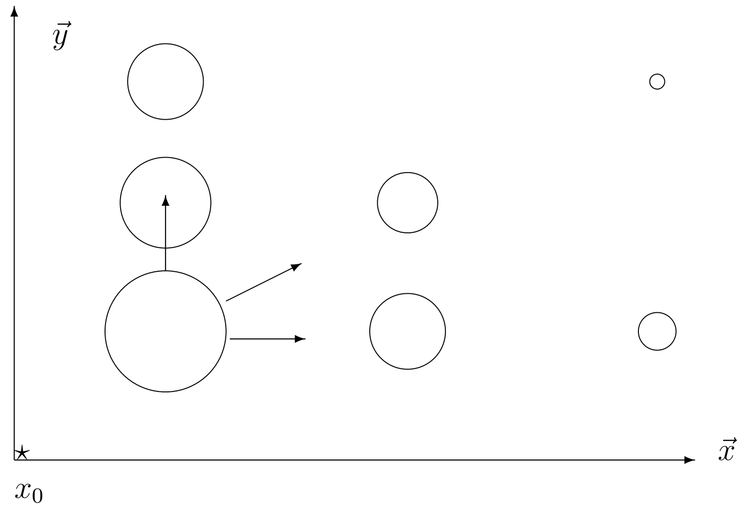

Figure 1. The apparent measure of the disc for n=2

2.5 Properties of the measure

Theorem 1 *1)* $\mu_{x_0}^{\alpha,\delta}$ *is an outer measure*

2) $\mu_{x_0}^{\alpha}$ *is an outer measure*

3) $\mu_{x_0}^{\alpha}$ *is a Borel measure.*

Proof: 1) $\mu_{x_0}^{\alpha,\delta}$ is an outer measure, indeed:

i) We have well $\mu_{x_0}^{\alpha,\delta}(B) \in \bar{\mathbb{R}}_+$ for all $B \subset \mathbb{R}^n$.

ii)

$$\mu_{x_0}^{\alpha,\delta}(\emptyset) = \inf\left\{\sum_{i=1}^{\infty}\mu_{x_0}^{\alpha}(P_i), \quad \emptyset \subset \bigcup_{i=1}^{\infty} P_i,\ diam(P_i) \leq \delta\right\} = 0.$$

Indeed it is enough to take $P_i = \{x \in \mathbb{R}^n / \quad a_i \leq x_i \leq a_i\}$.

iii) If $(A_k)_{k\geq 1}$ is a sequence of a subset of $\mathbb{R}^n$, let show that

$$\mu_{x_0}^{\alpha,\delta}(\bigcup_{k=1}^{\infty} A_k) \leq \sum_{k=1}^{\infty}\mu_{x_0}^{\alpha,\delta}(A_k).$$

If $\sum_{k=1}^{\infty}\mu_{x_0}^{\alpha,\delta}(A_k) = +\infty$, the inequality is obvious, if not we have $\mu_{x_0}^{\alpha,\delta}(A_k) < +\infty$ for all $k \geq 1$. There exists a sequence $(P_{i,k})_{i\geq 1}$ of a brick of $\mathbb{R}^n$ such as $A_k \subset \bigcup_{i=1}^{\infty} P_{i,k}$, and for all $N > 0$ we obtain

$$\sum_{i=1}^{\infty}\mu_{x_0}^{\alpha}(P_{i,k}) < \mu_{x_0}^{\alpha,\delta}(A_k) + \frac{1}{N2^k}.$$

Thus we have

$$\bigcup_{k=1}^{\infty} A_k \subset \bigcup_{i,k=1}^{\infty} P_{i,k}$$

a countable covering of a brick of $\mathbb{R}^n$. Then we have for all $N > 0$

$$\mu_{x_0}^{\alpha,\delta}\Big(\bigcup_{k=1}^{\infty} A_k\Big) \leq \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} \mu_{x_0}^{\alpha}(P_{i,k}) < \sum_{k=1}^{\infty} \mu_{x_0}^{\alpha,\delta}(A_k) + \frac{1}{N},$$

we then deduce the inequality.

2) $\mu_{x_0}^{\alpha}$is an outer measure, indeed, let $(A_k)_{k\geq 1} \subset \mathbb{R}^n$. Since

$$\mu_{x_0}^{\alpha,\delta}\Big(\bigcup_{1}^{\infty} A_n\Big) \leq \sum_{n=1}^{\infty} \mu_{x_0}^{\alpha,\delta}(A_n),$$

and since $\sup_{\delta>0} \mu_{x_0}^{\alpha,\delta}(A_k) = \mu_{x_0}^{\alpha}(A_k)$ we have then

$$\mu_{x_0}^{\alpha,\delta}\Big(\bigcup_{1}^{\infty} A_n\Big) \leq \sum_{n=1}^{\infty} \mu_{x_0}^{\alpha}(A_n).$$

When δ tends to 0, we obtain

$$\mu_{x_0}^{\alpha}\Big(\bigcup_{1}^{\infty} A_n\Big) \leq \sum_{n=1}^{\infty} \mu_{x_0}^{\alpha}(A_n).$$

3) $\mu_{x_0}^{\alpha}$ is a Borel measure, indeed, let A, B be a subset of $\mathbb{R}^n$, such as $d(A,B) > 0$, with

$$d(A,B) = inf\Big\{ \quad d(x,y), \forall x \in A, \forall y \in B \quad \Big\}$$

then let us just consider $d(A,B) > 4\delta$ for example, to get a right covering for equality.

Properties 2 *Let $\alpha \in]0,1]$, $x_0 \in \mathbb{R}^n$, and let $\mu_{x_0}^{\alpha}$ be a measure on $\mathbb{R}^n$. We have*

1) $\mu_{x_0}^{\alpha}(B) = 0$ for all $B \subset \mathbb{R}^{n'}$ and $n' < n$, where $\mathbb{R}^n$ is isomorphic to $\mathbb{R}^{n'} \times \mathbb{R}^{n-n'}$ for any integer $n' < n$.

2) $\mu_{x_0}^{1}$ coincide with the Lebesgue's measure on $\mathbb{R}^n$.

3) For all $\lambda > 0$, and all $B \subset \mathbb{R}^n$, we have $\mu_{\lambda x_0}^{\alpha}(\lambda B) = \lambda^n \mu_{x_0}^{\alpha}(B)$.

Proof: 1) We have

$$\mu_{x_0}^{\alpha,\delta}(B) = \inf\Big\{\sum_{i=1}^{\infty}\prod_{j=1}^{n} \mu_{x_{0,j}}^{\alpha}([a_{i,j}, b_{i,j}]), \quad B \subset \bigcup_{i=1}^{\infty} P_i,\ diam(P_i) \leq \delta\Big\},$$

with $\alpha \in]0,1]$. Since $n' < n$, P_i can be chosen in $\mathbb{R}^{n'}$, the $n-n'$ remaining coordinated are null, we have then the result.

This property means that the measure of a set depends on the dimension of the space in which is considered.

2) On $\mathbb{R}$ and for $\alpha = 1$ we have:

$$\mu^{\alpha}_{x_0}(P_i) = diam(P_i),$$

with $\mu^1_{x_0}(A) = H^1(A)$, and since the Hausdorff's measure, H^1 coincide with the Lebesgue's measure on $\mathbb{R}$, we deduce the result.

3/ For all $\lambda > 0$, we have

$$\mu^{\alpha,\delta}_{\lambda x_0}(\lambda B) = \inf\Big\{ \sum_{i=1}^{\infty}\prod_{j=1}^{n} \mu^{\alpha}_{\lambda x_{0,j}}([\lambda a_{i,j}, \lambda b_{i,j}]), \quad B \subset \bigcup_{i=1}^{\infty} P_i,\ diam(P_i) \leq \delta \Big\}.$$

Since

$$\sum_{i=1}^{\infty}\prod_{j=1}^{n} \mu^{\alpha}_{\lambda x_{0,j}}([l a_{i,j}, \lambda b_{i,j}]) = \lambda^n \sum_{i=1}^{\infty}\prod_{j=1}^{n} \mu^{\alpha}_{x_{0,j}}([a_{i,j}, b_{i,j}]),$$

we deduce that

$$\mu^{\alpha,\delta}_{\lambda x_0}(\lambda B) = \lambda^n \mu^{\alpha,\delta}_{x_0}(B),$$

which by letting δ tend to 0, gives

$$\mu^{\alpha}_{\lambda x_0}(\lambda B) = \lambda^n \mu^{\alpha}_{x_0}(B).$$

Proposition 1 *Let $\mathcal{P}_j$ be the projection of $\mathbb{R}^n$ according to the j-th coordinates. Let $\alpha \in]0,1]$, $A \subset \mathbb{R}^n$ and let $x_0, x_1 \in \mathbb{R}^n$,*

If $\forall j \in \{1,2,3\ldots,n\}$ we have $\inf \mathcal{P}_j(A) \geq x_{0,j} \geq x_{1,j}$,

then we have $\mu^{\alpha}_{x_0}(A) \geq \mu^{\alpha}_{x_1}(A)$.

Proof: Let P_i be a brick of $\mathbb{R}^n$ defined by the n inequality of the form

$$a_{i,j} \leq x \leq b_{i,j} \qquad (j = 1,\ldots,n)$$

with $a_{i,j} \leq b_{i,j}$ for all $j = 1,\ldots,n$, such as $A \subset \bigcup_{i=1}^{\infty} P_i$ and $diam(P_i) \leq \delta$ for all i. If $\forall j \in \{1,2,3\ldots,n\}$ we have $\inf \mathcal{P}_j(A) \geq x_{0,j} > x_{1,j}$, we deduce that $\forall i$ $\inf \mathcal{P}_j(P_i) \geq x_{0,j} > x_{1,j}$ then according to the properties of measure of real order in $\mathbb{R}$, we have

$$\mu^{\alpha}_{x_{0,j}}([a_{i,j}, b_{i,j}]) \geq \mu^{\alpha}_{x_{1,j}}([a_{i,j}, b_{i,j}])$$

for all $(i,j) \in \mathbb{N}^* \times \{1,\ldots,n\}$, of which

$$\sum_{i=1}^{\infty}\prod_{j=1}^{n} \mu^{\alpha}_{x_{0,j}}([a_{i,j}, b_{i,j}]) \geq \sum_{i=1}^{\infty}\prod_{j=1}^{n} \mu^{\alpha}_{x_{1,j}}([a_{i,j}, b_{i,j}]).$$

and we obtain

$$\mu_{x_0}^{\alpha,\delta}(A) \geq \mu_{x_1}^{\alpha,\delta}(A),$$

which completes the proof.

Theorem 2 *Let $f : B \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a function defined algebraically by*

$$f(x) = \Lambda x + C, \tag{21}$$

with C is a constant vector, Λ is a matrix $\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{pmatrix}$ with λ_i a real positive for all $i = 1, \ldots, n$. Let $x_0 \in B$, and let $A \subset B$. We have

$$\mu_{f(x_0)}^{\alpha}(f(A)) = (\prod_{j=1}^{n} \lambda_j^{\alpha})\ \mu_{x_0}^{\alpha}(A). \tag{22}$$

Proof: Let P_i be a brick of $\mathbb{R}^n$ defined by the n following inequality

$$a_{i,j} \leq x_j \leq b_{i,j} \qquad (j = 1, \ldots, n)$$

with $a_{i,j} \leq b_{i,j}$ for all $j = 1, \ldots, n$, such as $A \subset \bigcup_{i=1}^{\infty} P_i$ and $diam(P_i) \leq \delta$ for all i.

$f(P_i)$ is a brick of $\mathbb{R}^n$ defined by the n inequality of the form

$$\lambda_j a_{i,j} + c_j \leq x_j \leq \lambda_j b_{i,j} + c_j \qquad (j = 1, \ldots, n)$$

$$\mu_{f(x_0)}^{\alpha} f(P_i) = \prod_{j=1}^{n} \mu_{f(x_0)_j}^{\alpha}\Big([\lambda_j a_{i,j} + c_j, \lambda_j b_{i,j} + c_j]\Big) \tag{23}$$

$$= \prod_{j=1}^{n} \Big|\Big|f(x_0)_j - (\lambda_j b_{i,j} + c_j)\Big|^{\alpha} - \Big|f(x_0)_j - (\lambda_j a_{i,j} + c_j)\Big|^{\alpha}\Big| \tag{24}$$

$$= \prod_{j=1}^{n} \lambda_j^{\alpha}\Big|\Big|x_{0,j} - b_{i,j}\Big|^{\alpha} - \Big|x_{0,j} - a_{i,j}\Big|^{\alpha}\Big| \tag{25}$$

$$= (\prod_{j=1}^{n} \lambda_j^{\alpha})\ \mu_{x_0}^{\alpha}(P_i) \tag{26}$$

what it enables us to obtain the result.

3 Measure and dimension of real order

The measure defined in the preceding paragraph can not define a concept of dimension, so following the concept of Hausdorff measure let us build up the following measure:

Definition 6 *Let $(P_i)_{i\in\mathbb{N}^*}$ be a sequence of a parallelepiped of $\mathbb{R}^n$ covering $B \subset \mathbb{R}^n$ and $x_0 \in \mathbb{R}^n$. Let $\beta \in [0, +\infty[$ and $\alpha \in]0, 1]$. We define*

$$H_{x_0}^{\beta(\alpha),\delta}(B) = \inf\Big\{ \sum_{i=1}^{\infty} \Big(\sup_{1\leq j\leq n} \mu_{x_0,j}^{\alpha}([a_{i,j}, b_{i,j}])\Big)^{\beta}, \quad B \subset \bigcup_{i=1}^{\infty} P_i,\ diam(P_i) \leq \delta \Big\} \tag{27}$$

the measure of order (α, β) of B compared to x_0 is given by

$$H_{x_0}^{\beta(\alpha)}(B) = \lim_{\delta\longrightarrow\ 0} H_{x_0}^{\beta(\alpha),\delta}(B) = \sup_{\delta>0} H_{x_0}^{\beta(\alpha),\delta}(B). \tag{28}$$

Theorem 3 *1) $H_{x_0}^{\beta(\alpha),\delta}$ is an outer measure*

2) $H_{x_0}^{\beta(\alpha)}$ is a measure

3) $H_{x_0}^{\beta(\alpha)}$ coincide with the Hausdorff's measure of order β on $\mathbb{R}$ for $\alpha = 1$.

Proof: For 1)and 2) the proof is similar to the proof of the theorem 1.

3) When $\alpha = 1$, we have $\mu_{x_0,j}^{\alpha}([a_{i,j}, b_{i,j}]) = Diam([a_{i,j}, b_{i,j}])$.

If we denote A° the interior of A, we have the following lemma

Lemma 1 *Let A be a subset of $\mathbb{R}^n$, $x_0 \in \mathbb{R}^n \setminus A^{\circ}$, $\alpha \in]0, 1]$ and let $0 \leq \beta < \gamma < \infty$.*

If $\quad H_{x_0}^{\beta(\alpha)}(A) < +\infty \qquad \Longrightarrow \qquad H_{x_0}^{\gamma(\alpha)}(A) = 0.$

Proof: If $\quad H_{x_0}^{\beta(\alpha)}(A) < +\infty$, then there exists a sequence $(P_n)_{n\geq 1}$ of brick of $\mathbb{R}^n$ such that $A \subset \bigcup_{k=1}^{\infty} P_n$ with $diam P_n \leq \delta$, and we have for $k \in \mathbb{N}^*$

$$\sum_{i=1}^{\infty} \Big(\sup_{1\leq j\leq n} \mu_{x_0,j}^{\alpha}([a_{i,j}, b_{i,j}])\Big)^{\beta} \leq H_{x_0}^{\beta(\alpha),\delta}(A) + \frac{1}{k} \leq \sup_{\delta>0} H_{x_0}^{\beta(\alpha),\delta}(A) + \frac{1}{k}.$$

According to the definition of measure we have

$$H_{x_0}^{\gamma(\alpha),\delta}(A) \leq \sum_{i=1}^{\infty} \Big(\sup_{1\leq j\leq n} \mu_{x_0,j}^{\alpha}([a_{i,j}, b_{i,j}])\Big)^{\gamma}$$

$$= \sum_{i=1}^{\infty} \Big(\sup_{1\le j\le n} \mu^{\alpha}_{x_{0,j}}([a_{i,j}, b_{i,j}]) \Big)^{\beta} \Big(\sup_{1\le j\le n} \mu^{\alpha}_{x_{0,j}}([a_{i,j}, b_{i,j}]) \Big)^{\gamma-\beta}.$$

Since

$$\sup_{1\le j\le n} \mu^{\alpha}_{x_{0,j}}([a_{i,j}, b_{i,j}]) \le Diam(P_j), \quad \forall 1 \le j \le n,$$

then we obtain

$$H^{\gamma(\alpha),\delta}_{x_0}(A) \le \delta^{\gamma-\beta} \sum_{i=1}^{\infty} \Big(\sup_{1\le j\le n} \mu^{\alpha}_{x_{0,j}}([a_{i,j}, b_{i,j}]) \Big)^{\beta} \le \delta^{\gamma-\beta} \Big(H^{\beta(\alpha)}_{x_0}(A) + \frac{1}{k} \Big)$$

we deduce from it the result when δ tends to 0.

Thanks to this lemma we introduce the definition of relative dimension:

Definition 7 *Let $A \subset \mathbb{R}^n$, $\alpha \in]0,1]$. We define the relative dimension of a set A by*

$$\dim(A) = \inf \Big\{ \beta : \ H^{\beta(\alpha)}_{x_0}(A) = 0 \Big\} = \inf \Big\{ \beta : \ H^{\beta(\alpha)}_{x_0}(A) < \infty \Big\}. \tag{29}$$

$$= \sup \Big\{ \gamma : \ H^{\gamma(\alpha)}_{x_0}(A) = \infty \Big\} = \sup \Big\{ \gamma : \ H^{\gamma(\alpha)}_{x_0}(A) > 0 \Big\}. \tag{30}$$

Remark. We have the following

i) If $A \subset B \subset \mathbb{R}^n$, $x_0 \in \mathbb{R}^n$ and $x_0 \notin B$, we have $H^{\beta(\alpha)}_{x_0}(A) \le H^{\beta(\alpha)}_{x_0}(B)$.

ii) If $d(x_{0,i}, P_i(A)) \neq 0$ for all $i = 1, 2, ..., n$, then

$$\dim_{H^{\beta(\alpha)}_{x_0}}(A) \le \dim_{\mathcal{H}}(A).$$

Until now, our "apparent" measure depends on the axis in which the object can be plunged. To change this dependence we propose the following definition

Definition 8 *Let $(O_i)_{i\in\mathbb{N}^*}$ be a sequence of open sets of $\mathbb{R}^n$ covering $B \subset \mathbb{R}^n$ and let $x_0 \in \mathbb{R}^n$. Let $\beta \in [0, +\infty[$ and $\alpha \in]0,1]$. We define*

$$\mathcal{H}^{\beta(\alpha),\delta}_{x_0}(B) = \inf \Big\{ \sum_{i=1}^{\infty} \Big((D_{x_0}(O_i))^{\alpha} - (d_{x_0}(O_i))^{\alpha} \Big)^{\beta}, \quad B \subset \bigcup_{i=1}^{\infty} O_i, \ diam(O_i) \le \delta \Big\} \tag{31}$$

where $D_{x_0}(O_i) = \sup_{y\in O_i} d(x_0, y)$ and $d_{x_0}(O_i) = \inf_{y\in O_i} d(x_0, y)$. The measure of order (α, β) of B with respect to x_0 is given by

$$\mathcal{H}^{\beta(\alpha)}_{x_0}(B) = \sup_{\delta>0} \mathcal{H}^{\beta(\alpha),\delta}_{x_0}(B) = \lim_{\delta\longrightarrow\ 0} \mathcal{H}^{\beta(\alpha),\delta}_{x_0}(B). \tag{32}$$

As for the Hausdorff measure we have the following theorem

Theorem 4 *1) $\mathcal{H}_{x_0}^{\beta(\alpha),\delta}$ is an outer measure*

2) $\mathcal{H}_{x_0}^{\beta(\alpha)}$ is a measure

3) $\mathcal{H}_{x_0}^{\beta(\alpha)}$ coincide with the Hausdorff's measure of order β on $\mathbb{R}$ for $\alpha = 1$.

4) $\mathcal{H}_{x_0}^{\beta(\alpha)}(A) = 0$ if $A \subset \mathbb{R}^n$ and $\alpha\beta > n$.

Proof 4) Let m be an integer and let us consider the unit cube $Q \subset \mathbb{R}^n$. This cube can be decomposed into m^n cubes with side $\dfrac{1}{m}$.

Therefore

$$\mathcal{H}_{x_0}^{\beta(\alpha),\delta}(Q) \leq \sum_{i=1}^{m^n} \Big((\frac{1}{m})^\alpha \Big)^\beta = m^n \Big((\frac{1}{m})^\alpha \Big)^\beta = m^{n-\alpha\beta} \tag{33}$$

and we can then conclude.

Then we get the relative dimension given by the definition 7. Although the Hausdorff dimension measures the metric size of any subset in a metric space. We can find an "apparent" Hausdorff measure if we put the "apparent" diameter given by

$$\mu_{x_0}^\alpha(O_i) = \Big(D_{x_0}(O_i)\Big)^\alpha - \Big(d_{x_0}(O_i)\Big)^\alpha, \tag{34}$$

in place of $diam(O_i)$, and we have

Proposition 2 *Let $\alpha \in]0,1]$, $A \subset \mathbb{R}^n$ and let $x_0, x_1 \in \mathbb{R}^n$.*

Let $(O_i)_{i\in\mathbb{N}^*}$ be a sequence of open sets of $\mathbb{R}^n$ covering A,

$$\text{If} \quad \forall i \in \mathbb{N}^*,\ d(x_0, O_i) > d(x_1, O_i) \quad \text{then we have} \quad \mathcal{H}_{x_0}^{\beta(\alpha)}(A) \leq \mathcal{H}_{x_1}^{\beta(\alpha)}(A), \tag{35}$$

where $d(x_i, O_i) = \inf_{y\in A} d(x_i, y)$.

Proof: Let $(O_i)_{i\in\mathbb{N}^*}$ be a sequence of an open set of $\mathbb{R}^n$ covering $A \subset \mathbb{R}^n$ such as $diam(O_i) \leq \delta$, thanks to the properties of the apparent measure defined in the beginning of this article it can be said that if we have $\forall i \in \mathbb{N}^*$, $d(x_0, O_i) > d(x_1, O_i)$ then

$$\Big((D_{x_0}(O_i))^\alpha - (d_{x_0}(O_i))^\alpha \Big)^\beta < \Big((D_{x_1}(O_i))^\alpha - (d_{x_1}(O_i))^\alpha \Big)^\beta \tag{36}$$

which yields

$$\sum_{i=1}^{\infty} \Big((D_{x_0}(O_i))^\alpha - (d_{x_0}(O_i))^\alpha \Big)^\beta < \sum_{i=1}^{\infty} \Big((D_{x_1}(O_i))^\alpha - (d_{x_1}(O_i))^\alpha \Big)^\beta, \tag{37}$$

and then we have the result.

4 Application: The Cantor set in $\mathbb{R}$

Let $0 < \lambda < \frac{1}{2}$. Denote $I_{0,1} = [0, 1]$, and let $I_{1,1}$ and $I_{1,2}$ be the intervals $[0, \lambda]$ and $[1 - \lambda, 1]$, respectively, obtained by taking of from the interval $[0, 1]$ the interval $]\lambda, 1 - \lambda[$ of length $1 - 2\lambda$ and centered at the medium (Fig.2). We continue this process of selecting two subintervals of each already given interval, and so on. If we have defined intervals $I_{k-1,1}$,....,$I_{k-1,2^{k-1}}$, we define the intervals $I_{k,1}$,....,$I_{k,2^k}$ by deleting from the middle of each $I_{k-1,j}$ an interval of length

$$(1 - 2\lambda)d(I_{k-1,j}) = (1 - 2\lambda)\lambda^{k-1}.$$

All the intervals $I_{k,j}$ thus obtained have length λ^k. We define a kind of limit set of this construction by

$$C(\lambda) = \cap_{k=0}^{\infty} \cup_{j=1}^{2^k} I_{k,j}.$$

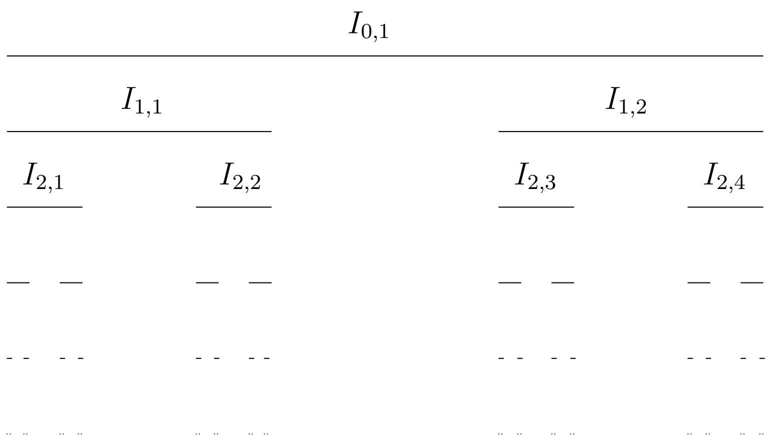

Figure 2. Cantor set

Then $C(\lambda)$ is an uncountable compact set without interior points and with zero Lebesgue measure, and one Hausdorff's measure, and the Hausdorff's dimension is equal to

$$\dim_{\mathcal{H}} = \frac{log2}{log\frac{1}{\lambda}}. \tag{38}$$

To determine the relative dimension of this set and to discuss its apparent measure, there are two cases:

i) Case where $x_0 = 0$, or where $x_0 = 1$, or more generally if there exists $(k, j) \in \mathbb{N} \times \mathbb{N}^*$ such as x_0 belongs to the boundary of $I_{k,j}$.

For every $k = 1, 2, ...$, we have $C(\lambda) \subset \cup_j I_{k,j}$, and then

$$H_{x_0}^{\beta(\alpha),\lambda^k}(C(\lambda)) \leq \sum_{j=1}^{2^k} \left(\mu_{x_0}^{\alpha}(I_{k,j})\right)^{\beta} \leq \sum_{j=1}^{2^k} (\lambda^{k\alpha})^{\beta} = (2\lambda^{\beta\alpha})^k, \tag{39}$$

In order for this upper bound to be useful, it should stay bounded as k tends to ∞. The smallest value of β for which this happens is given by $2\lambda^{\beta\alpha} = 1$, that is $\beta = \frac{\log 2}{\alpha \log(\frac{1}{\lambda})}$ for $\alpha \in]0, 1]$. Then for this choice we have

$$H_{x_0}^{\beta(\alpha)}(C(\lambda)) \leq \lim_{k\to\infty} \sum_{j=1}^{2^k} (\lambda^{k\alpha})^{\beta} = 1, \tag{40}$$

thus

$$\dim_{H_{x_0}^{\beta(\alpha)}}(C(\lambda)) \leq \frac{\log 2}{\alpha \log(\frac{1}{\lambda})}. \tag{41}$$

Since for the Hausdorff's measure see ([6], [9]), we can find a lower bounds for the $H_{x_0}^{\beta(\alpha)}(C(\lambda))$, indeed, let us consider an open recovering $(I_i)_{i\in\mathbb{N}}$ of $C(\lambda)$. Since $C(\lambda)$ is compact, a finite recovering can be chosen. We may assume that there were only $(I_j)_{j=1,..,n}$ to begin with. Since $C(\lambda)$ has no interior points, we can making I_j slightly larger if necessary, assume that the end-points of each I_j are outside $C(\lambda)$. Then there is $\delta > 0$ such as the distance from all these end-points to $C(\lambda)$ is at least δ. Choosing k so large that $\delta > \lambda^k = \mathcal{L}(I_{k,i})$, it follows that every interval $I_{k,i}$ is contained in some I_j.

Since for $\beta = \dfrac{\ln 2}{\ln \frac{1}{\lambda}}$ we can find a finite constant $c > 1$ such that

$$c(\mu_{x_0}^{\alpha}(I_j))^{\beta} \geq \sum_{I_{k,i}\subset I_j} (\mu_{x_0}^{\alpha}(I_{k,i}))^{\beta} \tag{42}$$

whence

$$c\sum_j (\mu_{x_0}^{\alpha}(I_j))^{\beta} \geq \sum_j \sum_{I_{k,i}\subset I_j} (\mu_{x_0}^{\alpha}(I_{k,i}))^{\beta} \geq \sum_{i=1}^{2^k} (\mu_{x_0}^{\alpha}(I_{k,i}))^{\beta}, \tag{43}$$

or

$$\sum_{i=1}^{2^k} (\mu_{x_0}^{\alpha}(I_{k,i}))^{\beta} \geq \sum_{i=1}^{2^k} \left(1 - (1-\lambda^k)^{\alpha}\right)^{\beta} = 2^k \left(1 - (1-\lambda^k)^{\alpha}\right)^{\beta} \simeq 2^k \alpha^{\beta} \lambda^{\beta k} \tag{44}$$

this lower bounds should remain bounded as k tends to infinity because $2\lambda^{\beta} = 1$, for $\beta = \dfrac{\ln 2}{\ln \frac{1}{\lambda}}$, and we have for this value

$$\sum_j (\mu_{x_0}^{\alpha}(I_j))^{\beta} \geq \frac{\alpha^{\beta}}{c}, \tag{45}$$

which allows us to say that

$$H_{x_0}^{\beta(\alpha)}(C(\lambda)) \geq \frac{\alpha^\beta}{c} \quad \text{for} \quad \beta = \frac{\ln 2}{\ln \frac{1}{\lambda}}, \tag{46}$$

which gives

$$\dim_{H_{x_0}^{\beta(\alpha)}}(C(\lambda)) \geq \frac{\log 2}{\log(\frac{1}{\lambda})} \tag{47}$$

Conclusion, if there is $(j,k) \in \mathbb{N} \times \mathbb{N}^*$ such as $x_0 \in \partial I_{k,j}$, then

$$\frac{\log 2}{\log(\frac{1}{\lambda})} \leq \dim_{H_{x_0}^{\beta(\alpha)}}(C(\lambda)) \leq \frac{\log 2}{\alpha \log(\frac{1}{\lambda})}. \tag{48}$$

ii) Case where $x_0 \in [0,1] \setminus C(\lambda)$, a similar demonstration of a) leads to the same conclusion.

iii) Case where $x_0 \notin [0,1]$, let us suppose $x_0 < 0$, the case $x_0 > 1$ will be treated by the same way. To find the upper bounds, we have

$$H_{x_0}^{\beta(\alpha),\lambda^k}(C(\lambda)) \leq$$

$$\sum_{j=1}^{2^k} \Big(\mu_{x_0}^\alpha(I_{k,j})\Big)^\beta \leq \sum_{j=1}^{2^k} \Big((\lambda^k - x_0)^\alpha - (-x_0)\Big)^\beta = 2^k\Big((\lambda^k - x_0)^\alpha - (-x_0)\Big)^\beta,$$

the last term $2^k\Big((\lambda^k - x_0)^\alpha - (-x_0)\Big)^\beta$ is equivalent to $\frac{\alpha^\beta}{(-x_0)^{(1-\alpha)\beta}}(2\lambda^\beta)^k$.

The smallest value of β for which the upper bounds of $H_{x_0}^{\beta(\alpha)}C(\lambda)$ should stay bounded as k tends to infinity, is given by $2\lambda^\beta = 1$, that is $\beta = \dfrac{\log 2}{\log \frac{1}{\lambda}}$.

For this choice we have

$$H_{x_0}^{\beta(\alpha)}(C(\lambda)) \leq \frac{\alpha^\beta}{(-x_0)^{(1-\alpha)\beta}} \tag{49}$$

therefore $\dim_{H_{x_0}^{\beta(\alpha)}}(C(\lambda)) \leq \beta$. The same covering chosen in i) will give us a lower estimate of the relative dimension $\dim_{H_{x_0}^{\beta(\alpha)}}(C(\lambda))$ by $\dfrac{\alpha^\beta}{4(-x_0)^{(1-\alpha)\beta}}$.

For this over coverings, we just need to see that for $\beta = \dfrac{\ln 2}{\ln \frac{1}{\lambda}}$, we can find a finite constant $c > 1$ such that

$$c \sum_j (\mu_{x_0}^\alpha(I_j))^\beta \geq \sum_j \sum_{I_{k,i} \subset I_j} (\mu_{x_0}^\alpha(I_{k,i}))^\beta \geq \sum_{i=1}^{2^k} (\mu_{x_0}^\alpha(I_{k,i}))^\beta, \tag{50}$$

with

$$\sum_{i=1}^{2^k}(\mu_{x_0}^{\alpha}(I_{k,i}))^{\beta} \geq \sum_{i=1}^{2^k}\Big((1-x_0)^{\alpha} - ((1-x_0)-\lambda^k)^{\alpha}\Big)^{\beta} \simeq \frac{\alpha^{\beta}(2\lambda^{\beta})^k}{(1-x_0)^{(1-\alpha)\beta}} \tag{51}$$

which enables us to have

$$\dim_{H_{x_0}^{\beta(\alpha)}}(C(\lambda)) \geq \frac{\log 2}{\log(\frac{1}{\lambda})} \qquad \text{and} \qquad H_{x_0}^{\beta(\alpha)}(C(\lambda)) \geq \frac{\alpha^{\beta}}{c(1-x_0)^{(1-\alpha)\beta}}, \tag{52}$$

for $\beta = \frac{\log 2}{\log(\frac{1}{\lambda})}$.

For Example: 1) If $x_0 = -1$, we find

$$\dim_{H_{x_0}^{\beta(\alpha)}}(C(\lambda)) \geq \frac{\log 2}{\log(\frac{1}{\lambda})} \qquad \text{and} \qquad \frac{\alpha^{\beta}}{c2^{(1-\alpha)\beta}} \leq H_{-1}^{\beta(\alpha)}(C(\lambda)) \leq \frac{\alpha^{\beta}}{c}, \tag{53}$$

for $\beta = \frac{\log 2}{\log(\frac{1}{\lambda})}$.

2) If $x_0 = -\frac{1}{2}$, we find

$$\dim_{H_{x_0}^{\beta(\alpha)}}(C(\lambda)) \geq \frac{\log 2}{\log(\frac{1}{\lambda})} \qquad \text{and} \qquad \frac{\alpha^{\beta}}{c(\frac{3}{2})^{(1-\alpha)\beta}} \leq H_{-\frac{1}{2}}^{\beta(\alpha)}(C(\lambda)) \leq \frac{\alpha^{\beta}}{c(\frac{1}{2})^{(1-\alpha)\beta}} \tag{54}$$

For the same β.

Conclusion, If $x_0 \notin [0,1]$ the relative dimension of a Cantor set is constant and coincide with Hausdorff's one, and if $x_0 \in [0,1] \cap \partial I_{k,j}$ for $(k,j) \in \mathbb{N} \times \mathbb{N}^*$, or if $x_0 \in [0,1] \setminus C(\lambda)$, the relative dimension of a Cantor set check

$$\frac{\log 2}{\log(\frac{1}{\lambda})} \leq \dim_{H_{x_0}^{\beta(\alpha)}}(C(\lambda)) \leq \frac{\log 2}{\alpha \log(\frac{1}{\lambda})}$$

which is a value grater or equal to the Hausdorff's dimension according to the value of $\alpha \in]0,1]$, if $\alpha = 1$ we find

$$\dim_{H_{x_0}^{\beta(1)}}(C(\lambda)) = \dim_{\mathcal{H}^{\beta}}(C(\lambda)). \tag{55}$$

Else the relative dimension of a Cantor set is greater than the Hausdorff's dimension and we have the following theorem

Theorem 5 *The relative dimension β of the Cantor set is*

i) If $x_0 \in [0,1] \setminus \mathcal{C}(\lambda)$, or $x_0 \in [0,1] \cap \partial I_{k,j}$ for $(k,j) \in \mathbb{N} \times \mathbb{N}^*$, then

$$\frac{\log 2}{\log(\frac{1}{\lambda})} \leq \beta \leq \frac{\log 2}{\alpha \log(\frac{1}{\lambda})}. \tag{56}$$

ii) If $x_0 \notin [0,1]$, then

$$\beta = \frac{\log 2}{\log(\frac{1}{\lambda})}. \tag{57}$$

Remark 1) For $x_0 = 0$ we have

$$\sum_{i=1}^{2^k} \mu_{x_0}^{\alpha}(I_{k,i}) = \lambda^{\alpha\beta k} + \Big[\Big(2\lambda^k + (1-2\lambda)\lambda^{k-1}\Big)^{\alpha} - \Big(\lambda^k + (1-2\lambda)\lambda^{k-1}\Big)^{\alpha}\Big]^{\beta}$$

$$+\Big[\Big(5\lambda^k + 2(1-2\lambda)\lambda^{k-1}\Big)^{\alpha} - \Big(4\lambda^k + 2(1-2\lambda)\lambda^{k-1}\Big)^{\alpha}\Big]^{\beta} + \ldots\ldots \tag{58}$$

$$= \lambda^{\alpha\beta k}\Big(1 + \Big[\Big(2 + \frac{(1-2\lambda)}{\lambda}\Big)^{\alpha} - \Big(1 + \frac{(1-2\lambda)}{\lambda}\Big)^{\alpha}\Big]^{\beta}$$

$$+\Big[\Big(5 + \frac{2(1-2\lambda)}{\lambda}\Big)^{\alpha} - \Big(4 + \frac{2(1-2\lambda)}{\lambda}\Big)^{\alpha}\Big]^{\beta} + \ldots\ldots \tag{59}$$

and then

$$\sum_{i=1}^{2^k} \mu_{x_0}^{\alpha}(I_{k,i}) = \lambda^{\alpha\beta k}\Big(u_1 + u_2 + \ldots + u_{2^k}\Big) \tag{60}$$

with u_n the sequence given by

$$u_n = \begin{cases} \Big((1 + P_n(\frac{1}{\lambda}))^{\alpha} - (P_n(\frac{1}{\lambda}))^{\alpha}\Big)^{\beta} & \text{if } n \text{ is an odd number} \\ \Big((P_n(\frac{1}{\lambda}))^{\alpha} - (P_n(\frac{1}{\lambda}) - 1)^{\alpha}\Big)^{\beta} & \text{if } n \text{ is an even number.} \end{cases}$$

Where P_n is a polynomial that is difficult to obtain his general form, but we can see that for an even number of the form 2^k, we have $P_{2^k}(\frac{1}{\lambda}) = \dfrac{1}{\lambda^k}$ and this my lead us to get a conclusion

i) If $\beta = \beta_2 = \dfrac{\ln 2}{\alpha \ln \frac{1}{\lambda}}$ then

$$\sum_{i=1}^{2^k} \Big(\mu_{x_0}^{\alpha}(I_{k,i})\Big)^{\beta_2} = \lambda^{\alpha\beta_2 k}\Big(u_1 + u_2 + \ldots + u_{2^k}\Big) \to 0, \tag{61}$$

by Cesaro's mean for the sequence u_n if it converge, we have

$$\frac{u_1 + u_2 + \ldots + u_{2^k}}{2^k} \approx u_{2^k} \to 0, \tag{62}$$

and we conclude $\dim_{H_{x_0}^{\beta_2(\alpha)}} C(\lambda) \leq \beta_2$.

ii) If $\beta = \beta_1 = \dfrac{\ln 2}{\ln \frac{1}{\lambda}}$ then

$$\sum_{i=1}^{2^k} \Big(\mu_{x_0}^{\alpha}(I_{k,i})\Big)^{\beta_1} = \lambda^{\alpha\beta_1 k}\Big(u_1 + u_2 + \ldots + u_{2^k}\Big) \to \alpha^{\beta_1} \tag{63}$$

indeed, we have

$$\frac{u_1 + u_2 + \ldots + u_{2^k}}{2^{k\alpha}} \approx u_{2^k}\ 2^{k(1-\alpha)}, \tag{64}$$

where $u_{2^k} = \left(\left(\frac{1}{\lambda^k}\right)^{\alpha} - \left(\frac{1}{\lambda^k} - 1\right)^{\alpha}\right)^{\beta_1}$, then

$$\lim_{k\to\infty} u_{2^k}\ 2^{k(1-\alpha)} \approx \lim_{k\to\infty} 2^{k(1-\alpha)}\lambda^{k\beta_1(1-\alpha)}\alpha^{\beta_1}.$$

or $\lambda^{\beta_1} = \dfrac{1}{2}$, then we conclude

$$\lim_{k\to\infty} u_{2^k}\ 2^{k(1-\alpha)} = \alpha^{\beta_1}.$$

we have then

$$\sum_{i=1}^{2^k} \Big(\mu_{x_0}^{\alpha}(I_{k,i})\Big)^{\beta_1} \to \alpha^{\beta_1} > 0. \tag{65}$$

This means that for $x_0 = 0$ and $\alpha < 1$ we have $\dim_{H_{x_0}^{\beta(\alpha)}}(C(\lambda)) \geq \dfrac{\log 2}{\log(\frac{1}{\lambda})}$.

2) To verify the formula (50) for a special case, suppose that each interval I_j cover two intervals of the k-th generation intervals of the Cantor, and then by construction, we have

$$\mu_{x_0}^{\alpha}(I_j) \geq \mu_{x_0}^{\alpha}(I_{k,i}) \tag{66}$$

for all $I_{k,i} \subset I_j$, thus we get

$$\mu_{x_0}^{\alpha}(I_j) \geq \frac{1}{2} \sum_{I_{k,i}\subset I_j} \mu_{x_0}^{\alpha}(I_{k,i}). \tag{67}$$

The convexity of the function $f(x) = x^{\beta}$ for $\beta \in]0,1[$ gives

$$\Big(\mu_{x_0}^{\alpha}(I_j)\Big)^{\beta} \geq \Big(\frac{1}{2} \sum_{I_{k,i}\subset I_j} \mu_{x_0}^{\alpha}(I_{k,i})\Big)^{\beta} \geq \frac{1}{2} \sum_{I_{k,i}\subset I_j} \Big(\mu_{x_0}^{\alpha}(I_{k,i})\Big)^{\beta} \tag{68}$$

then

$$4\Big(\mu_{x_0}^{\alpha}(I_j)\Big)^{\beta} \geq 2 \sum_{I_{k,i}\subset I_j} \Big(\mu_{x_0}^{\alpha}(I_{k,i})\Big)^{\beta} \geq \sum_{I_{k,i}\subset I_j} \Big(\mu_{x_0}^{\alpha}(I_{k,i})\Big)^{\beta}. \tag{69}$$

and we have a finite constant $c = 4$.

5 Application: Integral of real order

5.1 Integral and derivative of order $\alpha \in]0,1]$

In 1847 Riemann introduces the definition called today the Riemann-Liouville's integral of a real order ([12],[15], [10], [4]).

Definition 9 *Let f be a continuous function in $[a,b]$ with $-\infty < a < b < \infty$, the integral of order $\alpha \in]0,\infty[$ of the function f at the left of the point $x \in]a,b[$, if it exists, is defined by*

$$I_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \tag{70}$$

and the integral of order α at the right of the point $x \in]a,b[$, if it exists, is given by

$$I_b^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt. \tag{71}$$

Using the concept of measure of a real order that we introduced, we can rewrite the integral of a real order of Riemann-Liouville of the form:

$$I_a^\alpha f(x) = \frac{1}{\Gamma(\alpha+1)} \int_a^x f(t)d\mu_x^\alpha(t) = \int_a^x f(t)d\eta_x^\alpha(t), \tag{72}$$

with $d\eta_x^\alpha(t) = \dfrac{1}{\Gamma(\alpha+1)} d\mu_x^\alpha(t)$, and

$$I_b^\alpha f(x) = \frac{1}{\Gamma(\alpha+1)} \int_x^b f(t)d\mu_x^\alpha(t) = \int_x^b f(t)d\eta_x^\alpha(t), \tag{73}$$

with $d\eta_x^\alpha(t) = \dfrac{1}{\Gamma(\alpha+1)} d\mu_x^\alpha(t)$, and we can say that the measure η_x^α is also a density measure which satisfies the same properties that the "apparent" measure μ_x^α.

5.2 Integral of real order of a constant function

Let $f(x) = C$ be a constant for all $x \in [a,b]$, and let $x_0 \in]a,b[$. We have

$$\begin{aligned} I_a^\alpha f(x_0) &= \int_a^{x_0} f(t)d\eta_{x_0}^\alpha(t) = C \int_a^{x_0} d\eta_{x_0}^\alpha(t) \\ &= C\ \eta_{x_0}^\alpha(]a,x_0[) = C\ \frac{(x_0-a)^\alpha}{\Gamma(\alpha+1)}. \end{aligned} \tag{74}$$

To understand the distribution of this area, let us consider a subdivision of interval $[a,x_0]$ of n sub intervals of equal size

$$a = x_1 < x_2 < ..... < x_n = x_0, \tag{75}$$

we have then

$$I_a^\alpha f(x_0) = \int_a^{x_0} f(t)d\eta_{x_0}^\alpha(t) = \sum_{i=1}^n \int_{x_{i-1}}^{x_i} C\ d\eta_{x_0}^\alpha(t) = C\ \sum_{i=1}^n \eta_{x_0}^\alpha([x_{i-1},x_i]). \tag{76}$$

5.3 Comments

The integral of real order is a primitive, and the area gives a representation of this primitive. On the other hand, the area obtained depends on the value of α and of the interval length $[a, x_0]$, whether its length is greater or smaller than 1. In both cases there is always the following property

$$C\ \eta_{x_0}^{\alpha}([x_{n-1}, x_0]) > C\ \eta_{x_0}^{\alpha}([x_{n-2}, x_{n-1}]) > .... > C\ \eta_{x_0}^{\alpha}([x_{n-i}, x_{n-i+1}])$$

$$.... > C\ \eta_{x_0}^{\alpha}([x_{i-1}, x_i]).... > C\ \eta_{x_0}^{\alpha}([a, x_2]), \tag{77}$$

which means that the further x_0 is, the smaller the area of the rectangles obtained by the subdivision.

5.4 Integral of real order of a step function

Let f be a step function in $[a, b]$, taking values ξ_k in $]x_k, x_{k+1}[$, and let $x_0 \in]a, b[$. We can then write f of the form

$$f(x) = \sum_{i=1}^{n} \xi_i\ \mathbf{1}_{E_i}(x), \tag{78}$$

where $E_i =]x_i, x_{i+1}[$ is a subdivision of interval $[a, b]$.

The integral of real order of the function f is given by:

$$I_a^{\alpha} f(x_0) = \int_a^{x_0} f(t) d\eta_{x_0}^{\alpha}(t) = \sum_{i=1}^{n} \int_{x_i}^{x_{i+1}} \xi_i\ d\eta_{x_0}^{\alpha}(t) = \sum_{i=1}^{n} \xi_i\ \eta_{x_0}^{\alpha}(]x_{i-1}, x_i[). \tag{79}$$

Therefore the expression "f admits an integral of a real order on the left of the point x_0" means that the number

$$\sum_{i=1}^{n} \xi_i\ \eta_{x_0}^{\alpha}(]x_{i-1}, x_i[) \tag{80}$$

can be associated with the function f which represents a sum of area of rectangles with dimensions ξ_i and $\eta_{x_0}^{\alpha}(]x_{i-1}, x_i[)$ for $1 \leq i \leq n$ (Fig. 3).

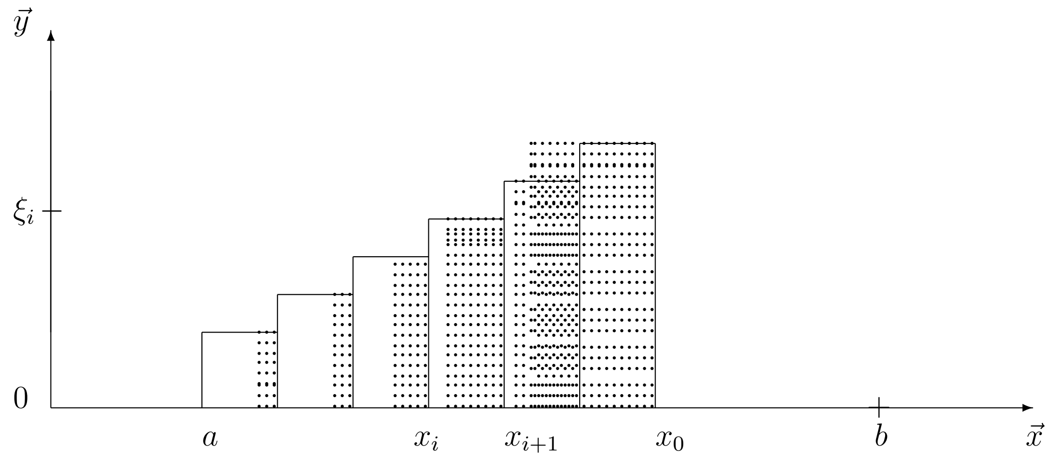

Figure 3.

Remarks 1) Intuitive considerations on the integral of real order

The summation from a to x_0 of the products $f(t)$ by the differential element $d\eta^{\alpha}_{x_0^+}$ represents a limit of the sum of n products

$$f(t_i) \times \eta^{\alpha}_{x_0}([t_i, t_{i+1}]) \tag{81}$$

obtained by dividing the interval $[a, x_0]$ into n sub intervals,

$$t_1 = a,\ t_2 = a + \frac{x_0 - a}{n}, ...,\ t_i = a + \frac{(i-1)(x_0 - a)}{n}, ...,\ t_n = x_0. \tag{82}$$

Let S be the sum of n products

$$f(t_i) \times \eta^{\alpha}_{x_0}([t_i, t_{i+1}])$$

representing the area of n right-angled basic

$$\eta^{\alpha}_{x_0}([t_i, t_{i+1}]) = \frac{1}{\Gamma(\alpha+1)}\Big((x_0 - t_i)^{\alpha} - (x_0 - t_{i+1})^{\alpha}\Big)$$
$$= \Big(\frac{x_0 - a}{n}\Big)^{\alpha}\Big((n-i+1)^{\alpha} - (n-i)^{\alpha}\Big)\frac{1}{\Gamma(\alpha+1)}, \tag{83}$$

and height $f(t_i)$, we have then

$$S = \sum_{i=1}^{n} f(t_i)\ \Big(\frac{x_0 - a}{n}\Big)^{\alpha}\Big((n-i+1)^{\alpha} - (n-i)^{\alpha}\Big)\frac{1}{\Gamma(\alpha+1)}, \tag{84}$$

(t_i vary from a to x_0).

Being at the extreme cases means that the larger n becomes the closer is the difference

$$\frac{1}{\Gamma(\alpha+1)}\Big((x_0 - t_i)^{\alpha} - (x_0 - i_{i+1})^{\alpha}\Big) \tag{85}$$

to the differential element

$$d\eta_x^\alpha(t) = d\frac{(x_0 - t)^\alpha}{\Gamma(\alpha+1)} \tag{86}$$

and that the sum of n produced becomes the integral of the "elementary products" of $f(t)$ by $d\eta_x^\alpha(t)$, where t vary from a to x_0.

2)Thanks to the measure $d\eta_{x_0}^\alpha$ in $\mathbb{R}$, we can rewrite the various integrals of a real order in fractional calculus, indeed:

5.4.1 Weyl's integral of real order (1917)

The Weyl integral of real order, [16], is given by

$$I_{-\infty}^\alpha f(x_0) = \frac{1}{\Gamma(\alpha)}\int_{-\infty}^{x_0}(x_0 - t)^{\alpha-1} f(t)dt = \int_{-\infty}^{x_0} f(t)d\eta_{x_0}^\alpha(t) \tag{87}$$

$$I_{+\infty}^\alpha f(x_0) = \frac{1}{\Gamma(\alpha)}\int_{x_0}^{+\infty}(t - x_0)^{\alpha-1} f(t)dt = \int_{x_0}^{+\infty} f(t)d\eta_{x_0}^\alpha(t), \tag{88}$$

and thus there is a same geometrical interpretation of Riemann-Liouville fractional integral.

5.4.2 Erdélyi's integral of real order (1964)

The Erdélyi's integral of real order, [5], is given by

$$I_0^\alpha f(x_0) = \frac{1}{\Gamma(\alpha)}\int_0^{x_0}(x_0 - t)^{\alpha-1} f(t)dt, \tag{89}$$

$$I_{+\infty}^\alpha f(x_0) = \frac{1}{\Gamma(\alpha)}\int_{x_0}^{+\infty}(t - x_0)^{\alpha-1} f(t)dt, \tag{90}$$

by using the concept of measurement of a real order, we have

$$I_0^\alpha f(x_0) = \int_0^{x_0} f(t)d\eta_{x_0}^\alpha(t), \qquad I_{+\infty}^\alpha f(x_0) = \int_{x_0}^{+\infty} f(t)d\eta_{x_0}^\alpha(t), \tag{91}$$

which enables us to obtain an equivalent geometrical interpretation.

5.4.3 Kober's integral of real order (1940)

The Kober's integral of real order, [7], is given by

$$I_0^{\beta,\alpha} f(x_0) = \frac{{x_0}^{-\beta-\alpha}}{\Gamma(\alpha)}\int_0^{x_0} t^\beta (x_0 - t)^{\alpha-1} f(t)dt \tag{92}$$

and

$$I_{+\infty}^{\beta,\alpha} f(x_0) = \frac{x_0^{\beta}}{\Gamma(\alpha)} \int_{x_0}^{+\infty} t^{-\beta-\alpha}(t-x_0)^{\alpha-1} f(t)dt, \tag{93}$$

by using the concept of measurement of a real order, the geometrical interpretation of this integral is more complicated

$$I_0^{\beta,\alpha} f(x_0) = x_0{}^{-\beta-\alpha} \int_0^{x_0} t^{\beta} f(t) d\eta_{x_0}^{\alpha}(t) \tag{94}$$

$$I_{+\infty}^{\beta,\alpha} f(x_0) = x_0^{\beta} \int_{x_0}^{+\infty} t^{-\beta-\alpha} f(t) d\eta_{x_0}^{\alpha}(t), \tag{95}$$

but in general we can have a geometric interpretation of the integral of real order of the function $x^{\delta} f(x)$ for all $\delta \in \mathbb{R}$.

5.4.4 Caputo's derivative of real order (1967)

The Caputo's derivative of real order ([2],[3]), is defined by an integral of the form

$$f^{(\beta)}(t) = \frac{1}{\Gamma(n-\beta)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{(\beta-1+n)}} d\tau \qquad n-1 < \beta < n,\ n \in \mathbb{N} - \{0\}. \tag{96}$$

It is noticed that if $\alpha = \beta - n + 1$ is posed, we have $f^{(\beta)} = (f^{(n)})^{(\alpha)}$ with $\alpha \in]0,1[$, this integral can be written with the concept of the measure of a real order like

$$f^{(\beta)}(t) = \int_a^t f^{(n)}(\tau) d\eta_{t^-}^{1-\alpha}(\tau) \qquad 0 < \alpha < 1. \tag{97}$$

that enables us to obtain a geometric interpretation of the Caputo's derivative of real order as being the area of real order of the derivative function $f^{(n)}(t)$.

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