Quaternionic-valued Gravitation in $8D$, Grand Unification and Finsler Geometry

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Abstract

A unification model of $4D$ gravity and $SU(3)\times SU(2)\times U(1)$ Yang-Mills theory is presented. It is obtained from a Kaluza-Klein compactification of $8D$ *quaternionic* gravity on an internal $CP^2 = SU(3)/U(2)$ symmetric space . We proceed to explore the *nonlinear* connection $A^a_\mu(\mathbf{x}, \mathbf{y})$ formalism used in Finsler geometry to show how ordinary gravity in $D = 4 + 2$ dimensions has enough degrees of freedom to encode a $4D$ gravitational and $SU(5)$ Yang-Mills theory. This occurs when the internal two-dim space is a sphere S^2 . This is an appealing result because $SU(5)$ is one of the candidate GUT groups. We conclude by discussing how the *nonlinear* connection formalism of Finsler geometry provides an infinite hierarchical extension of the Standard Model within a six dimensional gravitational theory due to the embedding of $SU(3)\times SU(2)\times U(1) \subset SU(5) \subset SU(\infty)$.

Keywords: Quaternions, Gravity, Grand Unification, Finsler Geometry, Kaluza-Klein.

1 Introduction

Exceptional, Jordan, Division, Clifford, noncommutative and nonassociative algebras are deeply related and are essential tools in many aspects in Physics, see [1], [2], [3], [4], [5] for references, among many others. In particular, a $15D$ model of a Chern-Simons E_8 Gauge theory of Gravity was proposed by [6] as a unified field theory of a Lanczos-Lovelock Gravitational Lagrangian with a E_8 Generalized Yang-Mills field theory in the $15D$ boundary of a $16D$ bulk space. More recently, a Clifford $Cl(5, C)$ Unified Gauge Field Theory of Conformal Gravity, Maxwell and $U(4) \times U(4)$ Yang-Mills in $4D$ was provided by [14]. For other results on grand unification based on Clifford algebras see [29], [30] and references therein.

It has been argued by [24] that a Kaluza-Klein compactification of $8D$ gravity on CP^2 involving a nontrivial *torsion* may bypass the no-go theorems [26] that one cannot obtain the group $SU(3) \times SU(2) \times U(1)$ from a Kaluza-Klein mechanism in $8D$. It was *assumed* by [24] that *if* the torsion components $T^a_{\mu\nu}$ were proportional to $F^I_{\mu\nu} e^a_I$, where e^a_I is a vielbein employed to change the $SU(2) \times U(1)$ group index $I = 1, 2, 3, 4$ to the internal four-dim space CP^2 index $a = 1, 2, 3, 4$, the $8D$ Lagrangian corresponding to the curvature scalar and associated with a connection with contorsion K : $\mathbf{R}(\Gamma + K) = R(\Gamma) + (K)^2 + \nabla K$ yields a gravitational and $SU(3) \times SU(2) \times U(1)$ Yang-Mills theory upon compactification on $CP^2 = SU(3)/SU(2) \times U(1)$. The problem was that no proof was presented in [24] which shows why $T^a_{\mu\nu}$ is proportional to $F^I_{\mu\nu} e^a_I$.

It is known that a horizontal-vertical splitting of the tangent space geometry using the canonical distinguished **d**-connection (instead of the torsionless Levi-Civita connection) within the formalism of Lagrange-Finsler spaces leads to nontrivial torsion $T^a_{\mu\nu}$ components and which are related to the generalized field strength $F^a_{\mu\nu}$ associated with a *nonlinear* connection $A^a_\mu(x^\nu, y^b)$. The coordinates x^ν, y^b are the horizontal base space and internal vertical space coordinates, respectively. A mapping between Finsler and Kaluza-Klein theories and the comparison of the Finslerian gauge approach to the Yang-Mills one can be found in [15], [16], respectively. In section **3** we will explore the *nonlinear* connection $A^a_\mu(x^\nu, y^b)$ formalism of Finsler geometry to show how gravity in $4 + 2$ dimensions has enough degrees of freedom to encompass $4D$ gravity and a $SU(5)$ Yang-Mills theory. This is an appealing result because $SU(5)$ is a candidate GUT group.

A complexification of ordinary gravity (not to be confused with Hermitian-Kahler geometry) has been known for a long time. Complex gravity requires that $g_{\mu\nu} = g_{(\mu\nu)} + i g_{[\mu\nu]}$ such that $g_{\nu\mu} = (g_{\mu\nu})^*$. A treatment of a non-Riemannan geometry based on a complex tangent space and involving a symmetric $g_{(\mu\nu)}$ plus antisymmetric $g_{[\mu\nu]}$ metric component was first advanced by Einstein-Strauss [7] (and later on by [8]) in their proposal to unify Electromagnetism (EM) with Gravity by identifying the EM field strength $F_{\mu\nu}$ with the antisymmetric metric $g_{[\mu\nu]}$ component. However this identification led to several problems.

Borchsenius [10] formulated the quaternionic extension of Einstein-Strauss unified theory of gravitation with EM by identifying the $SU(2)$ Yang-Mills field strength $F^i_{\mu\nu}, i = 1, 2, 3$ with the internal degrees of a freedom $(g_{[\mu\nu]})^i$ of a quaternionic-valued "metric tensor". Again this approach is problematic. For these reasons in section **2** we shall build an unification model of $4D$ gravity and $SU(3) \times SU(2) \times U(1)$ Yang-Mills theory obtained from a Kaluza-Klein compactification of $8D$ quaternionic gravity on CP^2, rather than introducing by hand the torsion squared terms [24]. In this way we avoid the problems encountered by [7], [10], and also construct unified theories that contain the electro-weak force and gravity in $4D$. Our results differ also from the construction in [25] to unify the electro-weak force with gravity in $4D$ after complexifying the de Sitter group.

The authors [11] much later provided the octonionic gravitational extension of Borchsenius theory involving two interacting $SU(2)$ Yang-Mills fields and where the exceptional group G_2 was realized naturally as the automorphism group of the octonions. The octonionic geometry (gravity) construction developed by [11] was extended further to spaces with noncommutative and nonassociative spacetime coordinates and momenta in [12] and which set the stage for the study of Exceptional Jordan Strings and Nonassociative Ternary Gauge Field Theories [13]. Having presented this very brief introduction we shall proceed with the main results of this work.

2 Gravity and Standard Model Unification from $8D$ Quaternionic Gravity

A geometrical treatment of a non-Riemannian geometry including an internal complex, quaternionic and octonionic space has been investigated by several authors [7], [10], [11], Castro-Jordan. A quaternionic-valued metric is defined as

$$\mathbf{g}_{\mu\nu} \ = \ g_{\mu\nu} \ e_o \ + \ \mathbf{g}^i_{[\mu\nu]} \ e_i, \quad e_i e_j = -\delta_{ij} e_o + \epsilon_{ijk} e_k, \quad i,j,k = 1,2,3 \tag{2.1}$$

obeying the symmetry condition $\mathbf{g}^\dagger_{\mu\nu} = \mathbf{g}_{\nu\mu}$ where the Hermitian conjugation is taken in the internal quaternionic space. Namely, one can represent the generators of the quaternionic algebra in terms of the Hermitian Pauli spin 2×2 matrices σ_i and the unit 2×2 matrix as $e_o = \mathbf{1}_{2\times 2}; e_i = -i\sigma_i$. Hence the Hermitian conjugation is carried on the 2×2 matrices. The physical distance is

$$ds^2 \ = \ \frac{1}{2} \ Trace \ (\mathbf{g}_{\mu\nu} dx^\mu dx^\nu \ ) \ = \ g_{(\mu\nu)} \ dx^\mu \ dx^\nu \tag{2.2}$$

due to the traceless condition of the Pauli spin matrices and commuting nature of the coordinates. One may choose $g_{\mu\nu} = g_{(\mu\nu)} + ig_{[\mu\nu]}$ and maintain the Hermiticity condition $\mathbf{g}^\dagger_{\mu\nu} = \mathbf{g}_{\nu\mu}$ if $(ig_{[\mu\nu]} e_o)^\dagger = -ig_{[\mu\nu]} e_o$; i.e. if one includes a complex conjugation on i as well and which is compatible with the fact that $(e_i)^\dagger = (-i\sigma_i)^\dagger = +i\sigma_i = -e_i$ since the Pauli spin 2×2 matrices σ_i are taken to be Hermitian.

The quaternionic-valued connection is

$$\Upsilon^\sigma_{\mu\rho} \ = \ (\ \Gamma^\sigma_{(\mu\rho)} \ + i \ \Gamma^\sigma_{[\mu\rho]} \ ) e_o \ + \ (\Theta^\sigma_{[\mu\rho]})^i e_i \tag{2.3}$$

we explicitly write $(\mu\rho), [\mu\rho]$ to denote the symmetry and antisymmetry properties of the connection components. We will show how a Kaluza-Klein compactification in the internal space CP^2, from $8D$ to $4D$, yields a gravitational, $SU(3) \times SU(2) \times U(1)$ Yang-Mills theory in $4D$.

The gravitational and $U(1)$ Maxwell's EM sector are encoded, respectively, in the symmetric piece $\Gamma^{\sigma}_{(\mu\rho)} e_o$ and antisymmetric piece $i\Gamma^{\sigma}_{[\mu\rho]} e_o$ corresponding to the unit element e_o of the quaternionic-algebra-valued connection. The $SU(2)$ sector is encoded in the internal part $(\Theta^{\sigma}_{[\mu\rho]})^i e_i$. The $SU(3)$ Yang-Mills sector arises upon the Kaluza-Klein compactification resulting from the isometry group of the CP^2 internal space. Therefore, from a pure quaternionic gravity in $8D$ one can obtain a grand unified field theory of gravity and the standard model group $SU(3) \times SU(2) \times U(1)$ in $4D$.

This result can be attained by restricting $\Gamma^{\sigma}_{[\mu\rho]} = \delta^{\sigma}_{\rho} A_{\mu} - \delta^{\sigma}_{\mu} A_{\rho}$ to be the Einstein-Schrodinger connection, where A_{μ} is the EM field. Due to the antisymmetry, $\Gamma^{\sigma}_{[\mu\rho]}$ transforms as a tensor. This is not the case with $\Gamma^{\sigma}_{(\mu\rho)}$. The internal part of the connection $\Theta^{\sigma}_{[\mu\rho]}$ is restricted to be of the form $(\delta^{\sigma}_{\rho}\ \Theta^{i}_{\mu} - \delta^{\sigma}_{\mu} \Theta^{i}_{\rho})\ e_i, i = 1, 2, 3$, such that the commutator becomes $[\Theta_{\mu}, \Theta_{\nu}] = 2\ \Theta^{i}_{\mu}\ \Theta^{j}_{\nu}\ \epsilon_{ijk}\ e_k$. The quaternionic-valued curvature

$$\mathbf{R}^{\sigma}_{\mu\nu\rho} = \partial_{\mu}\ \Upsilon^{\sigma}_{\nu\rho} - \partial_{\nu}\ \Upsilon^{\sigma}_{\mu\rho} + \Upsilon^{\sigma}_{\mu\tau}\ \Upsilon^{\tau}_{\nu\rho} - \Upsilon^{\sigma}_{\nu\tau}\ \Upsilon^{\tau}_{\mu\rho} =$$

$$(\ R^{\sigma}_{\mu\nu\rho} + i\ F^{\sigma}_{\mu\nu\rho}\ ) e_o + (\ \mathbf{P}^{\sigma}_{\mu\nu\rho}\ )^k\ e_k + \ extra\ terms \tag{2.4}$$

has for components the following terms : the standard Riemannian curvature tensor written in terms of the Christoffel symbols as

$$R^{\sigma}_{\mu\nu\rho} = \partial_{\mu}\ \Gamma^{\sigma}_{(\nu\rho)} - \partial_{\nu}\ \Gamma^{\sigma}_{(\mu\rho)} + \Gamma^{\sigma}_{(\mu\tau)}\ \Gamma^{\tau}_{(\nu\rho)} - \Gamma^{\sigma}_{(\nu\tau)}\ \Gamma^{\tau}_{(\mu\rho)} \tag{2.5}$$

The tensor containing the Maxwell field strength is

$$F^{\sigma}_{\mu\nu\rho} = \delta^{\sigma}_{\rho}\ (\partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}) + \delta^{\sigma}_{\mu}\ \partial_{\nu} A_{\rho} - \delta^{\sigma}_{\nu}\ \partial_{\mu} A_{\rho} \tag{2.6}$$

such that the contraction $F^{\sigma}_{\mu\nu\sigma} = (D-1) F_{\mu\nu}$ in D-dim is proportional to the $U(1)$ EM field strength $F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$. And, finally, the $SU(2)$ field strength is encoded in the internal part of the curvature tensor which can be written as

$$\mathbf{P}_{\mu\nu} = \partial_{\mu}\ \Theta_{\nu} - \partial_{\nu}\ \Theta_{\mu} + [\ \Theta_{\mu},\ \Theta_{\nu}\ ] =$$

$$(\ \partial_{\mu}\ \Theta^{k}_{\nu} - \partial_{\nu}\ \Theta^{k}_{\mu}\ )\ e_k + 2\ \Theta^{i}_{\mu}\ \Theta^{j}_{\nu}\ \epsilon_{ijk}\ e_k. \tag{2.7}$$

leading to

$$\mathbf{P}^{\sigma}_{\mu\nu\rho} = (\ \mathbf{P}^{\sigma}_{\mu\nu\rho}\ )^k\ e_k = \delta^{\sigma}_{\rho}\ (\mathbf{P}_{\mu\nu})^k\ e_k =$$

$$\delta^{\sigma}_{\rho}\ (\ \partial_{\mu}\ \Theta_{\nu} - \partial_{\nu}\ \Theta_{\mu} + [\ \Theta_{\mu},\ \Theta_{\nu}\ ]\ )^k\ e_k \tag{2.8}$$

There are *extra* terms in eq-(2.4) involving products of the form

$$\Gamma^{\sigma}_{(\mu\tau)}\ \Gamma^{\tau}_{[\nu\rho]},\ \ \Gamma^{\sigma}_{(\mu\tau)}\ (\Theta^{\tau}_{[\nu\rho]})^k,\ \ \Gamma^{\sigma}_{[\mu\tau]}\ \Gamma^{\tau}_{[\nu\rho]},\ \ \Gamma^{\sigma}_{[\mu\tau]}\ (\Theta^{\tau}_{[\nu\rho]})^k \tag{2.9}$$

and for simplicity are not written down. The first two terms in (2.9) can be reabsorbed inside the ordinary derivatives to yield "covariantized" $SU(2) \times U(1)$ field strengths involving the *analog* of covariant-like derivatives ∇_{μ} acting on the gauge fields; and the last two terms are *analogous* (but *not* identical) to torsion-squared terms and products of torsion terms.

If one has quaternionic gravity in $8D$, the indices are $M, N, L = 1, 2, 3, ..., 8$ and, if one wishes, one may build a Lagrangian out of the following tensorial quantities found *within* the quaternionic-valued curvature above : namely the $8D$ Riemannian scalar curvature $\mathcal{R} = g^{(MN)} R_{MN}$, the $U(1)$ and $SU(2)$ field strengths F_{MN}, F^i_{MN}. In particular, let us start with a standard Lagrangian for gravity plus $SU(2) \times U(1)$ Yang-Mills in $8D$ given by

$$\mathcal{L} \ = \ \mathcal{R} \ - \frac{1}{4} \ (F_{MN})^2 \ - \frac{1}{4} \ (F^i_{MN})^2, \quad M, N = 1, 2, 3, ..., 8 \qquad (2.10)$$

where we set the numerical couplings to unity. The components of the Ricci tensors after a Kaluza-Klein compactification are given by [23]

$$\mathcal{R}_{\mu\nu} \ = \ R_{\mu\nu} \ - \frac{1}{2} \ K^a_I \ K_{aJ} \ F^I_{\mu\rho} \ F^{J\rho}_\nu, \quad \mathcal{R}_{\mu a} \ = \ \frac{1}{2} \ K^I_a \ D_\nu F^{I\nu}_\mu \qquad (2.11a)$$

$$\mathcal{R}_{ab} \ = \ R_{ab} \ + \ \frac{1}{4} \ K^I_a \ K^J_b \ F^I_{\mu\nu} \ F^{J\mu\nu} \qquad (2.11b)$$

where K^{aI} are the Killing vectors associated with the $SU(3)$ isometry group (metric preserving symmetry) of the internal space $CP^2 = SU(3)/SU(2) \times U(1)$. The range of the indices is $\mu, \nu = 1, 2, 3, 4$; $a, b = 1, 2, 3, 4$ and $I, J = 1, 2, 3, ..., 8$. Eqs-(2.11a, 2.11b) lead to the following decomposition of the $8D$ scalar curvature

$$\mathcal{R} \ = \ R[g_{\mu\nu}] \ - \frac{1}{4} \ F^I_{\mu\nu} \ F^{\mu\nu}_I \ + \ g^{ab} \ R_{ab} \ + \ .... \qquad (2.11c)$$

so that the Lagrangian (2.10) furnishes a four-dim theory of gravity and $SU(3)$ Yang-Mills interacting with a non-linear sigma model scalar field stemming from the metric degrees of freedom in the internal space. The indices $I = 1, 2, 3, ..., 8$ span the 8 generators of the $SU(3)$ algebra and $R = g^{(\mu\nu)} R_{\mu\nu}$ is the four-dim scalar curvature.

Concluding, from a quaternionic-valued gravitational theory in $8D$, one has the necessary field ingredients to build the Lagrangian (2.10) and generate a gravitational and $SU(3) \times SU(2) \times U(1)$ Yang-Mills theory in $4D$ after a Kaluza-Klein compactification on CP^2. For this reason, this kind of grand unification program warrants further investigation. Closely related to the Lagrangian in eq-(2.10) is that the authors [22] have shown how an Einstein-Yang-Mills theory in $4 + d$ dimensions admits a solution (an spontaneous compactification) of the form $M_4 \times G/H$, where M_4 is an Einstein space and G/H is a symmetric space, if the gauge group of the Yang-Mills theory in $4 + d$ dimensions is H or larger. This agrees with our results above which are based on having $H = SU(2) \times U(1)$ and $d = 4$. The authors [22] focused in particular in the case when the four-dim base space M_4 is Minkowski or Anti de Sitter space.

Instead of having for Lagrangian the one provided by eq-(2.10), let us begin with the real part of the quaternionic-valued scalar curvature, and set the numerical physical (coupling) constants to unity, for simplicity

$$\mathcal{L} \; = \; \frac{1}{2} \, \mathbf{g}^{MN} \, \mathbf{R}_{MN} \; + \; \textit{quaternionic complex conjugate} \; =$$

$$g^{(MN)}\mathcal{R}_{MN} \; - \; g^{[MN]} \; F_{MN} \; - \; (\mathbf{g}^{[MN]})_j \; F^j_{MN} \; + \; .... \tag{2.12}$$

the Lagrangian (2.12) is the quaternionic version of the Einstein-Hilbert gravitational one. If one imposes the correspondence $g_{[MN]} \leftrightarrow F_{MN}$ and $(\mathbf{g}_{[MN]})_j \leftrightarrow F^j_{MN}$ in (2.12) then one recovers the correspondence with the gravity-Yang-Mills Lagrangian in (2.10). However, we must emphasize that the Lagrangians of eqs-(2.10, 2.12) are *not* the same because an antisymmetric metric tensor is *not* physically the same as a gauge field strength. Similarly, one could have interpreted the Born-Infeld actions for EM and gravity, respectively

$$\int \; \sqrt{\; det|g_{\mu\nu} + F_{\mu\nu}|}, \quad \int \; \sqrt{\; det|g_{\mu\nu} + R_{\mu\nu}|} \tag{2.13}$$

as if one had the determinants of an effective "metric" given by $g_{\mu\nu} + F_{\mu\nu}$ and $g_{\mu\nu} + R_{\mu\nu}$.

Before studying the quaternionic version of the Einstein-Hilbert Lagrangian (2.12), let us focus now on the Kaluza-Klein compactification of complex gravity. Earlier on we *restricted* the connection to be given as $\Gamma^\sigma_{[\mu\rho]} = \delta^\sigma_\rho A_\mu - \delta^\sigma_\mu A_\rho$, and similarly, the internal part of the quaternionic connection $\Theta^\sigma_{[\mu\rho]} = (\delta^\sigma_\rho \; \Theta^i_\mu - \delta^\sigma_\mu \Theta^i_\rho) e_i$. One may relax these restrictions and focus solely on a complex gravitational theory (without including the imaginary quaternionic part) where the antisymmetric part of the connection $\Gamma^\sigma_{[\mu\rho]}$ is now *unrestricted* and ask how a Kaluza-Klein compactification of complex gravity might look like. Because real gravity in $8D$ yields ordinary gravity and $SU(3)$ Yang-Mills theory in $4D$, upon a compactification on CP^2, one may wonder if *complex* gravity might furnish a *complex* gravitational and $SU(3)$ Yang-Mills theory in $4D$ upon compactification on CP^2.

It is known that the complexification of the $\mathbf{su}(N), \mathbf{u}(N)$ algebras are respectively the algebras $\mathbf{sl}(N, C), \mathbf{gl}(N, C)$. Rather than focusing on $SL(N, C)$ Yang-Mills theories (involving noncompact groups) we shall concentrate on complex-valued $SU(N)$ Yang-Mills fields $(A^I_\mu + i\tilde{A}^I_\mu)T_I$ where T_I are the $N^2 - 1$ generators of $SU(N)$. The symmetric and antisymmetric metric components $g_{AB} = g_{(AB)} + ig_{[AB]}$ in $8D$ admit the following $4 + 4$ decomposition

$$g_{(\mu\nu)} \; + i \; g_{[\mu\nu]} \; = \; \gamma_{(\mu\nu)} \; + i \; \gamma_{[\mu\nu]} \; + \; \phi_{(ab)} \; A^a_\mu \; A^b_\nu \; + i \; \phi_{[ab]} \; \tilde{A}^a_\mu \; \tilde{A}^b_\nu \tag{2.14a}$$

$$g_{ab} \; = \; \phi_{(ab)} \; + i \; \phi_{[ab]}, \quad g_{\mu a} \; = \; \phi_{(ab)} \; A^b_\mu \; + i \; \phi_{[ab]} \; \tilde{A}^b_\mu, \;\; g_{a\mu} = (g_{\mu a})^* \tag{2.14b}$$

and such that the interval ds^2 is the *same* as in ordinary real gravity with symmetric metrics

$$ds^2 \; = \; \phi_{(ab)} \; dy^a \; dy^b \; + \; (\gamma_{(\mu\nu)} + \phi_{(ab)} \; A^a_\mu \; A^b_\nu) \; dx^\mu \; dx^\nu \; + \; 2 \; \phi_{(ab)} \; A^b_\mu \; dx^\mu \; dy^a. \tag{2.15}$$

The antisymmetric metric components do not contribute to the distance due to the commutativity of the coordinate differentials. Another geometrical setting where the $4+4$ decomposition of symmetric and nonsymmetric metrics is relevant is in the study of Finsler geometry. A very rigorous treatment of nonsymmetric theories of gravity [8] involving nonsymmetric metrics in Finsler geometry was undertaken by [20] following the early work of Eisenhart [21]. Nonsymmetric metrics were very relevant in the study of Born's deformed reciprocal complex gravity and Noncommutative gravity [9].

It is reasonable to expect that a Kaluza-Klein compactification scenario should yield a *complexified* $SU(3)$ Yang-Mills theory based on the following complex-valued field strength

$$F^k_{\mu\nu} \ + i \ \tilde F^k_{\mu\nu} \ =$$

$$\partial_\mu \ (A^k_\nu \ + i \ \tilde A^k_\nu) - \partial_\nu \ (A^k_\mu \ + i \ \tilde A^k_\mu) \ + f_{lj}{}^k \ (A^l_\mu \ + i \ \tilde A^l_\mu) \ (A^j_\nu \ + i \ \tilde A^j_\nu); \ \ k = 1,2,3,....,8 \tag{2.16}$$

One may note that now one ends up with field strength components whose fields $A^a_\mu, \tilde A^a_\mu$ *mix* due to the nonabelian nature of $SU(3)$

$$F^k_{\mu\nu} \ = \ \partial_\mu \ A^k_\nu \ - \ \partial_\nu \ A^k_\mu \ + f_{lj}{}^k \ (\ A^l_\mu \ A^j_\nu \ - \ \tilde A^l_\mu \ \tilde A^j_\nu) \tag{2.17a}$$

$$\tilde F^k_{\mu\nu} \ = \ \partial_\mu \ \tilde A^k_\nu \ - \ \partial_\nu \ \tilde A^k_\mu \ + f_{lj}{}^k \ (A^l_\mu \ \tilde A^j_\nu \ + \ \tilde A^l_\mu \ A^j_\nu) \tag{2.17b}$$

These expressions should be compared with the standard $SU(3) \times SU(3)$ field strength, where the respective fields $A^{'a}_\mu$ and $\tilde A^{'a}_\mu$ do *not* mix

$$F^{'k}_{\mu\nu} \ = \ \partial_\mu \ A^{'k}_\nu \ - \ \partial_\nu \ A^{'k}_\mu \ + f_{lj}{}^k \ A^{'l}_\mu \ A^{'j}_\nu \tag{2.18a}$$

$$\tilde F^{'k}_{\mu\nu} \ = \ \partial_\mu \ \tilde A^{'k}_\nu \ - \ \partial_\nu \ \tilde A^{'k}_\mu \ + f_{lj}{}^k \ \tilde A^{'l}_\mu \ \tilde A^{'j}_\nu \tag{2.18b}$$

If one equates

$$F^k_{\mu\nu} \ = \ F^{'k}_{\mu\nu}, \quad \tilde F^k_{\mu\nu} \ = \ \tilde F^{'k}_{\mu\nu} \tag{2.19}$$

it imposes a relationship among the gauge fields $A^k_\mu, \tilde A^k_\mu, A^{'k}_\mu, \tilde A^{'k}_\mu$ of the form

$$A^k_\mu \ - \ A^{'k}_\mu \ = \ \partial_\mu \Lambda^k \ \equiv \ \Lambda^k_\mu, \quad \tilde A^k_\mu \ - \ \tilde A^{'k}_\mu \ = \ \partial_\mu \tilde\Lambda^k \ \equiv \ \tilde\Lambda^k_\mu \ \Rightarrow \tag{2.20a}$$

$$\partial_{[\nu} A^k_{\mu]} \ = \ \partial_{[\nu} A^{'k}_{\mu]}, \ \ \partial_{[\nu} \tilde A^k_{\mu]} \ = \ \partial_{[\nu} \tilde A^{'k}_{\mu]}, \quad k = 1,2,3,....,8 \tag{2.20b}$$

where the derivatives $\partial_\mu \Lambda^k = \Lambda^k_\mu, \partial_\mu \tilde\Lambda^k = \tilde\Lambda^k_\mu$ are not arbitrary functions but are *constrained* to satisfy the following set of stringent relations obtained by imposing the equalities

$$f_{lj}{}^k \ (\ A^l_\mu \ A^j_\nu \ - \ \tilde A^l_\mu \ \tilde A^j_\nu \ ) \ = \ f_{lj}{}^k \ A^{'l}_\mu \ A^{'j}_\nu, \quad k = 1,2,3,......,8 \tag{2.21a}$$

$$f_{lj}{}^k \ (\ A^l_\mu \ \tilde{A}^j_\nu \ + \ \tilde{A}^l_\mu \ A^j_\nu \ ) \ = \ f_{lj}{}^k \ \tilde{A}'^l_\mu \ \tilde{A}'^j_\nu, \quad k = 1,2,3,......,8 \qquad (2.21b))$$

resulting from eqs-(2.19), and after using eqs-(2.20), leading to a complicated functional relation among $\Lambda^k_\mu, \tilde{\Lambda}^k_\mu$ and $A'^k_\mu, \tilde{A}'^k_\mu$ of the form $\Lambda^k_\mu[A'^k_\mu, \tilde{A}'^k_\mu]; \tilde{\Lambda}^k_\mu[A'^k_\mu, \tilde{A}'^k_\mu]$. Hence, eqs-(2.20,2.21) provide a functional relation among the prime and unprimed fields of the form

$$A^k_\mu \ = \ A'^k_\mu \ + \ \Lambda^k_\mu[A'^k_\mu, \tilde{A}'^k_\mu], \quad \tilde{A}^k_\mu \ = \ \tilde{A}'^k_\mu \ + \ \tilde{\Lambda}^k_\mu[A'^k_\mu, \tilde{A}'^k_\mu] \qquad (2.22)$$

which permits us to equate the Yang-Mills kinetic terms

$$(F'^k_{\mu\nu})^2 \ + \ (\tilde{F}'^k_{\mu\nu})^2 \ = \ (F^k_{\mu\nu})^2 \ + \ (\tilde{F}^k_{\mu\nu})^2 \qquad (2.23)$$

To sum up, by taking the real part of the complex scalar curvature for the $8D$ Lagrangian : $\frac{1}{2}[g^{AB}R_{AB} + (g^{AB}R_{AB})^*]$, and after performing a Kaluza-Klein compactification on CP^2, one would expect to generate the real part of the complexified $SU(3)$ Yang-Mills kinetic terms in $4D$

$$\frac{1}{2} \ [\ (F^k_{\mu\nu} \ + i \ \tilde{F}^k_{\mu\nu})^2 \ + \ (F^k_{\mu\nu} \ - i \ \tilde{F}^k_{\mu\nu})^2 \ ] \ =$$

$$(F^k_{\mu\nu} + i \ \tilde{F}^k_{\mu\nu}) \ (F^{\mu\nu}_k - i \ \tilde{F}^{\mu\nu}_k) \ = \ (F^k_{\mu\nu})^2 \ + \ (\tilde{F}^k_{\mu\nu})^2 \ = \ (F'^k_{\mu\nu})^2 \ + \ (\tilde{F}'^k_{\mu\nu})^2 \quad (2.24)$$

Therefore, after recurring to the relations (2.19-2.22), one could extract in principle the $SU(3) \times SU(3)$ $4D$ Yang-Mills Lagrangian (2.24) from the complex gravitational theory in $8D$ after a Kaluza-Klein compactification on the internal space CP^2. Because the latter group $SU(3) \times SU(3)$ contains the standard model group $SU(3) \times SU(2) \times U(1)$, this is an appealing construction in $8D$ that does not require an ordinary gravitational theory in $D = 11$ [26] and which differs from the proposals in [24], [25].

Having described heuristically the Kaluza-Klein compactification of complex gravity and how a complex $SU(3)$ Yang-Mills might arise, we now turn to the *pure* quaternionic gravity case based on the metric $g_{(\mu\nu)}e_o + \mathbf{g}^i_{[\mu\nu]}e_i$, when the internal part of the connection is *unrestricted* $(\Theta^\sigma_{[\mu\rho]})^i e_i \neq \delta^\sigma_{[\rho}\Theta^i_{\mu]}e_i$, and ask firstly what a quaternionic analog of a $SU(3)$ Yang-Mills theory might look like. In general, a quaternionic-valued and $SU(N)$-valued gauge field can be written as $\mathbf{A}_\mu = A^{am}_\mu(e_a \otimes T_m)$ involving the $SU(N)$ algebra generators $T_m, m = 1,2,3,....,N^2-1$ and the quaternion algebra generators (including the unit generator) $e_a = e_0, e_1, e_2, e_3$; i.e. one has quaternionic-valued components for the $SU(N)$ gauge fields. The quaternionic-valued $SU(N)$ commutator is defined by

$$[\ \mathbf{A}_\mu, \ \mathbf{A}_\nu \ ] \ = \ [\ A^{am}_\mu \ (e_a \otimes T_m), \ A^{bn}_\nu \ (e_b \otimes T_n) \ ] \ =$$

$$\frac{1}{2} \ A^{am}_\mu \ A^{bn}_\nu \ \{e_a, \ e_b\} \otimes [T_m, \ T_n] \ + \frac{1}{2} \ A^{am}_\mu \ A^{bn}_\nu \ [e_a, \ e_b] \otimes \{T_m, \ T_n\} \qquad (2.25)$$

where

$$\{e_a, \ e_b\} \ = \ -2 \ \delta_{ab} \ e_o, \quad [e_a, \ e_b] \ = \ 2 \ c_{abc} \ e_c \qquad (2.26)$$

and

$$\{T_m,\ T_n\} \ = \ \frac{1}{N}\ \delta_{mn} \ + \ d_{mnp}\ T_p, \quad [T_m,\ T_n] \ = \ f_{mnp}\ T_p \tag{2.27}$$

From eqs-(2.25-2.27) one arrives at the different components of the field strengths

$$F^k_{\mu\nu} \ = \ \partial_\mu\ A^k_\nu \ - \partial_\nu\ A^k_\mu \ + \ f_{lj}{}^k\ A^l_\mu\ A^j_\nu \ - \ \delta_{ab}\ f_{lj}{}^k\ A^{al}_\mu\ A^{bj}_\nu \tag{2.28a}$$

$$F^c_{\mu\nu} \ = \ \partial_\mu\ A^c_\nu \ - \partial_\nu\ A^c_\mu \ + \ c_{ab}{}^c\ \frac{\delta_{lj}}{N}\ A^{al}_\mu\ A^{bj}_\nu \tag{2.28b}$$

$$F^{ck}_{\mu\nu} \ = \ \partial_\mu\ A^{ck}_\nu \ - \partial_\nu\ A^{ck}_\mu \ + \ c_{ab}{}^c\ d_{lj}{}^k\ A^{al}_\mu\ A^{bj}_\nu \ + \ f_{lj}{}^k\ A^l_\mu\ A^{cj}_\nu \tag{2.28c}$$

From the first three terms in the right hand side of eq-(2.28a) one learns that $F^k_{\mu\nu}$ does *contain* explicitly the $SU(N)$ field strength if $\delta_{ab} f_{lj}{}^k A^{al}_\mu A^{bj}_\nu \neq f_{lj}{}^k A^l_\mu A^j_\nu$. After requiring the summation over the l, j indices in eq-(2.28b) to be $\frac{\delta_{lj}}{N} A^{al}_\mu A^{bj}_\nu = A^a_\mu A^b_\nu$, in the special case that the indices span the range of values given by $a, b = 1, 2, 3;\ l, j = 1, 2, ...N^2 - 1$, the field strength $F^c_{\mu\nu}$ becomes

$$F^c_{\mu\nu} \ = \ \partial_\mu\ A^c_\nu \ - \partial_\nu\ A^c_\mu \ + \ c_{ab}{}^c\ \frac{\delta_{lj}}{N}\ A^{al}_\mu\ A^{bj}_\nu \ =$$

$$\partial_\mu\ A^c_\nu \ - \partial_\nu\ A^c_\mu \ + \ c_{ab}{}^c\ A^a_\mu\ A^b_\nu \tag{2.29}$$

and effectively behaves as a $SU(2)$-valued field strength since the quaternionic structure constants c_{abc} coincide with the epsilon symbols ϵ_{abc} when $a, b, c = 1, 2, 3$. When the index for c in eq-(2.28b) is $c = 0 \Rightarrow c_{ab}{}^0 = 0$, and eq-(2.28b) becomes in this case

$$F^0_{\mu\nu} \ = \ \partial_\mu\ A^0_\nu \ - \partial_\nu\ A^0_\mu \tag{2.30}$$

and which behaves effectively as an $U(1)$ field strength.

Therefore, when $N = 3$, the field strength components in eqs-(2.28a,2.28b) associated with a quaternionic-valued $SU(3)$ Yang-Mills theory, *contain* the $SU(3), SU(2), U(1)$ field strengths as special cases, and consequently, the Standard Model group. In this way one can see once more how a quaternionic-valued gravitational theory in $8D$ can furnish a gravitational and $SU(3) \times SU(2) \times U(1)$ Yang-Mills theory in $4D$ after a Kaluza-Klein compactification on CP^2. There are *additional* field strength components $F^{ck}_{\mu\nu}$ stemming from the *noncommutativity* of the quaternions which do *not* belong to the Standard Model group and which are given by eq-(2.28c). As expected, the structure of the quaternionic-valued $SU(3)$ Yang-Mills theory is richer than a one based on the standard model group $SU(3) \times SU(2) \times U(1)$,

One may compare these results with the Clifford $Cl(5, C)$ Unified Gauge Field Theory of Conformal Gravity, Maxwell and $U(4) \times U(4)$ Yang-Mills in 4D [14], the Kaluza-Klein theory without extra dimensions involving a curved Clifford space [30] and the Clifford $Cl(8)$ algebra models of [29]

3 Gravity and $SU(5)$ Yang-Mills Unification from Nonlinear Connections in Finsler Geometry

In this section we will explore a different approach than the standard Kaluza-Klein one to unification from gravity in higher dimensions. It will be based on the *nonlinear* connection formalism of Finsler geometry, [17], [18], [19]. Some time ago it was shown by [27] that a Kaluza-Klein-like formalism of Einstein's theory, based on the $(2+2)$-fibration of a generic 4-dimensional spacetime, describes General Relativity as a Yang-Mills gauge theory on the 2-dimensional base manifold, where the local gauge symmetry is the group of the diffeomorphisms of the 2-dimensional fibre manifold. They found the Schwarzschild solution by solving the field equations after a very laborious procedure. Their formalism was valid for any $m+n$ decomposition of the D-dim spacetime $D = m+n$. The line element in $m+n$ dimensions is parametrized as follows

$$ds^2 \ = \ \phi_{ab}\ dy^a\ dy^b \ + \ (\gamma_{\mu\nu} + \phi_{ab}\ A^a_\mu\ A^b_\nu)\ dx^\mu\ dx^\nu \ + \ 2\phi_{ab}\ A^b_\mu\ dx^\mu\ dy^a. \quad (3.1)$$

where *all* fields depend on the x^μ, y^a coordinates and the metric is symmetric. In particular, the nonlinear gauge connection is given by $A^a_\mu(x^\rho, y^b)$ and the span of indices is $\mu, \nu, \rho = 1, 2, \ldots.., m$ and $a, b, c = 1, 2, \ldots.., n$. To find the $(m+n)$-dimensional action principle of general relativity we must compute the scalar curvature of space-times in the $(m+n)$-decomposition. For this purpose it is convenient to introduce the following non-holonomic (non-coordinate basis) $\hat\partial_A = (\hat\partial_\mu, \hat\partial_a)$ where

$$\hat\partial_\mu \equiv \partial_\mu - A_\mu{}^a(x^\rho, y^b)\ \partial_a, \qquad \hat\partial_a \equiv \partial_a\ . \quad (3.2)$$

From this definition we have

$$[\ \hat\partial_A,\ \hat\partial_B\ ] \ = \ f_{AB}{}^C(x^\rho, y^b)\ \hat\partial_C,$$

where the structure coefficients (non-holonomic coefficients) $f_{AB}{}^C$ are given by

$$f_{\mu\nu}{}^a \ = \ -\,F_{\mu\nu}{}^a, \quad f_{\mu a}{}^b \ = \ -\,f_{a\mu}{}^b \ = \ \partial_a A_\mu{}^b, \quad f_{AB}{}^C = 0,\ otherwise \quad (3.3)$$

The virtue of this non-holonomic basis is that it brings the metric (3.1) into a block diagonal form

$$g_{AB} \ = \ \begin{pmatrix} \gamma_{\mu\nu} & 0 \\ 0 & \phi_{ab} \end{pmatrix}$$

which drastically simplifies the computation of the scalar curvature. In this non-holonomic (non-coordinate) basis the Levi-Civita connection is given by

$$\Gamma_{AB}{}^C = \frac{1}{2} g^{CD}(\hat\partial_A g_{BD} + \hat\partial_B g_{AD} - \hat\partial_D g_{AB}) + \frac{1}{2} g^{CD}(\ f_{ABD} - f_{BDA} - f_{ADB}\ ) \quad (3.4)$$

where $f_{ABC} = g_{CD} f_{AB}{}^{D}$. The field strength $F_{\mu\nu}{}^{a}$ corresponding to the (nonlinear) gauge connection $A_{\mu}{}^{a}$ is defined as

$$F_{\mu\nu}{}^{a} \;=\; \partial_\mu A^a_\nu \; - \partial_\nu A^a_\mu \; - \; A^c_\mu \partial_c A^a_\nu \; + A^c_\nu \partial_c A^a_\mu \tag{3.5}$$

For completeness, the connection coefficients components are [27]

$$\Gamma_{\mu\nu}{}^{\alpha} = \frac{1}{2}\gamma^{\alpha\beta}\Big(\hat\partial_\mu \gamma_{\nu\beta} + \hat\partial_\nu \gamma_{\mu\beta} - \hat\partial_\beta \gamma_{\mu\nu}\Big)$$

$$\Gamma_{\mu\nu}{}^{a} = -\frac{1}{2}\phi^{ab}\partial_b \gamma_{\mu\nu} - \frac{1}{2}F_{\mu\nu}{}^{a}$$

$$\Gamma_{\mu a}{}^{\nu} = \Gamma_{a\mu}{}^{\nu} = \frac{1}{2}\gamma^{\nu\alpha}\partial_a \gamma_{\mu\alpha} + \frac{1}{2}\gamma^{\nu\alpha}\phi_{ab}F_{\mu\alpha}{}^{b}$$

$$\Gamma_{\mu a}{}^{b} = \frac{1}{2}\phi^{bc}\hat\partial_\mu \phi_{ac} + \frac{1}{2}\partial_a A_\mu{}^{b} - \frac{1}{2}\phi^{bc}\phi_{ae}\partial_c A_\mu{}^{e}$$

$$\Gamma_{a\mu}{}^{b} = \frac{1}{2}\phi^{bc}\hat\partial_\mu \phi_{ac} - \frac{1}{2}\partial_a A_\mu{}^{b} - \frac{1}{2}\phi^{bc}\phi_{ae}\partial_c A_\mu{}^{e}$$

$$\Gamma_{ab}{}^{\mu} = -\frac{1}{2}\gamma^{\mu\nu}\hat\partial_\nu \phi_{ab} + \frac{1}{2}\gamma^{\mu\nu}\phi_{ac}\partial_b A_\nu{}^{c} + \frac{1}{2}\gamma^{\mu\nu}\phi_{bc}\partial_a A_\nu{}^{c}$$

$$\Gamma_{ab}{}^{c} = \frac{1}{2}\phi^{cd}\Big(\partial_a \phi_{bd} + \partial_b \phi_{ad} - \partial_d \phi_{ab}\Big). \tag{3.6}$$

The Torsion is defined as

$$T^A_{BC} \;=\; \Gamma^A_{BC} \; - \; \Gamma^A_{CB} \; - \; f^A_{BC} \tag{3.7a}$$

giving vanishing torsion components, consistent with the fact that the Levi-Civita connection is *torsionless* by definition. For example, from eqs-(3.3,3.6) one arrives at the vanishing values

$$T^a_{\mu\nu} \;=\; -F^a_{\mu\nu} + F^a_{\mu\nu} \;= 0, \quad T^b_{\mu a} \;=\; - \; T^b_{a\mu} \;=\; \partial_a A^b_\mu - \partial_a A^b_\mu = 0, \; ...... \tag{3.7b}$$

The curvature tensors are defined as

$$R_{ABC}{}^{D} \;=\; \hat\partial_A \Gamma_{BC}{}^{D} - \hat\partial_B \Gamma_{AC}{}^{D} + \Gamma_{AE}{}^{D}\Gamma_{BC}{}^{E} - \Gamma_{BE}{}^{D}\Gamma_{AC}{}^{E} - f_{AB}{}^{E}\Gamma_{EC}{}^{D}$$

$$R_{AC} = R_{ABC}{}^{B}, \quad R = g^{AC}R_{AC}. \tag{3.8a}$$

Explicitly, the scalar curvature R is given by

$$R = \gamma^{\mu\nu}(R_{\mu\alpha\nu}{}^{\alpha} + R_{\mu a\nu}{}^{a}) + \phi^{ab}(R_{acb}{}^{c} + R_{a\mu b}{}^{\mu}) \tag{3.8b}$$

which becomes, after a very lengthy computation

$$R \;=\; \gamma^{\mu\nu}\mathcal{R}_{\mu\nu} \;+\; \phi^{ac}\mathcal{R}_{ac} \;+\; \frac{1}{4}\phi_{ab}\gamma^{\mu\nu}\gamma^{\alpha\beta}F_{\mu\alpha}{}^{a}F_{\nu\beta}{}^{b} \;+$$

$$\frac{1}{4}\gamma^{\mu\nu}\phi^{ab}\phi^{cd}\Big\{(D_\mu\phi_{ac})(D_\nu\phi_{bd}) - (D_\mu\phi_{ab})(D_\nu\phi_{cd})\Big\} \;+$$

$$\frac{1}{4}\phi^{ab}\gamma^{\mu\nu}\gamma^{\alpha\beta}\Big\{(\partial_a\gamma_{\mu\alpha})(\partial_b\gamma_{\nu\beta}) - (\partial_a\gamma_{\mu\nu})(\partial_b\gamma_{\alpha\beta})\Big\} + \nabla_A j^A \tag{3.9}$$

where the "gauged" Ricci tensor $\mathcal{R}_{\mu\nu}$ in the base manifold and the internal space Ricci tensor $\mathcal{R}_{ac}$ are defined by

$$\mathcal{R}_{\mu\nu} = \hat{\partial}_\mu\Gamma_{\alpha\nu}{}^{\alpha} - \hat{\partial}_\alpha\Gamma_{\mu\nu}{}^{\alpha} + \Gamma_{\mu\beta}{}^{\alpha}\Gamma_{\alpha\nu}{}^{\beta} - \Gamma_{\beta\alpha}{}^{\beta}\Gamma_{\mu\nu}{}^{\alpha}$$

$$\mathcal{R}_{ac} = \partial_a\Gamma_{bc}{}^{b} - \partial_b\Gamma_{ac}{}^{b} + \Gamma_{ad}{}^{b}\Gamma_{bc}{}^{d} - \Gamma_{db}{}^{d}\Gamma_{ac}{}^{b}. \tag{3.10}$$

The derivative terms $\nabla_A j^A$ in (3.9) are

$$\nabla_A j^A = \nabla_\mu j^\mu + \nabla_a j^a, \;\; \nabla_\mu j^\mu = \Big(\hat{\partial}_\mu + \Gamma_{\alpha\mu}{}^{\alpha} + \Gamma_{c\mu}{}^{c}\Big) j^\mu$$

$$\nabla_a j^a = \Big(\partial_a + \Gamma_{ca}{}^{c} + \Gamma_{\alpha a}{}^{\alpha}\Big) j^a, \tag{3.11}$$

where j^μ and j^a are given by

$$j^\mu = \gamma^{\mu\nu}\Big(\phi^{ab}\hat{\partial}_\nu\phi_{ab} - 2\partial_a A_\nu{}^{a}\Big), \quad j^a = \phi^{ab}\gamma^{\mu\nu}\partial_b\gamma_{\mu\nu}. \tag{3.12}$$

Therefore, Einstein gravity in $D = m + n$ dimensions describes an m-dim generally invariant field theory under the gauge transformations corresponding to the Diffs $\mathcal{N}$ of the internal n-dim space $\mathcal{N}$. A^a_μ couples to the graviton $\gamma_{\mu\nu}$, meaning that the graviton is charged (gauged) in this theory and also to the ϕ_{ab} field on $\mathcal{N}$ which can be identified as a non-linear sigma field whose self interaction potential term is given by $\phi^{ab}\mathcal{R}_{ab}$ [27].

When the internal manifold $\mathcal{N}$ is a homogeneous compact space one can perform a harmonic expansion of the fields w.r.t the internal y^a coordinates, and after integrating the action w.r.t these y^a coordinates, one will generate an infinite-component field theory on the m-dimensional space represented by the x^μ coordinates. A reduction of the Diffs $\mathcal{N}$, via the inner automorphims of a subgroup G of the Diffs $\mathcal{N}$, yields the usual Einstein-Yang-Mills theory interacting with a nonlinear sigma field. But in general, the former theory described above is much *richer* than the latter one.

When the internal space $\mathcal{N}$ is two-dimensional, the area-preserving diffeomorphisms subalgebra of the Diffs $\mathcal{N}$ algebra is generated by those vector fields ξ^a which are tangent to the internal two-dim surface and are divergence-free $\partial_a\xi^a = 0$. If the internal surface is a sphere S^2 one may recur to the finding by Hoppe [28] showing that there exists a basis-dependent limit of $SU(N)$ such that $SU(N = \infty)$ is isomorphic to the algebra of area-preserving diffs of the sphere S^2. The $SU(\infty)$-valued gauge fields $\mathbf{A}_\mu(x^\mu) = \mathbf{A}_\mu^I T_I$, with $I = 1,2,3,......, N = \infty$, are mapped to c-functions depending on the x^μ, y^a coordinates $A_\mu(x^\mu, y^a)$. The Lie algebra $SU(\infty)$ commutators $[\mathbf{A}_\mu(x^\rho), \mathbf{A}_\nu(x^\rho)]$ are replaced by the Poisson brackets $\{A_\mu(x^\rho, y^a), A_\nu(x^\rho, y^a)\}_{PB}$ with respect to the internal $y^a = y^1, y^2$ coordinates of the sphere. In terms of the two angles like θ, ϕ, the Poisson brackets are

$$\{A_\mu(x^\rho, y^a), A_\nu(x^\rho, y^a)\}_{PB} \equiv \frac{\partial A_\mu}{\partial(cos\theta)} \frac{\partial A_\nu}{\partial\phi} - \frac{\partial A_\nu}{\partial(cos\theta)} \frac{\partial A_\mu}{\partial\phi} \tag{3.13}$$

The group trace operation in the $N \to \infty$ limit is replaced by an integral with respect to the internal y^a coordinates of the sphere such that

$$\int d^4x \int d^2y \sqrt{|det\ \gamma_{\mu\nu}|} \sqrt{|det\ \phi_{ab}|}\ \phi_{ab}\ \gamma^{\mu\nu}\ \gamma^{\alpha\beta}\ F_{\mu\alpha}{}^a\ F_{\nu\beta}{}^b =$$
$$\int d^4x \int d^2y \sqrt{|det\ \gamma_{\mu\nu}|} \sqrt{|det\ h_{ab}(y)|}\ \gamma^{\mu\nu}\ \gamma^{\alpha\beta}\ \mathcal{F}_{\mu\alpha}\ \mathcal{F}_{\nu\beta} \tag{3.14}$$

where $h_{ab}(y)$ is the standard metric on the sphere and the field strengths in the right hand side are defined in terms of the Poisson brackets as

$$\mathcal{F}_{\mu\nu}(x^\rho, y^a) = \partial_\mu A_\nu(x^\rho, y^a) - \partial_\nu A_\mu(x^\rho, y^a) + \{A_\mu(x^\rho, y^a), A_\nu(x^\rho, y^a)\}_{PB} \tag{3.15}$$

Therefore, by restricting to the area-preserving Diffs S^2 symmetry transformations ($(\partial_a \xi^a = 0)$ the above horizontal-vertical decomposition of six dimensional gravity (3.9), based on the *nonlinear* connection $A_\mu(x, y)$, yields a $4D$ theory of (gauged) gravity and $SU(\infty)$ Yang-Mills. Because $SU(5) \subset SU(\infty)$, a grand unification procedure from pure gravity in $D = 4 + 2$ dimensions is plausible, in principle, after one truncates (or breaks) the infinite-dim symmetry $SU(\infty)$ down to $SU(5)$. An spontaneous breakdown of the $SU(\infty)$ symmetry to $SU(5)$ via the Higgs mechanism leads to an infinite number of massive spin **1** fields and Higgs scalars along the infinite chain of steps from $SU(\infty) \to SU(5)$. A truncation of the $SU(\infty)$ down to $SU(5)$, rather than an infinite chain of spontaneous symmetry breaking processes, from very high energies to lower energies, is another possibility. Despite the fact that it does not seem very physically appealing to have an infinite hierarchy of massive spin **1** fields, and Higgs scalars along the infinite chain, one should not exclude this possibility from being realized in Nature. Concluding, the nonlinear connection formalism of Finsler geometry provides a hierarchical extension of the standard model, and the $SU(5)$ GUT, within a six dimensional gravitational theory in the form of the Lagrangian described by eq-(3.9), when the internal two-dim space is a sphere S^2.

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