Kaluza-Cartan Theory And A New Cylinder Condition

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Abstract

Kaluza's 1921 theory of gravity and electromagnetism using a fifth wrapped-up spatial dimension is at the root of many modern attempts to develop new physical theories. For a number of reasons the theory is incomplete and generally considered untenable. An alternative approach is presented that includes torsion. Focus is here placed on the variety of electromagnetic fields that result. Two connections are used, one with, and one without torsion. Coulomb's law in the form of the Lorentz force law is investigated starting with a non-Maxwellian definition of charge, this is shown to be related to Maxwellian charge. It is concluded that Kaluza's 5D space and torsion should go together in what is here called a Kaluza-Cartan theory in order to form a unified theory of gravity and electromagnetism. This is so as to have both sufficient electromagnetic fields and a derivation of the Lorentz force law at the same time. A new cylinder condition is proposed that takes torsion into account and is fully covariant. This is novel and surprisingly leads to the required results. Different formulations of the vacuum and matter models are called matter model regimes and are compared. The concept of general covariance is investigated with respect to global properties that can be modelled via a non-maximal atlas. Differences from existing theories, general relativity being a limiting case, suggest experimental tests may be possible.

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1 Conventions

The following conventions are adopted unless otherwise specified:

Five dimensional metrics, tensors and pseudo-tensors are given the hat symbol. Five dimensional indices, subscripts and superscripts are given capital Roman letters. So for example the five dimensional Ricci flat 5-dimensional superspace-time of Kaluza theory is given as: $\hat{g}_{AB}$, all other tensors and indices are assumed to be 4 dimensional, if a general non-specified dimensional case is not being considered, for which either convention can be used. Index raising is referred to a metric $\hat{g}_{AB}$ if 5-dimensional, and to g_{ab} if 4-dimensional. The domain of partial derivatives carries to the end of a term without need for brackets, so for example we have $\partial_a g_{db} A_c + g_{db} g_{ac} = (\partial_a (g_{db} A_c)) + (g_{db} g_{ac})$. Terms that might repeat dummy variables or are otherwise in need of clarification use additional brackets. Square brackets can be used to make dummy variables local in scope.

Space-time is given signature $(-, +, +, +)$, Kaluza space $(-, +, +, +, +)$ in keeping with [6], except where stated and an alternative from [1] is referred to. Under the Wheeler et al [6] nomenclature, the sign conventions (for metric signature, Riemannian curvature definition and Einstein tensor sign) used here as a default are $[+, +, +]$, as used throughout [6] itself. This is the usual modern convention used in general relativity. Note that torsion means that further conventions in the definition of the Riemannian curvature are required and thus the notation of Wheeler [6] is actually insufficient here, see the definition of the Ricci tensor below. The first dimension (index 0) is always time and the 5^{th} dimension (index 4) is always the topologically closed Kaluza dimension. Time and distance are geometrized throughout such that c = 1. **G** is the gravitational constant. The scalar field component is labelled ϕ^2 (in keeping with the literature) only as a reminder that it is associated with a spatial dimension, and to be taken as positive. The matrix of g_{cd} can be written as $|g_{cd}|$ when considered in a particular coordinate system to emphasize a component view. The Einstein summation convention may be used without special mention.

Some familiar defining equations consistent with [1] (using Roman lowercase for the general case only for ease of reference) define the Ricci tensor and Einstein tensors in terms of the Christoffel symbols along usual lines, noting that torsion will here be allowed and so the order of indices can not be carelessly interchanged:

$$R_{ab} = \partial_c \Gamma^c_{ba} - \partial_b \Gamma^c_{ca} + \Gamma^c_{ba} \Gamma^d_{dc} - \Gamma^c_{da} \Gamma^d_{bc} \tag{1.0.1}$$

$$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab} = 8\pi \mathbf{G} T_{ab} \tag{1.0.2}$$

For convenience we will define $\alpha = \frac{1}{8\pi\mathbf{G}}$.

$$F_{ab} = \nabla_a A_b - \nabla_b A_a = \partial_a A_b - \partial_b A_a \text{ equally } F = dA \tag{1.0.3}$$

Any 5D exterior derivatives and differential forms could also be given a hat, thus: $\hat{d}\hat{B}$. However, the primary interest here will be 4D forms. $\Box$ represents the 4D D'Alembertian [6], the relativistic analog of the Laplacian, a wave operator.

Torsion introduces non-obvious conventions in otherwise established definitions. The order of the indices in the Christoffel symbols comes to matter, and this includes in the Ricci tensor definition and the definition of the Christoffel symbols themselves:

$$\nabla_a w_b = \partial_a w_b - \Gamma^c_{ab} w_c \tag{1.0.4}$$

Christoffel symbols with torsion will take the usual form: Γ^c_{ab} or Γ^{abc} and so on. The metric for any given torsion tensor defines a unique connection. There are therefore two unique connections for a given metric: one with and one without the torsion. In the completely antisymmetric case, when the torsion tensor is completely anti-symmetric, we can write the without-torsion connection coefficients as: $\Gamma^c_{(ab)}$. The unique Levi-Civita connection is more explicitly written as: $\{^{\,c}_{ab}\}$ or $\mathit{\Gamma}^c_{ab}$, and the covariant Levi-Civita derivative operator (ie again without torsion): $\triangle_a$. In order to distinguish general G_{ab} and R_{ab} etc. which relate to the torsion connection from the Levi-Civita case we use cursive: $\mathcal{G}_{ab}$ and $\mathcal{R}_{ab}$.

2 Introduction

Kaluza's 1921 theory of gravity and electromagnetism [2][3][4] using a fifth wrapped-up spatial dimension is at the heart of many modern attempts to develop new physical theories [1][5]. From supersymmetry to string theories topologically closed small extra dimensions are used to characterize the various forces of nature. It is therefore at the root of many modern attempts and developments in theoretical physics. However it has a number of foundational problems and is often considered untenable in itself. This paper looks at these problems from a purely classical perspective, without involving quantum theory.

2.1 The Metric

The theory assumes a (1,4)-Lorentzian Ricci flat manifold to be the underlying metric, split as shown below (and for interest this can be compared to the later ADM formalism [9]). A_a is to be identified with the electromagnetic potential, ϕ^2 is to be a scalar field, and g_{ab} the metric of 4D space-time:

$$\hat{g}_{AB} = \begin{bmatrix} g_{ab} + \phi^2 A_a A_b & \phi^2 A_a \\ \phi^2 A_b & \phi^2 \end{bmatrix} \tag{2.1.1}$$

Note that a scaling factor has been set to $k = 1$ and so is not present, this will be reintroduced later in the text (3.3.1), it is mathematically arbitrary, but physically scales units when units are geometrized. By inverting this metric as a matrix (readily checked by multiplication) we get:

$$\hat{g}^{AB} = |\hat{g}_{AB}|^{-1} = \begin{bmatrix} g^{ab} & -A^a \\ -A^b & \frac{1}{\phi^2} + A_i A^i \end{bmatrix} \tag{2.1.2}$$

Maxwell's law are automatically satisfied: dF=0 follows from dd = 0. d*F= 4π*J can be set by construction. d*J=0, conservation of charge, follows also by dd=0 on most parts of the manifold.

However, in order to write the metric in this form there is a subtle assumption, that g_{ab}, which will be interpreted as the usual four dimensional space-time metric, is itself non-singular. However, this will always be the case for moderate or small values of A_x which will here be identified with the electromagnetic 4-vector potential. The raising and lowering of this 4-vector are defined in the obvious way in terms of g_{ab}. The 5D metric can be represented at every point on the Kaluza manifold in terms of this 4D metric g_{ab} (when it is non-singular), the vector potential A_x, and the scalar field ϕ^2. We have also assumed that topology is such as to allow the Hodge star operator and Hodge duality of forms to be well-defined (see [6] p.88). This means that near a point charge source the argument that leads to charge conservation potentially breaks down as the potential may cease to be well-defined. Whereas the Toth charge that will be defined in the sequel does not have this problem. So two different definitions of charge are to be given: the Maxwellian, and the Toth charge.

With values of ϕ^2 around 1 and relatively low 5-dimensional metric curvatures, we need not concern ourselves with this assumption beyond stating it on the basis that physically these parameters encompass tested theory. Given this proviso A_x is a vector and ϕ^2 is a scalar - with respect to the tensor system defined on any 4-dimensional submanifold that can take the induced metric g.

2.2 Kaluza's Cylinder Condition And The Original Field Equations

Kaluza's cylinder condition (KCC) is that all partial derivatives in the 5th dimension i.e. ∂_4 and $\partial_4\partial_4$ etc... of all metric components and of all tensors and their derivatives are zero. A perfect 'cylinder'. Here we extend it to torsion terms, and indeed all tensors and pseudo-tensors. This leads to constraints on g_{ab} given in [1] by three equations, the field equations of the original Kaluza theory, where the Einstein-Maxwell stress-energy tensor can be recognised embedded in the first equation:

$$G_{ab} = \frac{k^2\phi^2}{2}\left\{\frac{1}{4}g_{ab}F_{cd}F^{cd} - F^c_aF_{bc}\right\} - \frac{1}{\phi}\{\nabla_a(\partial_b\phi) - g_{ab}\Box\phi\} \tag{2.2.1}$$

$$\nabla^a F_{ab} = -3\frac{\partial^a\phi}{\phi}F_{ab} \tag{2.2.2}$$

$$\Box\phi = \frac{k^2\phi^3}{4}F_{ab}F^{ab} \tag{2.2.3}$$

Note that there is both a sign difference and a possible factor difference with respect to Wald [7] and Wheeler [6]. The sign difference appears to be due to the mixed use of metric sign conventions in [1] and can be ignored. A k factor is present and scaling will be investigated. These will be referred to as the first, second and third torsionless field equations, or original field equations, respectively. They are valid only in Kaluza vacuum, that is, outside of matter and charge models, and when there is no torsion. This requires Kaluza's original cylinder condition and the usual conception of matter models.

2.3 Some Definitions

Definition 2.3.1: Perpendicular and null electromagnetic fields.

Fields for which the following equation hold will be called *perpendicular* electromagnetic fields, and likewise those that do not satisfy this: non-perpendicular. *Null* solutions are perpendicular solutions with a further constraint. But perpendicular solutions will be of most interest here.

$$F_{ab}F^{ab} = 0 \tag{2.3.1}$$

By looking at the third field equation (2.2.3) it can be seen that if the scalar field does not vary then only a limited range of solutions result, that have perpendicular electric and magnetic fields, for example null solutions. The second field equation (2.2.2) then also imposes no charge sources. Here the scalar term could be allowed to vary in order to allow for non-zero $F_{ab}F^{ab}$. This falls within Kaluza's original theory. This potentially allows for more electromagnetic solutions, but there are problems to overcome: the field equations cease being necessarily electrovacuum. Null electromagnetic fields require perpendicularity plus the following condition, where the star is the Hodge star operator:

$$F_{ab}(*F^{ab}) = 0 \tag{2.3.2}$$

2.4 Foundational Problems

A key problem addressed in this paper is the variety of electromagnetic solutions that are a consequence of Kaluza theory, whilst maintaining the Lorentz force law. This is not usually considered such a big problem, but that's just not really correct: a sufficient variety of electromagnetic fields must be available. This is surely the real problem with Kaluza theory, as it prevents a geometric unification of gravity and electromagnetism. The missing solutions are the non-perpendicular solutions. Important examples include static electric fields, so they include really important solutions!

One inadequate and arbitrary fix in standard Kaluza theory is to set the scalar field term large to ensure that the second field equation (2.2.2) is identically zero despite scalar fluctuations. This approach will not be taken here as it seems arbitrary. The stress-energy tensor under scalar field fluctuations is different from the Einstein-Maxwell tensor [6][7] and the accepted derivation of the Lorentz force law (for electrovacuums [6]) can not be assumed. A variable scalar field also implies non-conservation of Maxwell charge via the third field equation (2.2.3). Attempts to loosen constraints such as the KCC have also not been successful so far. Thus the scalar field will later be fixed, and therefore the non-perpendicular solutions will need reintroducing - and this will be done via the introduction of torsion.

Another foundational issue of Kaluza theory is that even with a scalar field it does not have convincing sources of mass or charge built in. The second field equation (2.2.2) has charge sources, but it's unlikely that realistic sources are represented by this equation. Matter and charge models in this work will be governed by different constraints and definitions in the Kaluza-Cartan space called 'matter model regimes', just as matter/energy is analogously assumed to be the conserved Einstein tensor in general relativity, and vacuum the state where the Ricci tensor is zero. For example vacuum in the 5D Kaluza space of the original Kaluza theory yields electromagetic fields in the corresponding 4D space-time. There is room here for some experimentation with different notions.

As mentioned, charge will be given a possible alternative definition, Toth charge: as 5-dimensional momentum, following a known line of reasoning [8] within Kaluza theory. This will enable a derivation of Coulomb's law, via the Lorentz force law - leading to a genuine mathematical unification of electromagnetism and gravity. As momentum, the Toth charge is of necessity locally conserved, provided there are no irregularities in the topology of the Kaluza 5th dimension. Similarly the conservation of Maxwellian charge is normally guaranteed by the existence of the potential, except that this may not be valid in extreme curvatures where the values here associated with the 4-potential may cease to be a vector.

We will also assume global hyperbolicity in the sense of the existence of a Cauchy surface as is often done in general relativity to ensure 4 dimensional causality. Though this will not necessarily guarantee 5D causality in the event that either KCC is weakened or the theory presented here breaks down due to singularities. It also requires the usual energy conditions on the resulting 4D space-time and fields, or a 5D equivalent generalization.

One leading issue is that Kaluza theory offers limited electromagnetic solutions. Non-perpendicular electrovacuums more generally are not so easily supported as changes in the scalar field may force divergence of the field equations from those of the electrovacuum (see the first field equation (2.2.1)). This will lead to the potential failure of the Lorentz Force Law, in effect Coulomb's law. This is resolved by fixing the scalar field constant and adding extra degrees of freedom thus lost via torsion.

2.5 A Solution?

The Lorentz force law/Coulomb's law is to be derived from the theory independently of the electrovacuum solutions of general relativity, and the missing non-perpendicular solutions included at the same time to create a more complete theory. Note that in addition the derivation of the Lorentz force law within general relativity (from an assumed Einstein-Maxwell stress-energy tensor) is not without problems of principle [6], so it is not the Einstein-Maxwell stress-energy tensor that is necessarily here being sought. The stress-energy tensor that defines the electrovacuum geometry has to be assumed in classical electrodynamics within general relativity, whereas in this Kaluza-Cartan theory it is not. The Lorentz force law is instead derived from first principles. A link between the Toth charge (momentum in the fifth dimension) and Maxwellian charge (defined in terms of the vector potential) is required to do this.

2.6 A Torsion Connection And A New Cylinder Condition

A further major change is the definition of a new cylinder condition (postulate K3) based on the to-be-introduced torsion connection. Matter, charge and spin sources will also be investigated under the discussion about matter model regimes. To obtain the sought for range of electromagnetic solutions a particular constraint will be weakened: that the torsion tensor is necessarily vanishing. Torsion will be allowed to vary allowing greater scope for solutions. This is necessary to ensure the presence of non-perpendicular solutions given that the scalar field will also be constant.

The combination of torsion and the 5th spatial dimension justifies the label Kaluza-Cartan theory.

2.7 The Program

A new cylinder condition will be imposed as with Kaluza's original theory, but one based on the covariant derivative and associated with a metric torsion connection (ie compatible in the usual sense, with the covariant derivative of the metric vanishing). It will be a generally covariant definition. The missing non-perpendicular electromagnetic fields will thus be sought. A new definition of charge, the momentum in the fifth dimension, will be introduced and Maxwellian charge will be shown to coincide at an appropriate limit. The collocation of torsion with electromagnetism is different from other Einstein-Cartan theories where the torsion is limited to within matter models. Here certain specific components of torsion are an essential counterpart of electromagnetism, and other components of torsion can exist outside of matter models, perhaps with detectable consequences.

Restrictions to the geometry and certain symmetries will be handled by reducing the maximal atlas to a reduced Kaluza atlas that automatically handled the restrictions and symmetries without further deferment to general covariance. Physically this represents the idea that in 4D, charge is a generally covariant scalar, whereas in 5D, charge is entirely dependent on the frame. That this is meaningful stems from the global property of a small wrapped-up fifth spatial dimension with new cylinder condition. Mathematically the Kaluza atlas is a choice of subatlas for which the partial derivatives in the Kaluza direction are vanishing. This led to useful constraints on the Christoffel symbols for all coordinate systems in the Kaluza atlas. The 5D metric decomposes into a 4D metric and the electromagnetic vector potential and the scalar field that will be set constant.

Given certain assumptions about matter-charge models (various matter model regimes), the entirety of classical electrodynamics is rederived. Gravity and electromagnetism are unified in a way not fully achieved by general relativity, Einstein-Cartan theory or Kaluza's original 5D theory. This is the objective of this research and the reason for introducing torsion.

Due to the lack of realistic charge models (and some resulting imaginary numbers) this theory remains incomplete, though many essential ingredients are present.

Why go to all this effort to unify electromagnetism and gravitation and to make electromagnetism fully geometric? Because experimental differences should be detectable given sufficient technology on the one hand, or, on the other, and equally, simply because such an attempt at unification might be right. This theory differs from both general relativity and Einstein-Cartan theory, and this may be empirically fruitful.

3 Overview Of Kaluza-Cartan Theory

The new theory (or theories) presented here, the Kaluza-Cartan theory (or Kaluza-Cartan theories), purports to resolve the foundational problems of the original Kaluza theory - even if presenting some new issues of its own.

3.1 The New Cylinder Condition

Throughout this work the limit of a new cylinder condition (postulate K3) will be taken to be that the covariant derivative of all tensors in the Kaluza direction are to be zero, and that the covariant derivative depends on torsion. The 5D metric generally decomposes into 4D metric, vector potential and scalar field, at least when the embedded 4D metric is non-singular.

The new cylinder condition by construction allows for an atlas of charts wherein also the partial derivatives are zero. This is true for a subatlas covering the 5D Lorentz manifold. But charts may exist in the maximal atlas for which these constraints are not possible. The atlas that is compliant is restricted. This means that the new cylinder condition can be represented by a subatlas of the maximal atlas for the manifold. The set of local coordinate transformations that are compliant with this atlas (call it the Kaluza atlas) is in general non-maximal by design.

A further reduction in how the atlas might be interpreted is also implied by setting c=1, and constant **G**. The existence of a single unit for space and time can be assumed, and this must be scaled in unison for all dimensions. Consistently with cgs units we can choose either centimetres or seconds. This would leave velocities (and other geometrically unitless quantities) unchanged in absolute magnitude. This doesn't prevent reflection of an axis however, and indeed reflection of the Kaluza dimension will be equivalent to a charge inversion. However, given an orientation we can also remove this.

Space-time can not be an arbitrary 4D Lorentz submanifold as it must be one that is normal to a Kaluza axis and that satisfies certain constraints. This will generally have the interpretation best visualized as a cylinder with a longitudinal space-time and a perpendicular Kaluza dimension. However this will not be so simple when considering more than one connection.

We can further reduce the Kaluza atlas by removing boosts in the Kaluza dimension. Why? This requires the new cylinder condition, as it significantly reduces the possible geometries. Space-time is taken to be a subframe within a 5D frame within the Kaluza subatlas wherein uncharged matter can be given a rest frame via a 4D Lorentz transformation. Boosting uncharged matter along the Kaluza axis will give it kinetic Toth charge (as described in the Introduction, and as detailed shortly). The Kaluza atlas represents the 4D view that charge is 4D covariant. Here we require that the Toth charge coincides with Maxwellian charge in some sense. The justification for this assertion will be clarified later. Rotations into the Kaluza axis can likewise be omitted. These result in additional constraints on the Christoffel symbols associated with charts of this subatlas, and enable certain geometrical objects to be more easily interpreted in space-time. The use of this subatlas does not prevent the theory being generally covariant, but simplifies the way in which we look at the Kaluza space through a 4D physical limit and worldview.

Preliminary Description 3.1.1: The Kaluza-Cartan space-time.

In the most general case a *Kaluza-Cartan space-time* will be a 5D Lorentz manifold with metric and metric torsion connection (ie compatible in the usual sense, with the covariant derivative of the metric vanishing). As mentioned above the new cylinder condition will be added (see postulate K3 later) and the topology of the Kaluza dimension will be closed and geometrically small. A global Kaluza direction will be defined as normal (relative to the new cylinder condition) to a 4D Lorentz submanifold. That submanifold, and all parallel submanifolds as a set, will constitute space-time. The torsionless Levi-Civita connection will also continue to be available. Charged particles will be those that are not restricted in movement to within space-time but traverses the Kaluza dimension in some sense laterally. Uncharged particles will be restricted to motion within each slice of the set of parallel space-time slices. The new cylinder condition will ensure that all the parallel space-times are equivalent. The rigidity of this is expressed via the definition of the Kaluza atlas.

A complete definition of the global Kaluza-Cartan space-time is given later in postulates K1-K6.

Definition 3.1.1: The Kaluza atlas.

The *Kaluza atlas* is a subatlas of the maximal atlas of Kaluza-Cartan space-time where boosts and rotations into the Kaluza dimension (as defined by the new cylinder condition) are explicitly omitted.

(3.1.1)

Mathematically this is also an atlas of charts for which the partial derivatives of tensors and pseudo-tensors in the Kaluza dimension vanish.

It can be used to decompose the entire 5D geometry into a 4D metric, a vector potential and a scalar field when curvatures are not so extreme as to lead to a singularity in the 4D metric. This represents the physical interpretation of charge as a covariant property of space-time even if it is not a covariant property of the 5D Kaluza-Cartan space. It can be given geometrized units when interpreted physically.

3.2 Kinetic Toth Charge

Kinetic Toth charge is defined as the 5D momentum component in terms of the 5D Kaluza rest mass of a hypothesised particle: ie (i) its rest mass in the 5D Lorentz manifold (m_{k0}) and (ii) its proper Kaluza velocity ($dx_4/d\tau^*$) with respect to a frame in the maximal atlas that follows the particle. And equally it can be defined in terms of (i) the relativistic rest mass (m_0), relative to a projected frame where the particle is stationary in space-time, but where non-charged particles are stationary in the Kaluza dimension, and in terms of (ii) coordinate Kaluza velocity (dx_4/dt_0):

Definition 3.2.1: Toth charge (scalar).

$$Q^* = m_{k0} dx_4/d\tau^* = m_0 dx_4/dt_0 \tag{3.2.1}$$

This makes sense because mass can be written in fundamental units (i.e. in distance and time). And the velocities in question defined relative to particular frames. It is not a generally covariant definition but it is nevertheless mathematically meaningful. In the Appendix (9.6.1) it is shown that this kinetic Toth charge can be treated in 4D Space-Time, and the Kaluza atlas, as a scalar: the first equation above is covariant with respect to the Kaluza atlas. It can be generalized to a 4-vector as follows, and it is also conserved:

In general relativity at the special relativistic Minkowski limit the conservation of momentum-energy/stress-energy can be given in terms of the stress-energy tensor as follows [9]:

$$\frac{\partial \hat{T}^{00}}{\partial t} + \frac{\partial \hat{T}^{i0}}{\partial x_i} = 0 \tag{3.2.2}$$

Momentum in the j direction:

$$\frac{\partial \hat{T}^{0j}}{\partial t} + \frac{\partial \hat{T}^{ij}}{\partial x_i} = 0 \tag{3.2.3}$$

This is approximately true at a weak field limit and can be applied equally to Kaluza theory, in the absence of torsion. We have a description of conservation of momentum in the 5th dimension as follows:

$$\frac{\partial \hat{T}^{04}}{\partial t} + \frac{\partial \hat{T}^{i4}}{\partial x_i} = 0 \tag{3.2.4}$$

We also have i=4 vanishing by KCC. Thus the conservation of Toth charge becomes (when generalized to different space-time frames) the property of a 4-vector current, which we know to be conserved:

$$(\hat{T}^{04}, \hat{T}^{14}, \hat{T}^{24}, \hat{T}^{34}) \tag{3.2.5}$$

$$\partial_0 \hat{T}^{04} + \partial_1 \hat{T}^{14} + \partial_2 \hat{T}^{24} + \partial_3 \hat{T}^{34} = 0 \tag{3.2.6}$$

As in relativity this can be extended to a definition that is valid even when there is curvature. Nevertheless the original Toth charge definition (3.2.1) has meaning in all Kaluza atlas frames as a scalar.

Kinetic Toth charge current is the 4-vector, induced from 5D Kaluza-Cartan space as follows (using the Kaluza atlas to ensure it is well-defined as a 4-vector):

$$J^{*a} = -\alpha \hat{G}^{a4} \tag{3.2.7}$$

Noting that,

$$\hat{\nabla}_A \hat{G}^{AB} = \hat{\nabla}_a \hat{G}^{a4} = 0 \tag{3.2.8}$$

Using Wheeler et al [6] p.131, and selecting the correct space-time (or Kaluza atlas) frame, we have:

$$Q^* = J^*_a (1, 0, 0, 0)^a \tag{3.2.9}$$

So we have a scalar, then a vector representation of relativisitic invariant charge current, and finally a 2-tensor unification with mass-energy.

When torsion is introduced this will not prevent the Levi-Civita connection having meaning and application on the same manifold contemporaneously. Kinetic Toth charge is defined in the same way even when torsion is present - via the Einstein tensor without torsion, and applying conservation of mass-energy relative to the torsionless connection. The new cylinder condition is defined, on the other hand, using the *torsion* connection. Distinction is therefore necessary between connections. So the 5D geoemtry depends on the torsion connection, but here a conservation still depends on the Levi-Civita connection.

The definition of kinetic charge and the conservation law of mass-energy-charge need to be written using the appropriate notation when torsion is used on the same manifold:

Definition 3.2.10: Toth charge current.

Toth charge current is defined to be the 4-vector $J^{*a} = -\alpha \hat{\mathcal{G}}^{a4}$, with respect to the Kaluza atlas, and noting:

$$\hat{\triangle}_A \hat{\mathcal{G}}^{AB} = 0 \tag{3.2.10}$$

3.3 Two Types Of Geometrized Charge

The metric components used in [1] as the 5D Kaluza metric, defined in terms of the original KCC follow. It will be equally used here in its new context, where the geometry of this space will depend on the new cylinder condition defined in terms of a metric torsion tensor. It is called here the Kaluza-Cartan metric to remind us of this context. The vector potential and electromagnetic fields formed via the metric are sourced in Maxwell charge Q_M.

Definition 3.3.1: The 5D Kaluza-Cartan metric.

$$\hat{g}_{AB} = \begin{bmatrix} g_{ab} + k^2\phi^2 A_a A_b & k\phi^2 A_a \\ k\phi^2 A_b & \phi^2 \end{bmatrix} \tag{3.3.1}$$

This gives inverse as follows:

$$\hat{g}^{AB} = |\hat{g}_{AB}|^{-1} = \begin{bmatrix} g^{ab} & -kA^a \\ -kA^b & \frac{1}{\phi^2} + k^2 A_i A^i \end{bmatrix} \tag{3.3.2}$$

This gives (with respect to space-time) perpendicular solutions [1] under the original KCC, such that $G_{ab} = -\frac{k^2}{2}\ F_{ac}F^c_b$. Compare this with [7] where we have $G_{ab} = 2F_{ac}F^c_b$ in geometrized units we would need to have $k = 2$ or $k = -2$ for compatibility of results and formulas. Noting the sign change introduced by [1] - where it appears that the Einstein tensor was defined relative to (+, −, −, −), despite the 5D metric tensor being given in a form that can only be (−, +, +, +, +), which is confusing. Approximately the same result, but with consistent sign conventions, is achieved here in (5.2.3).

The geometrized units, Wald [7], give a relation for mass in terms of fundamental units. This leads to an expression for Toth charge in terms of Kaluza momentum when $k = 2$ and $\mathbf{G} = 1$. $\mathbf{G}$ and k are not independent however. If we fix one the other is fixed too, as a consequence of requiring the Lorentz force law written in familiar form. The relation between $\mathbf{G}$ and k is given in equation (6.5.5). Simple compatibility with Wald [7], where $k = 2$ and $\mathbf{G} = 1$, results however. The sign of k is also fixed by (6.1.4). The result as given in the Appendix, written in terms of the Toth charge Q^*, is:

$$Q^* = \frac{c}{\sqrt{\mathbf{G}}} P_4 \tag{3.3.3}$$

Generally speaking the approach here will be to do the calculations using $k = 1$ and then add in the general k term later, as and when needed, simply to ease calculation.

An important part of this theory is the nature of the relationship between these two types of charge: Q^* and Q_M - to be dealt with later.

3.4 Consistency With Special Relativity

Toth charge is identified with 5D momentum in a space-time rest frame. This is already known in the original Kaluza theory to obey a Lorentz-like force law, but will be extended here in scope.

That this is consistent with special relativity can be investigated. What this consistency means is that the relativistic mass created by momentum in the 5th dimension is kinematically identical to the relativistic rest mass.

The additions of velocities in special relativity is not obvious. Assume a flat 5D Kaluza space (i.e without geometric curvature or torsion, thus analogously to special relativity at a flat space-time limit, a 5D Minkowski limit). Space-time can be viewed as a 4D slice (or series of parallel slices) perpendicular to the 5th Kaluza dimension that minimizes the length of any loops that are perpendicular to it. Taking a particle and an inertial frame, the relativistic rest frame where the particle is stationary with respect to space-time but moving with velocity u in the 5th dimension, and a second frame where the charge is now moving in space-time at velocity v, but still with velocity u in the 5th dimension, then the total speed squared of the particle in the second frame is according to relativistic addition of orthogonal velocities:

$$s^2 = u^2 + v^2 - u^2v^2 \tag{3.4.1}$$

The particle moving in the Kaluza dimension with velocity u, but stationary with respect to 4D space-time, will have a special relativistic 4D rest mass (m_0) normally greater than its 5D Kaluza rest mass (m_{k0}). We can see that the Kaluza rest mass definition (m_{k0}) is consistent with the orthogonal addition of velocities as follows:

$$m_0 = \frac{m_{k0}}{\sqrt{(1-u^2)}} \text{ where } u = \tanh[\sinh^{-1}(Q^*/(m_{k0}))] \tag{3.4.2}$$

$$m_{rel} = \frac{m_0}{\sqrt{(1-v^2)}} = \frac{m_{k0}}{\sqrt{(1-u^2)}} \times \frac{1}{\sqrt{(1-v^2)}} = \frac{m_{k0}}{\sqrt{(1-u^2-v^2+u^2v^2)}} \tag{3.4.3}$$

By putting $u = \tanh[\sinh^{-1}(Q^*/(m_{k0}))]$ (keeping the hyperbolics to recall the conversion between unidirectional proper and coordinate velocities) into the definition of relativistic rest mass in terms of Kaluza rest mass and solving, we get that charge, whether positive or negative, is related to the relativistic rest mass according to the following formula:

$$\cosh[\sinh^{-1}(Q^*/(m_{k0}))] = m_0/m_{k0} = \frac{dt_0}{d\tau^*}$$
$$= \sqrt{(Q^*/(m_{k0}))^2 + 1} \tag{3.4.4}$$

Using $k = 2$ we also have, for a typical unit charge:

$$m_e = 9.1094 \times 10^{-28} g \tag{3.4.5}$$

$$Q^* = 4.8032 \times 10^{-10} statcoulomb = 4.8032 \times 3.87 \times 10^{-10+3} g = 1.859 \times 10^{-6} g \tag{3.4.6}$$

If we take these figures and equate $m_e = m_0$ then we end up with imaginary m_{k0} and imaginary proper Kaluza velocity. Obviously to detail this the Kaluza-Cartan space-time would have to be adapted further in some way. But on the other hand it causes no causality problems provided the net result is compliant with any energy conditions being applied. And what is important in this respect is that the figures we know to be physical in 4D remain so.

Further issues are pertinent.

Observed electrons have static charge, angular momentum, a magnetic moment, and a flavor. The only thing distinguishing the electron from the muon is the flavor. The mass difference between the muon and the electron is about 105 MeV, perhaps solely due to this difference in flavor. The issue of modeling particles within a classical theory is, not surprisingly, a difficult one! Thus at this stage the idealized hypothetical charges used here, and real particles, can only be tentatively correlated.

It is possible to proceed without concern for the foundational issues of such charge models or attempting to interpret this quandary, instead simply developing the mathematics as is and seeing where it leads without judging it a priori.

3.5 Matter And Charge Models, A Disclaimer

This theory assumes some sort of particle model of matter and charge is possible, that it can be added to the original theory without significantly changing the ambient space-time solution and thus its own path, which is approximate here as it is also in general relativity. Here however there are more complications such as the lack of an explicit matter-charge model, and the presence of torsion. Secondly we might imagine that what has been described is a particle whizzing around the fifth dimension like a roller coaster on its spiralled tracks. The cylinder conditions can in fact be maintained if, instead of a 5D particle, the matter and charge sources were rather a ‘solid’ ring, locked into place around the 5th dimension, rotating at some predetermined proper Kaluza velocity (albeit imaginary). An exact solution could even involve changes in the size of the 5th dimension. None of that is investigated here, the aim was originally just to see whether non-perpendicular solutions can be found in a variant Kaluza theory.

In Einstein-Cartan theory geodesics, or extremals, are followed by spinless particles in 4D Einstein-Cartan theory [11]. Other particles follow different paths when interaction with torsion is present. Auto-parallels and extremals are two analogs of geodesics used when torsion is present, but neither of which in the most general case determine the paths followed by all particles. Note that spinless particles according to [11] follow extremals. Extremals coincide with auto-parallels when torsion is completely antisymmetric. Particles with spin may interact in other ways. So the assumption is that torsion-spin coupling does not significantly effect the path of the particle, at least to some approximation. The approximate closeness of these two geodesic analogs, and variants, is supported later by limit postulate L3. For this reason we switch here from the more obvious use of extremals to auto-parallels. Exactly how sensitive this assumption is requires further research. Here however it is packaged into an assumption.

An exact differential geometrical model of a matter and charge source is presumed too difficult to produce here, even if possible, especially given the previous discussion about imaginary masses and velocities. In addition, the fact that real charge sources are quantum mechanical may also be discouraging, though a classical limit interpretation should be possible regardless. The philosophy here has not been to provide a Lagrangian for a hypothesised charge model, but instead to delimit what may constitute such models.

The following assumption summarizes the preceding:

Geodesic Assumption: That any particle-like matter and charge model approximately follow auto-parallels.

Further we shall introduce various definitions that delimit both matter regions and vacuum regions called 'matter model regimes'. Different matter model regimes are not necessarily mutually compatible. That the vacuum consists of Ricci flat regions with respect to the Levi-Civita connection is the matter model regime of general relativity (we might further add a chosen energy condition to the matter model regime as required). This works well as the divergence of the Einstein tensor (without torsion) is zero, and the Einstein tensor (without torsion) is zero in vacuum as defined within general relativity. Thus we can say that the vacuum (within that conception of matter and vacuum) has some sort of 'integrity' in that they are well-defined and comply with observation, having both special relativistic and Newtonian limits. Alternatives are possible, and Kaluza's original theory without torsion requires that 5D (Kaluza space) Ricci flatness implies 4D (space-time) electromagnetic fields - it is interesting that a so-called 'vacuum' in one situation can be interpreted as having 'matter' or energy in another. The definitions cease to follow the basic intuition of general relativity, and a formal regime, or the possibility to define different regimes and give these or similar words precise meaning in any given context is needed.

To show the potential of Kaluza-Cartan theory (or theories) does not require that the 'best' matter model regime be identified, but only that reasonable matter model regimes can be presented that make sense of both the theory and a wide range of phenomena.

3.6 Field Equations And Torsion

The detailed formulas for dealing with Christoffel symbols, torsion and other mathematical necessities are in the Appendix. Exploration of various new field equations is undertaken in the sequel given the new cylinder condition which is very tight and the extra degrees of freedom given by torsion.

Torsion is necessary to free up degrees of freedom after they have been reduced substantially by the new cylinder condition.

4 Postulates Of The Kaluza-Cartan Theory

In this section an axiomatization of the theory in terms of postulates, assumptions, notes and other definitions is presented. K1-K6 constitute the core of the theory proper. A further postulate B1 is also needed, and a way to deal with weak field limits given by L1-L2, and L3 having its application as required. The matter model regimes then constitute interpretations of matter, charge and spin in the theory, with specific matter models not the subject of investigation.

4.1 Kaluza-Cartan Space-Time Definition

A definition of Kaluza-Cartan Space-Time, or Kaluza-Cartan Space, follows. K3 is the new cylinder condition:

Core Definitions and Postulates:

POSTULATE (K1): A Kaluza-Cartan manifold is a 5D smooth Lorentzian manifold.

POSTULATE (K2): There is a connection that is a metric torsion connection with respect to the geometry, this is the torsion connection. Metric here means the compatibility condition that the covariant derivative (with torsion) of the metric tensor is vanishing.

POSTULATE (K3): One spatial dimension is topologically closed and 'small', the Kaluza dimension. There is a global unit vector that defines this direction and approximately forms closed non-intersecting loops around the Kaluza direction. This is given mathematical meaning as follows: The covariant derivative (with respect to the torsion tensor) of all tensors and pseudotensors in the Kaluza direction are zero. This is the covariant derivative with lower index: $\hat{\nabla}_4$. This is the *new cylinder condition* .

POSTULATE (K4): The other spatial dimensions and time dimension are 'large'. 'large' here simply means that the considerations given to 'small' in K3 does not apply.

POSTULATE (K5): Kaluza-Cartan space is assumed globally hyperbolic in the sense that there exists a 3D spatial cauchy surface plus time, extended in the obvious way via the new cylinder condition into 5D (see K3).

POSTULATE (K6): Kaluza-Cartan space is oriented.

The new cylinder condition given here by K3 determines that local charts are possible with vanishing partial derivatives in the Kaluza direction for all tensors and that a Kaluza atlas (3.1.1) is possible. That is, partial derivatives with lower index ∂_4 are all zero. What we can do now is take a loop by following the Kaluza direction and forming a 1-dimensional submanifold for every point of space-time approaching the limit defined by K3. By inspecting the bundles inherited by each loop submanifold we can observe that at every point they are necessarily static. As a result ∂^4 and $\hat{\nabla}^4$ must be vanishing for all tensors.

4.2 The Scalar Field

In the sequel (equation 6.4.1) it will be shown that restricting the scalar field and accordingly the metric, maximal atlas and Kaluza atlas, is necessary, and as follows:

PROVISIONAL POSTULATE (B1): $\phi^2 = 1$

The use of B1 will be minimized so as to make the need for its application clearer. It has been separated from the main list of postulates as potentially a limit.

The scalar field results from the the decomposition of the Kaluza metric (see Note S3). It is contained within the metric explicitly in (3.3.1). Thus (B1) is a constraint on the 5D metric.

4.3 Weak Field Limits

The following additional postulates L1-L3 constitute a *weak field limit* that will be applied by way of approximation for the 'classical' limit of behaviour:

A weak field limit for the metric and curvatures will be assumed. The torsionless metric will be approximately flat, yet some curvature will be required as with general relativity - essentially a weak field limit. The diagonal metric components will approximate 1 or -1, and other components 0. The deviation from the 5D-Minkowski metric is given by a tensor $\hat{h}_{AB}$. This tensor belongs to a set of small tensors that we might label $O(h)$. Whilst this uses a notation similar to orders of magnitude, and is indeed analogous, the meaning here is a little different. This is the weak field approximation of general relativity using a more flexible notation. Partial derivatives, to whatever order, of metric of terms in a particular set $O(x)$ will be in that same set at the weak field limit.

In the weak field approximation of general relativity, terms that consist of two $O(h)$ terms multiplied together get discounted and are treated as vanishing at the limit. We might use the notation $O(h^2)$ to signify such terms. There is the weak field approximation given by discounting $O(h^2)$ terms. But we might also have a less aggressive limit given by, say, discounting $O(h^3)$ terms, and so on and so forth. We can talk about weak field limits (plural) that discount

$O(h^n)$ terms for $n > 1$, but they are based on the same underlying construction. Following the weak field limit [6] of general relativity we have:

LIMIT POSTULATE (L1): The metric can be written as follows in terms of the 5D Minkowski tensor and $\hat{h} \in O(h)$:

$$\hat{g}_{AB} = \hat{\mu}_{AB} + \hat{h}_{AB}$$

This is the same method as the weak field approximation in general relativity.

Torsion will also be considered a weak field under normal observational conditions, similarly to L1. Torsion is defined in terms of the Christoffel symbols. Christoffel symbols are in part constructed from the partial derivatives of the metric and that part is constrained by L1 to be $O(h)$. The contorsion term being the difference. See eqn(9.2.3). The contorsion (and therefore the torsion) will treated as $O(h)$ accordingly.

LIMIT POSTULATE (L2): The contorsion (and therefore the torsion) will be an $O(h)$ term at the weak field limits.

One further constraint is required at the weak field limit. Its use will be minimized (both the application of the antisymmetry and the allowance for some small symmetry terms) so that its use demonstrates its sphere of application (and limitations) more clearly. In L3, symmetric parts of the torsion and contorsion tensor (and their derivatives) are treated as particularly 'small' in that they are small relative to any antisymmetric parts of the torsion and contorsion tensor, torsion already assigned to $O(h)$ by L2.

The torsion tensor will be given the following limit: It is to be weakly (completely) antisymmetric - *a weak antisymmetric limit*, this will be so even with respect to L1 and L2. Thus the symmetric parts of the contorsion and torsion tensors will be $O(h^2)$ at the weak field limit. All derivatives thereof follow the same rule.

LIMIT POSTULATE (L3): The symmetric parts of the contorsion and torsion tensors will be $O(h^2)$ at the weak field limits.

It is claimed that such a limit may be approached without loss of generality of the solutions from a physical perspective. In other words at the L1-L2 weak field limit equation (5.1.9) is compatible with the weak antisymmetric limit L3, and poses no constraint due to the product of the potential and field also being discounted at the weak field limit via L1. We may approximately apply certain results associated with complete antisymmetry at this L3 limit, noting that such results must be taken to be approached via the limit and that some contorsion symmetry components (analogous how curvature is needed but can still have a weak field limit) are in fact required for example by equation (5.1.9).

As the failure of B1 may indicate the presence of a scalar field, and disruption to classical electrodynamics, likewise symmetric torsion and contorsion components of any greater significance than that prescribed here by L3 would likewise constitute a torsion field of experimental significance and imply alteration of the usual dynamics. This situation will be relevant in the sequel as the limits will be seen to have some disadvantages. These limits are therefore not written in stone.

4.4 Further Definitions And Notes

Some useful observations, definitions and terminology that will be needed later or are useful to bear in mind:

NOTE (S1): The *Kaluza-Cartan vacuum* is a Ricci flat region of a Kaluza-Cartan manifold with respect to the torsion connection definition of the Ricci tensor. Similarly the *Kaluza vacuum* is a Ricci flat region with respect to the Levi-Civita connection. They are different: $\hat{R}_{ab} = 0$ and $\hat{\mathcal{R}}_{ab} = 0$ respectively. Here they are both defined in terms of the geometry implied by the new cylinder condition. There is also the Kaluza space (and hence Kaluza vacuum) of the original Kaluza theory. There ceases to be a single definition of vacuum. The Kaluza vacuum in the presence of torsion typically contains fields when the 4D metric is inspected, and will most often not be Ricci flat in 4D space-time. Likewise for the Kaluza vacuum under the original KCC, and for the Kaluza-Cartan vacuum under the new cylinder condition. Kaluza vacuum and Kaluza-Cartan vacuums may be different.

NOTE (S2): We will define *matter model regimes* and attempt to pick out the definitions we wish to use and that delimit matter and charge models. Note that the cosmological constant of general relativity could perhaps be included as part of an alternative regime, although it is not used here. Energy conditions may potentially also be included. There is scope for alternatives. The objective here is to show the potential of Kaluza-Cartan theories, not to give the last word. A range of global constraints could be experimented with, and applied to matter models. The aim here is to keep the options as broad as possible. Assumptions pertaining to matter models are therefore defined separately within matter model regimes.

NOTE (S3): K3 can be used to decompose the entire 5D geometry into a 4D metric using the Kaluza atlas, a vector potential and a scalar field when curvatures are not so extreme as to lead to a singularity in the 4D metric. It defines how space-time is a parallel set of submanifolds. Singularities resulting from break-down of the decomposition will be regions where the theory as presented also breaks down. This could however also be useful in extending the theory towards the quantum scale. This is not dealt with further.

NOTE (S4): What is allowed as a physical solution needs to be delimited in some way, at the very least to avoid acausality as in classical physics. This can be done by using energy conditions as in relativity for the resultant 4D space-time, at least as long as the decomposition mentioned in S3 doesn't break-down.

The cylinder condition can then extrapolate that to the whole of the Kaluza-Cartan manifold. Here this problem will be ignored and it is assumed that it can ultimately be subsumed into an experimentally correct matter model regime.

NOTE (S5): Note that in the general case with torsion, whilst every tensor and pseudo-tensor in sight has covariant derivative in the Kaluza direction of zero with respect to the torsion connection, and similarly partial derivatives likewise, the covariant derivative with respect to the non-torsion derivative is not so constrained. Thus Kaluza-Cartan theory ceases to obey the original Kaluza cylinder condition.

NOTE (S6): Matter and charge models, and matter model regimes, must also be consistent with K3. A realistic elementary charge model would have imaginary Kaluza rest mass and Kaluza velocity, but this does not prevent it satisfying the new cylinder condition.

NOTE (S7): The new cylinder condition is defined in terms of covariant derivatives instead of the usual frame dependent partial derivatives, and then a restricted Kaluza subatlas is used in which the partial derivatives are also zero. In particular it takes account of torsion.

4.5 Matter And Charge Models

Some notes on matter and charge models:

MATTER (M1): Matter and charge models must be consistent with the preceding postulates, at least at any classical limit.

MATTER (M2): Simple examples could include black-hole singularities, though these are not good examples for the general case.

MATTER (M3): Net positivity of mass and energy may be imposed by the usual energy conditions or similar extensions in 5D as required. This may be included in the matter model regime definition, or in a particular matter model.

MATTER (M4): No comment is passed on the fact that the proportions of mass and charge of a realistic elementary charge model yield imaginary Kaluza rest mass and Kaluza proper velocity - this paper proceeds without judging this peculiar result.

MATTER (M5): There is a need to make an assumption about the paths of particles (ie 'small' matter and charge models): the Geodesic Assumption. This states that particle-like solutions are possible and follow 5D auto-parallels. I.e. geodesics when torsion is not present. This should in any case be the case for spinless particles, as seen in 4D Einstein-Cartan theory. As applied to particles with spin we are making an assumption about spin-torsion coupling that may well be incorrect, though likely correct at a limit. This is further motivated by the local L3 limit, as in 4D Einstein-Cartan theory the difference lies in non-completely antisymmetric torsion.

MATTER (M6): Quantization of charge is not dealt with - no less than quantization of energy or momentum. The proper place for this is a quantum theory, or a theory that encompasses the quantum. This is not the purpose of this paper.

MATTER (M7): The exact definition of matter and charge in 5D, matter-charge models, may be different from the related concept in general relativity, and, as with concepts of vacuum, will depend on the matter model regimes selected.

MATTER (M8): That L1-L3 are limit postulates, not postulates written in stone, plays a role in the sequel.

4.6 Matter Model Regimes

This sections defines the different matter model regimes used in this work. These represent various interpretations of what may constitute a matter model in the broadest sense to be analysed. What we are really concerned about here is what constitutes matter (as opposed to energy or vacuum). It is a choice as to which are being used for any particular theory. Instead of the confusing (in this context) term 'vacuum', we will use here 'non-matter' as being here complementary to matter. 'Kaluza' and 'Kaluza-Cartan vacuums' will keep their meanings, however, as specific to different Ricci tensors. The clearest definition would be to limit the unqualified term 'vacuum' to the 4D torsionless space-time version.

REGIME (R0): The basic matter model regime defines matter-charge models as any part of the Kaluza-Cartan manifold that are not Kaluza vacuums (according to definition S1) - analogously to Kaluza's original theory. Although it is not defined in terms of torsion it lives in the presence of torsion with respect to the new cylinder condition. Non-matter is defined as $\hat{\mathcal{R}}_{AB} = 0$. This implies $\hat{\mathcal{G}}_{AB} = 0$. This has the advantage of a ready-made conservation law for the Einstein (without torsion) tensor. This non-matter is by definition free of Toth charge (3.2.1) and is shown to be free of local Maxwellian charge in (5.3.1).

REGIME (R1): The next most obvious definition of matter-charge models would be to define them as any regions that are not Kaluza-Cartan vacuums (according to definition S1). Non-matter is defined as $\hat{R}_{AB} = 0$. This implies in non-matter that $\hat{G}_{AB} = 0$. Matter models are then the complement of this instead. The conservation law of the torsionless Einstein tensor does not apply to the Einstein tensor with torsion however.

R0 and R1 provide the two most obvious definitions of non-matter. R0 will be shown to be too constraining. R1 will be claimed to be at least an example not so constraining. That R1's Kaluza-Cartan vacuum is also free of local Maxwell charge at the $O(h^2)$ L1-L3 limit is also shown (5.3.5).

REGIME (R2): This matter model regime proposes the alternative idea that instead of looking for matter or charge models, we might analogously look for spin current model. Non-spin regions are given by: $\hat{V}_{AB} = 0$, see (9.3.2). Spin current models are defined as the complement of this.

4.7 An Apology

Admittedly (M4) through (M7) and the complexity of the consideration of the different matter model regimes show that this theory is in an early state of development. Many theories pass through such a state, and complete realization from the outset can not be expected. An objective in the foregoing has been to present the postulates that are needed for the theory to be well-defined, or have well-defined parameters, even if incomplete. A further problem is that realistic matter models for charged particles imply imaginary proper velocities and Kaluza rest masses.

Further and better matter model regimes than those given here may well be possible. For example when dealing with energy conditions and causality, or even the cosmological constant.

Ultimately empirical observation, above and beyond the minimum requirements of consistency, determines validity.

5 The Field Equations

5.1 The Cylinder Condition And Scalar Field

Here we look at how K3 affects the Christoffel symbols of any coordinate system within the Kaluza atlas (using $k = 1$). Given that the Kaluza atlas is reduced in the following way: all partial derivatives of tensors in the Kaluza direction are set to zero. The Appendix (see section containing 9.4.1 and related) contains a reference for Christoffel symbol working both with and without the torsion component.

The following follow from: the selection of coordinates (the Kaluza atlas) that set the partial derivatives in the Kaluza dimension to zero; from K3, the new covariant version of the cylinder condition; and, from the relationship between these two and the Christoffel symbols given in Wald [7] p33 eqn (3.1.14) as applied to a number of test vectors. Note that there is no hint of symmetry of the (with torsion) Christoffel symbols suggested here. That is, these terms are forced zero by the fact that both the partial derivatives and the covariant derivatives in the Kaluza direction are zero. Cf equation (1.0.4), where the consequences of setting both the partial derivatives and the covariant derivative to zero can be seen on the Christoffel symbols.

$$0 = 2\hat{\Gamma}^A_{4c} = \sum_d \hat{g}^{Ad}(\partial_4 g_{cd} + \partial_4\phi^2 A_c A_d + \partial_c\phi^2 A_d - \partial_d\phi^2 A_c) + \hat{g}^{A4}\partial_c\phi^2 - 2\hat{K}_{4c}{}^{A} \tag{5.1.1}$$

$$0 = 2\hat{\Gamma}^A_{44} = 2\sum_d \hat{g}^{Ad}\partial_4\phi^2 A_d - \sum_d \hat{g}^{Ad}\partial_d\phi^2 + \hat{g}^{A4}\partial_4\phi^2 - 2\hat{K}_{44}{}^{A} \tag{5.1.2}$$

We have:

$$2\hat{K}_{4c}{}^{A} = \hat{g}^{Ad}(\partial_c\phi^2 A_d - \partial_d\phi^2 A_c) + \hat{g}^{A4}\partial_c\phi^2 \quad (5.1.3)$$

$$2\hat{K}_{44}{}^{A} = -\hat{g}^{Ad}\partial_d\phi^2 \quad (5.1.4)$$

Inspecting the first of these equations (5.1.1), and given that $K_{A(BC)} = 0$ (9.2.4), and further applying A=c without summing, we have a constraint on the scalar field in terms of the vector potential that further motivates postulate B1. Here, however, we make a priori use of postulate B1.

The immediate result of this is as follows (using $k = 1$):

$$2\hat{K}_{4c}{}^{A} = \hat{g}^{Ad}(\partial_c A_d - \partial_d A_c) \quad (5.1.5)$$

$$2\hat{K}_{44}{}^{A} = 0 \quad (5.1.6)$$

This gives the contorsion a very clear interpretation in terms of the electromagnetic field.

$$\hat{K}_{4c}{}^{a} = \frac{1}{2}F^a_c \quad (5.1.7)$$

$$\hat{K}_{4c}{}^{4} = -\frac{1}{2}A^d F_{cd} \quad (5.1.8)$$

In the case of complete antisymmetry of torsion/contorsion, again using (5.1.1), this specialises to:

$$\hat{K}_{4c}{}^{4} = \hat{K}_{c4}{}^{4} = \hat{\Gamma}^4_{c4} = \hat{\Gamma}^4_{4c} = A^d F_{cd} = 0 \quad (5.1.9)$$

This presents too tight a constraint on electromagnetism as it employs more degrees of freedom than any gauge conditions. For this reason non-completely antisymmetric torsion is allowed, yet constrained at the weak field limits by L3. In the general case:

$$\begin{aligned} \hat{K}_{[4c]}{}^{4} + \hat{K}_{(4c)}{}^{4} = \hat{K}_{4c}{}^{4} &= -\frac{1}{2}A^d F_{cd} \\ \hat{K}_{c4}{}^{4} = 0 &= -\hat{K}_{[4c]}{}^{4} + \hat{K}_{(c4)}{}^{4} \end{aligned} \quad (5.1.10)$$

$$\begin{aligned} \hat{K}_{[4c]}{}^{4} = \hat{K}_{(4c)}{}^{4} = \frac{1}{2}\hat{K}_{4c}{}^{4} &= -\frac{1}{4}A^d F_{cd} \\ \hat{K}_{c4}{}^{4} = \hat{\Gamma}^4_{4c} &= 0 \\ \hat{K}_{4c}{}^{4} = \hat{\Gamma}^4_{c4} &= -\frac{1}{2}A^d F_{cd} \end{aligned} \quad (5.1.11)$$

We can see from this section the limits within which postulate L3 must be employed.

5.2 The First Field Equation With Torsion, $k = 1$

The first field equation in this theory is somewhat complicated (5.2.3), but an analysis here will show that Kaluza-Cartan theory and the original Kaluza theory share a limit for certain perpendicular solutions. This analysis also investigates the comparative merits of R0 and R1 matter model regimes.

Looking at the Ricci tensor, but here with torsion (using equations 9.4.7 repeatedly, and the new cylinder condition as required):

$$\begin{aligned}
\hat{R}_{ab} &= \partial_C \hat{\Gamma}^C_{ba} - \partial_b \hat{\Gamma}^C_{Ca} + \hat{\Gamma}^C_{ba}\hat{\Gamma}^D_{DC} - \hat{\Gamma}^C_{Da}\hat{\Gamma}^D_{bC} \\
\hat{R}_{ab} &= \partial_c \hat{\Gamma}^c_{ba} - \partial_b \hat{\Gamma}^c_{ca} + \hat{\Gamma}^C_{ba}\hat{\Gamma}^D_{DC} - \hat{\Gamma}^C_{Da}\hat{\Gamma}^D_{bC} \\
\hat{R}_{ab} &= \partial_c \hat{\Gamma}^c_{ba} - \partial_b \hat{\Gamma}^c_{ca} + \hat{\Gamma}^C_{ba}\hat{\Gamma}^d_{dC} - \hat{\Gamma}^C_{da}\hat{\Gamma}^d_{bC} \qquad (5.2.1)
\end{aligned}$$

Doing the same for the without torsion definitions (using equations 9.4.6 repeatedly, and the new cylinder condition as required):

$$\begin{aligned}
\hat{\mathcal{R}}_{ab} &= \partial_C \hat{\mathit{\Gamma}}^C_{ba} - \partial_b \hat{\mathit{\Gamma}}^C_{Ca} + \hat{\mathit{\Gamma}}^C_{ba}\hat{\mathit{\Gamma}}^D_{DC} - \hat{\mathit{\Gamma}}^C_{Da}\hat{\mathit{\Gamma}}^D_{bC} \\
\hat{\mathcal{R}}_{ab} &= \partial_c \hat{\mathit{\Gamma}}^c_{ba} - \partial_b \hat{\mathit{\Gamma}}^c_{ca} + \frac{1}{2}\partial_b(A^d F_{ad}) + \hat{\mathit{\Gamma}}^c_{ba}\hat{\mathit{\Gamma}}^D_{Dc} - \hat{\mathit{\Gamma}}^C_{Da}\hat{\mathit{\Gamma}}^D_{bC} \qquad (5.2.2)
\end{aligned}$$

In the original Kaluza theory the Ricci curvature of the 5D space is set to 0. The first field equation (2.2.1) comes from looking at the Ricci curvature of the space-time that results. Here there is a choice: whether to base the vacuum on a Kaluza vacuum, R0, or a Kaluza-Cartan vacuum, R1, or even something else altogether. Taking as a lead the concept of matter-charge models according to R0 with conservation law (3.2.10), the Kaluza vacuum is investigated first. Setting $\hat{\mathcal{R}}_{ab} = 0$ (as would be required for matter model regime R0):

$$\begin{aligned}
\mathcal{R}_{ab} &= \mathcal{R}_{ab} - \hat{\mathcal{R}}_{ab} \\
&= \partial_c \mathit{\Gamma}^c_{ba} - \partial_b \mathit{\Gamma}^c_{ca} - \partial_c \hat{\mathit{\Gamma}}^c_{ba} + \partial_b \hat{\mathit{\Gamma}}^c_{ca} - \frac{1}{2}\partial_b(A^d F_{ad}) + \frac{1}{2}\partial_b(A^d F_{ad} + A_a F_c{}^c) \\
&\quad + \mathit{\Gamma}^c_{ba}\mathit{\Gamma}^d_{dc} - \mathit{\Gamma}^c_{da}\mathit{\Gamma}^d_{bc} - \hat{\mathit{\Gamma}}^c_{ba}\hat{\mathit{\Gamma}}^D_{Dc} + \hat{\mathit{\Gamma}}^C_{Da}\hat{\mathit{\Gamma}}^D_{bC} \\
&= -\frac{1}{2}\partial_c(A_b F_a{}^c + A_a F_b{}^c) + \mathit{\Gamma}^c_{ba}\mathit{\Gamma}^d_{dc} - \mathit{\Gamma}^c_{da}\mathit{\Gamma}^d_{bc} - \hat{\mathit{\Gamma}}^c_{ba}\hat{\mathit{\Gamma}}^D_{Dc} + \hat{\mathit{\Gamma}}^C_{Da}\hat{\mathit{\Gamma}}^D_{bC} \\
&= -\frac{1}{2}\partial_c(A_b F_a{}^c + A_a F_b{}^c) + \mathit{\Gamma}^c_{ba}\mathit{\Gamma}^d_{dc} - \mathit{\Gamma}^c_{da}\mathit{\Gamma}^d_{bc} \\
&\quad - \hat{\mathit{\Gamma}}^c_{ba}\hat{\mathit{\Gamma}}^D_{Dc} + \hat{\mathit{\Gamma}}^c_{Da}\hat{\mathit{\Gamma}}^D_{bc} + \hat{\mathit{\Gamma}}^4_{Da}\hat{\mathit{\Gamma}}^D_{b4} \\
&= -\frac{1}{2}\partial_c(A_b F_a{}^c + A_a F_b{}^c) + \mathit{\Gamma}^c_{ba}\mathit{\Gamma}^d_{dc} - \mathit{\Gamma}^c_{da}\mathit{\Gamma}^d_{bc} \\
&\quad - (\mathit{\Gamma}^c_{ba} + \frac{1}{2}(A_b F_a{}^c + A_a F_b{}^c))(\mathit{\Gamma}^d_{dc} + \frac{1}{2}(A_d F_c{}^d + A_c F_d{}^d)) - (\mathit{\Gamma}^c_{ba} + \frac{1}{2}(A_b F_a{}^c + A_a F_b{}^c))(-\frac{1}{2}A^d F_{cd}) \\
&\quad + (\mathit{\Gamma}^c_{da} + \frac{1}{2}(A_d F_a{}^c + A_a F_d{}^c))(\mathit{\Gamma}^d_{bc} + \frac{1}{2}(A_b F_c{}^d + A_c F_b{}^d)) + (\frac{1}{2}F_a{}^c)(-A_d \mathit{\Gamma}^d_{bc} + \frac{1}{2}(\partial_b A_c + \partial_c A_b))
\end{aligned}$$

$$
\begin{aligned}
&+(-A_c\Gamma^c_{\ da} + \frac{1}{2}(\partial_d A_a + \partial_a A_d))(\frac{1}{2}F_b{}^d) + (-\frac{1}{2}A^d F_{ad})(-\frac{1}{2}A^c F_{bc}) \\
&= -\frac{1}{2}\partial_c(A_b F_a{}^c + A_a F_b{}^c) \\
&-\frac{1}{2}(A_b F_a{}^c + A_a F_b{}^c)\Gamma^d_{\ dc} \\
&+\frac{1}{2}\Gamma^c_{\ da}(A_b F_c{}^d + A_c F_b{}^d) + \frac{1}{2}(A_d F_a{}^c + A_a F_b{}^c)\Gamma^d_{\ bc} + \frac{1}{2}(A_d F_a{}^c + A_a F_d{}^c)\frac{1}{2}(A_b F_c{}^d + A_c F_b{}^d) \\
&+\frac{1}{2}F_a{}^c(-A_d\Gamma^d_{\ bc} + \frac{1}{2}(\partial_b A_c + \partial_c A_b)) \\
&+(-A_c\Gamma^c_{\ da} + \frac{1}{2}(\partial_d A_a + \partial_a A_d))\frac{1}{2}F_b{}^d + \frac{1}{4}A^d F_{ad}A^c F_{bc} \\
&= -\frac{1}{2}A_b\partial_c F_a{}^c - \frac{1}{2}A_a\partial_c F_b{}^c - \frac{1}{2}(\partial_c A_b)F_a{}^c - \frac{1}{2}(\partial_c A_a)F_b{}^c - \frac{1}{2}(A_b F_a{}^c + A_a F_b{}^c)\Gamma^d_{\ dc} \\
&+\frac{1}{2}\Gamma^c_{\ da}A_b F_c{}^d + \frac{1}{2}A_a F_b{}^c\Gamma^d_{\ bc} + \frac{1}{4}(A_d F_a{}^c + A_a F_d{}^c)(A_b F_c{}^d + A_c F_b{}^d) \\
&+\frac{1}{4}F_a{}^c(\partial_b A_c + \partial_c A_b) + \frac{1}{4}(\partial_d A_a + \partial_a A_d)F_b{}^d + \frac{1}{4}A^d F_{ad}A^c F_{bc} \\
&= -\frac{1}{2}A_b\partial_c F_a{}^c - \frac{1}{2}A_a\partial_c F_b{}^c \\
&-\frac{1}{2}(\partial_c A_b)F_a{}^c - \frac{1}{2}(\partial_c A_a)F_b{}^c + \frac{1}{4}F_a{}^c(\partial_b A_c + \partial_c A_b) + \frac{1}{4}(\partial_d A_a + \partial_a A_d)F_b{}^d \\
&-\frac{1}{2}(A_b F_a{}^c + A_a F_b{}^c)\Gamma^d_{\ dc} + \frac{1}{2}\Gamma^c_{\ da}A_b F_c{}^d + \frac{1}{2}A_a F_b{}^c\Gamma^d_{\ bc} \\
&+\frac{1}{4}(A_d F_a{}^c + A_a F_d{}^c)(A_b F_c{}^d + A_c F_b{}^d) + \frac{1}{4}A^d F_{ad}A^c F_{bc} \\
&= -\frac{1}{2}A_b\partial_c F_a{}^c - \frac{1}{2}A_a\partial_c F_b{}^c + \frac{1}{2}F_{ac}F_b{}^c \\
&-\frac{1}{2}(A_b F_a{}^c + A_a F_b{}^c)\Gamma^d_{\ dc} + \frac{1}{2}\Gamma^c_{\ da}A_b F_c{}^d + \frac{1}{2}A_a F_b{}^c\Gamma^d_{\ bc} \\
&+\frac{1}{4}(A_d F_a{}^c + A_a F_d{}^c)(A_b F_c{}^d + A_c F_b{}^d) + \frac{1}{4}A^d F_{ad}A^c F_{bc} \qquad (5.2.3)
\end{aligned}
$$

The electrovacuum terms for a perpendicular electromagnetic field can be seen embedded in this equation as the third term, this shows that we are not producing a completely new theory from Kaluza's original theory. Kaluza-Cartan theory has a limit in common with Kaluza theory. Taking $O(h^3)$ L1-L3 weak field clarifies this. Only the first three terms (5.2.3) survive, of which the first two are charge terms and the latter is the stress-energy of perpendicular solutions. However, if the charge terms are ignored then there is a lack in the above equation of likely significant terms to provide any other type of solution, non-perpendicular electromagnetic fields in particular. It is therefore too restrictive when the scalar field is constant just like Kaluza's original theory.

For this reason we can try an alternative matter model regime such as R1 and the Kaluza-Cartan vacuum in order to obtain a fuller range of geometries.

Now, doing the same with respect to R1 so that we are defining a Kaluza-Cartan vacuum (S1) and matter models as strictly complement ($\hat{R}_{AB} = 0$) - and using (5.2.1) - gives:

$$\begin{aligned}\mathcal{R}_{ab} = \mathcal{R}_{ab} - \hat{R}_{ab} = \partial_c \varGamma^{c}_{\ ba} - \partial_b \varGamma^{c}_{\ ca} + \varGamma^{c}_{\ ba}\varGamma^{d}_{\ dc} - \varGamma^{c}_{\ da}\varGamma^{d}_{\ bc} \\ -\partial_c \hat{\Gamma}^{c}_{ba} + \partial_b \hat{\Gamma}^{c}_{ca} - \hat{\Gamma}^{C}_{ba}\hat{\Gamma}^{d}_{dC} + \hat{\Gamma}^{C}_{da}\hat{\Gamma}^{d}_{bC}\end{aligned} \tag{5.2.4}$$

Detailing each term here without a specific point to make is not profitable, is lengthy, and shall not be undertaken. There are more degrees of freedom than for R0 for electromagnetic fields.

There is a limit in common between R0 and R1 when there is no appreciable torsion (noting that perpendicular solutions under R0 need no such torsion). More generally allowing torsion terms allows for non-perpendicular electromagnetic fields under R1. Similarly other non-matter definitions than R0 or R1 are likely not to be as constrained as R0. However in all cases it is necessary to show that prospective electromagnetic solutions in any case obey the Lorentz force law as the general relativistic Einstein-Maxwell equation is probably not satisfied in the general case.

5.3 The Second Field Equation With Torsion

Rederivation of the second field equation under the new cylinder condition:

$$\begin{aligned}\hat{\mathcal{R}}_{a4} &= \partial_C \hat{\varGamma}^{C}_{\ 4a} - \partial_4 \hat{\varGamma}^{C}_{\ Ca} + \hat{\varGamma}^{C}_{\ 4a}\hat{\varGamma}^{D}_{\ DC} - \hat{\varGamma}^{C}_{\ Da}\hat{\varGamma}^{D}_{\ 4C} \\ &= \partial_c \hat{\varGamma}^{c}_{\ 4a} + \hat{\varGamma}^{c}_{\ 4a}\hat{\varGamma}^{D}_{\ Dc} - \hat{\varGamma}^{c}_{\ Da}\hat{\varGamma}^{D}_{\ 4c} = \partial_c \hat{\varGamma}^{c}_{\ 4a} + \hat{\varGamma}^{c}_{\ 4a}\hat{\varGamma}^{d}_{\ dc} - \hat{\varGamma}^{c}_{\ da}\hat{\varGamma}^{d}_{\ 4c} \\ &= \frac{1}{2}\partial_c F^c_a + \frac{1}{2}F^c_a \varGamma^{d}_{\ dc} + \frac{1}{4}F^c_a A^d F_{cd} - \frac{1}{2}(\varGamma^{c}_{\ da} + \frac{1}{2}(A_d F^c_a + A_a F^c_d))F^d_c\end{aligned}$$

Looking at this at an $O(h^2)$ L1-L3 weak field limit (re-inserting general k):

$$\hat{\mathcal{R}}_{a4} \to \frac{k}{2}\partial_c F^c_a \tag{5.3.1}$$

This couldn't be a clearer (albeit approximate at the $O(h^2)$ weak field limit) conception of Maxwell charge. This coincides with the Einstein (without torsion) tensor at the same limit providing an alternative conception of the conservation of Maxwell charge locally (cf 6.1.1):

$$\hat{\mathcal{G}}_{a4} \to \hat{\mathcal{R}}_{a4} \to \frac{k}{2}\partial_c F^c_a \tag{5.3.2}$$

On the other hand, by definition (and the new cylinder condition, and 9.4.7), we immediately get:

$$\hat{R}_{a4} = 0 \tag{5.3.3}$$

Whereas $\hat{R}_{4b}$ simplifies at the $O(h^2)$ weak field limit to:

$$\hat{R}_{4b} \to \frac{1}{2}\partial_c F^c_b - \partial_c \hat{K}_{b4}{}^c + \partial_b \hat{K}_{c4}{}^c \tag{5.3.4}$$

This is also approximately conserved Maxwell charge (re-inserting general k) given at the $O(h^2)$ L1-L3 weak field limit, explicitly using L3 and equation (5.1.7):

$$\hat{R}_{4b} \to k\partial_c F^c_b \tag{5.3.5}$$

This means that the Kaluza-Cartan vacuum may not have stray charges in it of any significance, which is a required quality of a pure electromagnetic field.

5.4 The Third Field Equation With Torsion, $k = 1$

This section shows how torsion releases the constraint of the third torsionless field equation (2.2.3), thus allowing non-perpendicular solutions. The constraint that the Ricci tensor be zero leads to no non-perpendicular solutions in the original Kaluza theory. This is caused by setting $\hat{R}_{44} = 0$ in that theory and observing the terms. The result is that (when the scalar field is constant) $0 = F_{cd}F^{cd}$ in the original Kaluza theory. The same issue arises here:

We have:

$$\begin{aligned}\hat{\mathcal{R}}_{44} &= \partial_C \hat{\digamma}^{C}_{44} - \partial_4 \hat{\digamma}^{C}_{C4} + \hat{\digamma}^{C}_{44}\hat{\digamma}^{D}_{DC} - \hat{\digamma}^{C}_{D4}\hat{\digamma}^{D}_{4C} \\ &= 0 - 0 + 0 - \hat{\digamma}^{C}_{D4}\hat{\digamma}^{D}_{4C} = -\hat{\digamma}^{c}_{d4}\hat{\digamma}^{d}_{4c} \\ &= -\frac{1}{4}F^c_d F^d_c \end{aligned} \tag{5.4.1}$$

The result is that whilst we can have non-perpendicular solutions, we can only have them outside of a Kaluza vacuum, for example in a Kaluza-Cartan vacuum.

By definition (and the new cylinder condition, and 9.4.7), we immediately get:

$$\hat{R}_{44} = 0 \tag{5.4.2}$$

Thus no such limit is placed on matter model regime R1. Similarly there is no reason to expect such constraints for any other conceptions of non-matter. There is no reason in general for equation (5.4.1) to be 0, and so non-perpendicular solutions are generally available.

6 The Lorentz Force Law

Toth [8] derives a Lorentz-like force law where there is a static scalar field and Kaluza's cylinder condition applies in the original Kaluza theory. The resulting 'charge' is the momentum term in the fifth dimension and it was not apparent how this related to the Maxwell current, except as Toth states via 'formal equivalence'. Toth's calculation is extended here and clarification obtained. Here we make use of the Geodesic Assumption M5. First the identification of Toth charge and Maxwell charge is investigated.

6.1 Toth Charge

Now to investigate the relationship between Toth charge and Maxwell charge. For this we need the $O(h^2)$ weak field limit defined by L1 (cf equation 5.3.2) and discounting $O(h^2)$ terms:

$$\hat{\mathcal{G}}^{a4} = \hat{\mathcal{R}}^{a4} - \frac{1}{2}\hat{g}^{a4}\hat{\mathcal{R}} = \hat{\mathcal{R}}^{a4} - \frac{1}{2}(-A^a)\hat{\mathcal{R}} \rightarrow \hat{\mathcal{R}}^{a4}$$

$$\hat{\mathcal{R}}^{a4} = \partial_C \hat{F}^{C4a} - \partial^4 \hat{F}^{C\ a}_{\ C} + \hat{F}^{Cba}\hat{F}^{D}_{\ DC} - \hat{F}^{C\ a}_{\ D}\hat{F}^{Db}_{\ \ C}$$

$$\hat{\mathcal{G}}^{a4} \rightarrow \hat{\mathcal{R}}^{a4} = \partial_c \hat{F}^{c4a} \tag{6.1.1}$$

Putting k back in, and by using Appendix equation (9.5.1) for the Christoffel symbol, we get:

$$\hat{\mathcal{R}}^{a4} \rightarrow \frac{1}{2}\partial_c k F^{ac} \tag{6.1.2}$$

And so by (3.2.10),

$$J^*_a \rightarrow -\frac{\alpha k}{2}\partial_c F_a^{\ c} \tag{6.1.3}$$

So Toth and Maxwell charges are related by a simple formula. The right hand side being Maxwell's charge current (see p.81 of [6]), and has the correct sign to identify a positive Toth charge Q^* with a positive Maxwell charge source $4\pi Q_M$, whenever $\alpha k > 0$. In the appropriate space-time frame, and Kaluza atlas frame, using (3.2.9), and approaching the $O(h^2)$ limit given by L1 (L2 and L3 weren't used):

$$4\pi Q_M \rightarrow +\frac{2}{\alpha k}Q^* \tag{6.1.4}$$

6.2 A Lorentz-Like Force Law

The Christoffel symbols and the geodesic equation are the symmetric ones defined in the presence of totally antisymmetric torsion. We will here initially use $k = 1$, a general k can be added in later.

$$\begin{aligned}
&\hat{\Gamma}^c_{(4b)} = \tfrac{1}{2}g^{cd}(\delta_4\hat{g}_{bd} + \delta_b\hat{g}_{4d} - \delta_d\hat{g}_{4b}) + \tfrac{1}{2}\hat{g}^{c4}(\delta_4\hat{g}_{b4} + \delta_b\hat{g}_{44} - \delta_4\hat{g}_{4b}) = \\
&\tfrac{1}{2}g^{cd}[\delta_b(\phi^2 A_d) - \delta_d(\phi^2 A_b)] + \tfrac{1}{2}g^{cd}\delta_4\hat{g}_{bd} + \tfrac{1}{2}\hat{g}^{c4}\delta_b\hat{g}_{44} = \\
&\tfrac{1}{2}\phi^2 g^{cd}[\delta_b A_d - \delta_d A_b] + \tfrac{1}{2}g^{cd}A_d\delta_b\phi^2 - \tfrac{1}{2}g^{cd}A_b\delta_d\phi^2 + \tfrac{1}{2}g^{cd}\delta_4\hat{g}_{bd} + \tfrac{1}{2}\hat{g}^{c4}\delta_b\phi^2 = \\
&\tfrac{1}{2}\phi^2 F^c_b + \tfrac{1}{2}g^{cd}A_d\delta_b\phi^2 - \tfrac{1}{2}g^{cd}A_b\delta_d\phi^2 + \tfrac{1}{2}g^{cd}\delta_4\hat{g}_{bd} + \tfrac{1}{2}\hat{g}^{c4}\delta_b\phi^2 =
\end{aligned}$$

$\frac{1}{2}\phi^2 F_b^c - \frac{1}{2}g^{cd}A_b\delta_d\phi^2 + \frac{1}{2}g^{cd}\delta_4\hat{g}_{bd} = \frac{1}{2}\phi^2 F_b^c \quad -\frac{1}{2}g^{cd}A_b\delta_d\phi^2$

(6.2.1)

$\hat{\Gamma}^c_{44} = \frac{1}{2}\hat{g}^{\mathrm{cD}}(\delta_4\hat{g}_{4D} + \delta_4\hat{g}_{4D} - \delta_D\hat{g}_{44}) =$ - $\frac{1}{2}g^{\mathrm{cd}}\delta_d\phi^2$

(6.2.2)

We have:

$$\begin{aligned}
&\hat{\Gamma}^c_{(ab)} = \tfrac{1}{2}g^{cd}(\delta_a g_{db} + \delta_b g_{da} - \delta_d g_{ab}) \\
&+\tfrac{1}{2}g^{cd}(\delta_a(\phi^2 A_d A_b) + \delta_b(\phi^2 A_a A_d) - \delta_d(\phi^2 A_a A_b)) + \tfrac{1}{2}\hat{g}^{c4}(\delta_a\hat{g}_{4b} + \delta_b\hat{g}_{4a} - \delta_4\hat{g}_{ab}) \\
&= \Gamma^c_{(ab)} \quad +\tfrac{1}{2}g^{cd}(\delta_a(\phi^2 A_d A_b) + \delta_b(\phi^2 A_a A_d) - \delta_d(\phi^2 A_a A_b)) \\
&\qquad -A^c(\delta_a\phi^2 A_b + \delta_b\phi^2 A_a)
\end{aligned}$$

(6.2.3)

So, for any coordinate system within the maximal atlas:

$$\begin{aligned}
0 &= \tfrac{d^2x^a}{d\tau^2} + \hat{\Gamma}^a_{(BC)}\tfrac{dx^B}{d\tau}\tfrac{dx^C}{d\tau} \\
&= \tfrac{d^2x^a}{d\tau^2} + \hat{\Gamma}^a_{(bc)}\tfrac{dx^b}{d\tau}\tfrac{dx^c}{d\tau} + \hat{\Gamma}^a_{(4c)}\tfrac{dx^4}{d\tau}\tfrac{dx^c}{d\tau} + \hat{\Gamma}^a_{(b4)}\tfrac{dx^b}{d\tau}\tfrac{dx^4}{d\tau} + \hat{\Gamma}^a_{44}\tfrac{dx^4}{d\tau}\tfrac{dx^4}{d\tau} \\
&= \tfrac{d^2x^a}{d\tau^2} + \hat{\Gamma}^a_{(bc)}\tfrac{dx^b}{d\tau}\tfrac{dx^c}{d\tau} + (\phi^2 F^a_b - g^{ad}A_b\delta_d\phi^2)\tfrac{dx^b}{d\tau}\tfrac{dx^4}{d\tau} - \tfrac{1}{2}g^{\mathrm{ad}}\delta_d\phi^2\tfrac{dx^4}{d\tau}\tfrac{dx^4}{d\tau}
\end{aligned}$$

(6.2.4)

Taking $\phi^2 = 1$ and the charge-to-mass ratio to be:

$$Q'/m_{k0} = \frac{dx^4}{d\tau} \tag{6.2.5}$$

We derive a Lorentz-like force law:

$$\frac{d^2x^a}{d\tau^2} + \hat{\Gamma}^a_{(bc)}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} = -(Q'/m_{k0})F^a_b\frac{dx^b}{d\tau} \tag{6.2.6}$$

Putting arbitrary k and variable ϕ back in we have:

$$\frac{d^2x^a}{d\tau^2} + \hat{\Gamma}^a_{(bc)}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} = -k(Q'/m_{k0})(\phi^2 F^a_b - g^{ad}A_b\delta_d\phi^2)\frac{dx^b}{d\tau} - \frac{1}{2}g^{\mathrm{ad}}\delta_d\phi^2\frac{dx^4}{d\tau}\frac{dx^4}{d\tau}$$

(6.2.7)

6.3 Constant Toth Charge

Having derived a Lorentz-like force law we look also at the momentum of the charge in the Kaluza dimension. We look at this acceleration as with the Lorentz force law. We have, with torsion (and $k = 1$):

$$0 = \frac{d^2x^4}{d\tau^2} + \hat{\Gamma}^4_{(BC)} \frac{dx^B}{d\tau} \frac{dx^C}{d\tau}$$

$$= \frac{d^2x^4}{d\tau^2} + \hat{\Gamma}^4_{(bc)} \frac{dx^b}{d\tau} \frac{dx^c}{d\tau} + \hat{\Gamma}^4_{(4c)} \frac{dx^4}{d\tau} \frac{dx^c}{d\tau} + \hat{\Gamma}^4_{(b4)} \frac{dx^b}{d\tau} \frac{dx^4}{d\tau} + \hat{\Gamma}^4_{44} \frac{dx^4}{d\tau} \frac{dx^4}{d\tau}$$

$$= \frac{d^2x^4}{d\tau^2} + \hat{\Gamma}^4_{(bc)} \frac{dx^b}{d\tau} \frac{dx^c}{d\tau} + 2\hat{\Gamma}^4_{(4c)} \frac{dx^4}{d\tau} \frac{dx^c}{d\tau} + \frac{1}{2} A^d \delta_d \phi^2 \frac{dx^4}{d\tau} \frac{dx^4}{d\tau} \tag{6.3.1}$$

6.4 Unitary Scalar Field And Torsion

Both equations above (6.2.7) and (6.3.1) have a term that wrecks havoc to any similarity with the Lorentz force law proper, the terms at the end. Both terms can however be eliminated by setting the scalar field to 1. This is postulate B1.

The two equations under B1 become (for all k):

$$\frac{d^2x^a}{d\tau^2} + \hat{\Gamma}^a_{(bc)} \frac{dx^b}{d\tau} \frac{dx^c}{d\tau} = -k(Q'/m_{k0}) F^a_b \frac{dx^b}{d\tau} \tag{6.4.1}$$

$$\frac{d^2x^4}{d\tau^2} + \hat{\Gamma}^4_{(bc)} \frac{dx^b}{d\tau} \frac{dx^c}{d\tau} = -k^2(Q'/m_{k0}) A_c F^c_b \frac{dx^b}{d\tau} \tag{6.4.2}$$

This certainly looks more hopeful. The more extreme terms have disappeared, the general appearance is similar to the Lorentz force law proper. The right hand side of (6.4.2) is small, but in any case the well-behaved nature of charge follows from local momentum conservation and the required integrity of charge models - that they do not lose charge to the vacuum.

6.5 The Lorentz Force Law

It is necessary to confirm that equation (6.4.1) not only looks like the Lorentz force law formally, but is indeed the Lorentz force law. Multiplying both sides of (6.4.1) by $\frac{d\tau}{d\tau'} \frac{d\tau}{d\tau'}$, where τ' is an alternative affine coordinate frame, gives:

$$\frac{d^2x^a}{d\tau'^2} + \hat{\Gamma}^a_{(bc)} \frac{dx^b}{d\tau'} \frac{dx^c}{d\tau'} = -k \frac{d\tau}{d\tau'} (Q'/m_{k0}) F^a_b \frac{dx^b}{d\tau'} \tag{6.5.1}$$

Given $Q^* = Q'\frac{d\tau}{d\tau^*}$ and therefore $\frac{m_{k0}}{m_0}Q^* = Q'\frac{d\tau}{dt_0}$ by definition, we can set the frame such that $\tau' = t_0$ via the projected 4D space-time frame of the charge. And the Lorentz force is derived:

$$\frac{d^2x^a}{d\tau'^2} + \hat{\Gamma}^a_{(bc)}\frac{dx^b}{d\tau'}\frac{dx^c}{d\tau'} = -k(Q^*/m_0)F^a_b\frac{dx^b}{d\tau'} \tag{6.5.2}$$

In order to ensure the correct Lorentz force law using the conventions of Wald [7] p69, this can be rewritten as follows, using the antisymmetry of $F^a_b = -F^a{}_b$:

$$= k(Q^*/m_0)F^a{}_b\frac{dx^b}{d\tau'} \tag{6.5.3}$$

Using (6.1.4) as its L1 weak field limit is approached, this can be rewritten again in terms of the Maxwell charge:

$$\to k(\frac{\alpha k}{2}(4\pi Q_M)/m_0)F^a{}_b\frac{dx^b}{d\tau'} \tag{6.5.4}$$

The result is that we must relate **G** and k to obtain the Lorentz force law in acceptable terms:

$$\frac{d^2x^a}{d\tau'^2} + \hat{\Gamma}^a_{(bc)}\frac{dx^b}{d\tau'}\frac{dx^c}{d\tau'} \to (Q_M/m_0)F^a_b\frac{dx^b}{d\tau'}$$

$$k = 2\sqrt{\mathbf{G}} \tag{6.5.5}$$

This shows that the Lorentz force law proper can be derived approaching the limit for (6.1.4), and provides a constraint in so doing.

7 Analysis Of Matter Model Regimes

R0 has already been discounted as not viable. Nevertheless the fundamental conservation law for mass-energy belongs to the torsionless Einstein tensor. R1 may define non-matter regions and matter regions, but this is then distinct from the torsionless Einstein mass-energy accordingly.

Under R2, spin current obeys the fundamental conservation law (9.3.5). Spin can therefore not be lost to or gained from the non-spin version of non-matter by a spin current model by definition, nor acquired or lost within either a non-spin region or spin model. This is as robust as the conservation law for the torsionless Einstein tensor (which is instead relative to the torsionless covariant derivative).

Charge requires spin, at least at a local, $O(h^2)$ L1-L3 weak field limit. This requires L3 and the consequential approximate complete antisymmetry of torsion. It follows from (9.3.9) and (5.3.5). Two approximative conservation laws result: (9.3.7), and that implied by applying this in turn to (9.3.9). The result is the appearance of Maxwell charge as a significant term in (9.3.9), via (5.3.5) and (5.3.3) - approximately conserved (relative to the torsion connection). Components of the spin current/charge also gets identified at this limit with the Maxwell current/charge. So R2 proposes that it is the spin that is fundamental here within the spin model, due to the fundamental conservation law (9.3.5). Any leakage of charge (below the threshold of the weak field limit to or from spin models to or from non-matter regions) can not be permanent or significant, and are restrained by this relation. In this sense it makes sense to substitute charge models with spin models as proposed by R2.

R1 non-matter regions at the weak field limit do not have significant spin by (9.3.9). So in this sense spin and R1 matter are bound together in R1 matter models and both excluded from R1 non-matter models at this limit, just as is the case with charge.

Often spins cancel out in bulk matter, leaving net charge. This situation seems contradictory as the bulk matter will then have 0 net spin. There is however a simple resolution: in these circumstances symmetric torsion terms of proportional significance may simply be present. The weak field limit may to some proportional extent not apply.

Matter models are therefore any region where a significant (in the sense that it can not be discounted by L1-L3) amount of charge, spin or R1 matter is present. The presence of charge ensures the presence of spin, both ensuring the presence of R1 matter at this limit, any divergence from this (such as spins netting out to 0) requiring divergence from the limit and symmetric torsion terms. Consistent with observation, R1 matter does not necessarily imply the presence of spin or charge.

So taking both R1 and R2 into consideration, and assuming that fundamental particle spin correlates meaningfully with the spin addressed here, we have that:

(i) Matter is defined by R1 (not necessarily conserved),
(ii) Mass-energy by the torsionless Einstein-tensor (conserved relative to the torsionless covariant derivative),
(iii) Spin is conserved by the torsion-bearing covariant derivative,
(iv) The presence of spin implies the presence of matter at the weak field limit (but not the other way round),
(v) Charge implies the presence of matter at the weak field limit (but not the other way round),
(vi) Charge implies the presence of spin at the weak field limit (but not the other way round),
(vii) Netting out of spins without the netting out of charges suggests the L1-L3 weak field limit no longer applies fully - that symmetric torsion terms are being generated, that L3 is no longer hold sufficiently.

The final concern, point (vii), does not in any case prevent the Kaluza-Cartan theory from being self-consistent and able to describe a wide variety of experimental phenomena. The L1-L3 limits and their application to actual experimental situations is an area for further research.

8 Conclusion

Kaluza's 1921 theory of gravity and electromagnetism using a fifth wrapped-up spatial dimension is at the root of many modern attempts to develop new physical theories. However for a number of reasons it is sometimes considered untenable.

A new cylinder condition was imposed as with Kaluza's original theory, but one based on the covariant derivative and associated with a metric torsion connection. A generally covariant definition. A number of other constraints and definitions were provided. The result was the appearance of the missing non-perpendicular electromagnetic fields and a new definition of charge in terms of the 5D momentum. The new definition of kinetic charge and the Maxwellian charge coincide at an appropriate limit. In order to obtain non-perpendicular electromagnetic fields it was necessary to generalize matter-charge models. A number of matter model regimes were used to analyse this.

Restrictions to the geometry and certain symmetries were handled by reducing the maximal atlas to a reduced Kaluza atlas that automatically handled the restrictions and symmetries without further deferment to general covariance. Physically this represents the idea that in 4D, charge is a generally covariant scalar, whereas in 5D, charge is entirely dependent on the frame. That this is meaningful stems from the global property of a small wrapped-up fifth spatial dimension with new cylinder condition. Mathematically the Kaluza atlas is a choice of subatlas for which the partial derivatives in the Kaluza direction are vanishing. This led to useful constraints on the Christoffel symbols for all coordinate systems in the Kaluza atlas.

Decomposition of the 5D metric into a 4D metric and a vector and scalar part is also possible.

Some interesting results such as that spin and charge can exist only in the presence of matter (at a particular limit) were derived, although an issue is raised as to when the limit postulates L1-L3 break-down and how that relates to real phenomena. The extent to which L1-L3 can be extended and applied needs investigation.

One outstanding issue is that realistic charge models are not possible without involving imaginary numbers (imaginary proper velocities in the Kaluza dimension and imaginary Kaluza rest mass). Obviously to detail this the Kaluza-Cartan space-time would have to be adapted further in some way, such as further dimensions. But on the other hand limiting the theory solely to the 4D resultant space-time manifold, and applying a realistic charge model by hand need cause no such problems provided the net result is compliant with any energy conditions, as all the figures in that context become real. Barring this failure to provide realistic charge models, which poses serious challenges to the 5D theory, the postulates currently required are straight forward. It is in this sense a simple theory. In effect all we have is a 5D manifold with a covariant cylinder condition on one spatial dimension defined with respect to an approximately completely antisymmetric metric torsion tensor limit, with weak fields and curvature.

Given certain assumptions about matter-charge models (various matter model regimes), the entirety of classical electrodynamics is rederived. Gravity and electromagnetism are unified in a way not fully achieved by general relativity, Einstein-Cartan theory or Kaluza's original 5D theory.

The collocation of torsion with electromagnetism is different from other Einstein-Cartan theories where the torsion is limited to within matter models. Here certain specific components of torsion are an essential counterpart of electromagnetism, and other components of torsion can exist outside of matter models, perhaps with detectable consequences. The scalar field (which is present in the original Kaluza theory), on the other hand, was fixed constant.

Due to the lack of realistic charge models, and certain other considerations, this theory remains at an early stage of development and incomplete, though many essential ingredients are present. Other issues that remain outstanding include the exact role of spin in the dynamics of particles, and a fuller exploration of the energy conditions and causality. Much work from Einstein-Cartan theory could probably be carried over or extended - similarly for Kaluza-Klein theories.

The full theory is given by K1-7 (core postulates), plus L1-L2 (more empirically justified limits), and L3 where applicable. The matter model regime discussion delimits the presumed nature of matter within the Kaluza theory framework, with further details coming from the particular matter model itself, not here having been specified.

Why go to all this effort to unify electromagnetism and gravitation and to make electromagnetism fully geometric? Because experimental differences should be detectable given sufficient technology on the one hand, or, on the other, and equally, simply because such an attempt at unification might be right. This theory differs from both general relativity and Einstein-Cartan theory, and this may be empirically fruitful. Also the expected ω-consistency of Einstein-Cartan theory together with the derivation of a Lorentz force law via the Kaluza part of the theory gives a unique theoretical motivation, as does the fact that the other approaches beyond general relativity have not fulfilled their promise. Further, attempting to extend and unify classical theory prior to a full unification with quantum mechanics may even be a necessary step in a future unification, whether this turns out to be the right way or not. It may be that current attempts are more difficult than necessary as the question may not yet have been framed correctly.

It is often asserted that the true explanation for gravitational theory and space-time curvatures will most likely, by reductionist logic, emerge out of its constituent quantum phenomena. Such an approach has merit, but is overly optimistic, and does not optimize the search [23]. Before constituent quantum parts can be properly defined, the larger scale whole must have been present at first to then be so divided. The dividing and putting together of parts assumes a context, and a context assumes a whole [22], though in daily life we take our conception of space-time for granted, we take the whole as given.

When comparing quantum mechanics and general relativity we should not do this. Something of the context is evidently missing from quantum mechanics on account of the difficulty in squaring the two worldviews, and therefore possibly from both. Thus reductionism assumes the whole, or contextual knowledge about that whole, before it is even applied. There is paradoxically an inherently non-reductionist assumption, or presumption, within that worldview. This is further justification for the here.

9 Appendix

9.1 Geometrized Charge

The geometrized units, Wald [7], give a relation for mass in terms of fundamental units. This leads to an expression for Toth charge in terms of Kaluza momentum when $k = 2$ and $\mathbf{G} = 1$.

$$\mathbf{G} = 1 = 6.674 \times 10^{-8} cm^3 g^{-1} s^{-2} = 6.674 \times 10^{-8} cm^3 g^{-1} \times (3 \times 10^{10} cm)^{-2}$$
$$= 6.674 \times 10^{-8} cm g^{-1} \times (3 \times 10^{10})^{-2}$$

$$1g \approx 7.42 \times 10^{-29} cm \text{ for c=1, } \mathbf{G}\text{=1} \tag{9.1.1}$$

$$1g \approx G/c^2 cm \text{ for c=1, } \mathbf{G}\text{=1} \tag{9.1.2}$$

For $k = 2$, c=1, **G**=1 we have:

$$1 statcoulomb = 1 cm^{3/2} s^{-1} g^{1/2} = cm^{1/2} \times (7.42 \times 10^{-29} cm)^{1/2} / (3.00 \times 10^{10})$$

$$= 8.61 \times 10^{-15} cm / (3 \times 10^{10}) \approx 2.87 \times 10^{-25} cm \approx 3.87 \times 10^3 g \tag{9.1.3}$$

Using cgs (Gaussian) units and the cgs versions of **G** and c, ie $\mathbf{G} = 6.67 \times 10^{-7} cm^3 g^{-1} s^{-1}$ and $c = 3 \times 10^{10} cms^{-1}$, the charge can be written in terms of 5D proper momentum P_4 as follows:

$$1 statcoulomb = 1 cm^{3/2} s^{-1} g^{1/2} = 1 (cm/s) cm^{1/2} g^{1/2} = \frac{c}{\sqrt{\mathbf{G}}} g.cm/s$$

$$Q^* = \frac{c}{\sqrt{\mathbf{G}}} P_4 \tag{9.1.4}$$

9.2 Introducing The Geometry Of Torsion

5D Cartan torsion will be admitted. This will provide extra and required degrees of freedom since the new cylinder condition (K3) would be too tight to yield interesting geometry otherwise. It is noted that Einstein-Cartan theory, that adds torsion to the dynamics of relativity theory is most probably a minimal ω-consistent extension of general relativity [13][14] and therefore the use of torsion is not only natural, but arguably a necessity on philosophical and physical grounds. That argument can also be applied here. What we have defined by this addition can be called Kaluza-Cartan theory as it takes Kaluza's theory and adds torsion. We assume that the torsion connection is metric.

For both 5D and 4D manifolds (i.e. dropping the hats and indices notation for a moment), torsion will be introduced into the Christoffel symbols as follows, using the notation of Hehl [11]. Metricity of the torsion tensor will be assumed [19], the reasonableness of which (in the context of general relativity with torsion) is argued for in [20] and [21]:

$$\frac{1}{2}(\Gamma^k_{ij} - \Gamma^k_{ji}) = S_{ij}{}^k \tag{9.2.1}$$

This relates to the notation of Kobayashi and Nomizu [12] and Wald [7] as follows:

$$T^i{}_{jk} = 2S_{jk}{}^i \equiv \Gamma^i_{jk} - \Gamma^i_{kj} \tag{9.2.2}$$

We have the contorsion tensor $K_{ij}{}^k$ [11] as follows, and a number of relations [11]:

$$\Gamma^k_{ij} = \frac{1}{2}g^{kd}(\partial_i g_{dj} + \partial_j g_{di} - \partial_d g_{ij}) - K_{ij}{}^k = \{^k_{ij}\} - K_{ij}{}^k \tag{9.2.3}$$

$$K_{ij}{}^k = -S_{ij}{}^k + S_j{}^k{}_i - S^k{}_{ij} = -K_i{}^k{}_j \tag{9.2.4}$$

Notice how the contorsion is antisymmetric in the last two indices.

With torsion included, the auto-parallel equation becomes [11]:

$$\frac{d^2x^k}{ds^2} + \Gamma^k_{(ij)}\frac{dx^i}{ds}\frac{dx^j}{ds} = 0 \tag{9.2.5}$$

$$\Gamma^k_{(ij)} = \{^k_{ij}\} + S^k{}_{(ij)} - S_{(j}{}^k{}_{i)} = \{^k_{ij}\} + 2S^k{}_{(ij)} \tag{9.2.6}$$

Only when torsion is completely antisymmetric is this the same as the extremals [11] which give the path of spinless particles and photons in Einstein-Cartan theory: extremals are none other than geodesics with respect to the Levi-Civita connection.

$$\frac{d^2x^k}{ds^2} + \{^k_{ij}\}\frac{dx^i}{ds}\frac{dx^j}{ds} = 0 \tag{9.2.7}$$

With complete antisymmetry we have many simplifications such as:

$$K_{ij}{}^k = -S_{ij}{}^k \tag{9.2.8}$$

9.3 Stress-Energy Tensors And Conservation Laws

The stress-energy tensor for a torsion bearing non-symmetric connection in Einstein-Cartan theory is usually labelled $\kappa\hat{P}_{AB}$, it need not be symmetric. In the literature the constant κ is included analogously to the 8π in general relativity. Here we will use the Einstein tensor $\hat{G}_{AB}$, taking a purely geometrical view.

The Belinfante-Rosenfeld [15] stress-energy tensor $\hat{B}$ is a symmetric adjustment of $\kappa\hat{P}$ that adjusts for spin currents as sources for Riemann-Cartan spaces. It can be defined equally for the 5D case. It is divergence free. It is the torsion equivalent according to Belinfante and Rosenfeld of the original Einstein tensor $\hat{G}$ [12] in some sense: the Einstein-Hilbert action. However the usual definition in terms of stress-energies and Noether currents, rather than the Einstein tensor, is not appropriate here. In effect repeating the Belinfante-Rosenfeld procedure, by defining the torsionless Einstein tensor in terms of torsion bearing components, yields what can be interpreted as extra spin-spin coupling term $\hat{X}_{AB}$:

$$\hat{\mathcal{G}}_{AB} = \hat{G}_{AB} + \hat{V}_{AB} + \hat{X}_{AB} \tag{9.3.1}$$

$$\hat{V}_{AB} = \frac{1}{2}\hat{\nabla}^C(\hat{\sigma}_{ABC} + \hat{\sigma}_{BAC} + \hat{\sigma}_{CBA}) \tag{9.3.2}$$

Where σ is defined as the spin tensor in Einstein-Cartan theory. However, here we do not start with spin (and some particle Lagrangians), but with the torsion tensor. So instead the spin tensor is defined in terms of the torsion tensor using the Einstein-Cartan equations. Here spin is explicitly defined in terms of torsion:

$$\hat{\sigma}_{ABC} = 2\hat{S}_{ABC} + 2\hat{g}_{AC}\hat{S}_{BD}{}^{D} - 2\hat{g}_{BC}\hat{S}_{AD}{}^{D} \tag{9.3.3}$$

This simplifies definition (9.3.2):

$$\hat{V}_{AB} = \frac{1}{2}\hat{\nabla}^C(\hat{\sigma}_{CBA}) = \hat{\nabla}^C(\hat{S}_{CBA} + \hat{g}_{CA}\hat{S}_{BD}{}^{D} - \hat{g}_{BA}\hat{S}_{CD}{}^{D}) \tag{9.3.4}$$

By considering symmetries and antisymmetries we get a conservation law:

$$\hat{\nabla}_B\hat{V}^{AB} = 0 \tag{9.3.5}$$

The Case Of Complete Anytisymmetry

Note that the mass-energy-charge conservation law for the torsionless Einstein tensor is in terms of the torsionless connection, but the spin source conservation law here is in terms of the torsion-bearing connection. However, for completely antisymmetric torsion we have:

$$\hat{\nabla}_C \hat{\mathcal{G}}_{AB} = \hat{\triangle}_C \hat{\mathcal{G}}_{AB} + \hat{K}_{CA}{}^{D} \hat{\mathcal{G}}_{DB} + \hat{K}_{CB}{}^{D} \hat{\mathcal{G}}_{AD}$$

So,

$$\hat{\nabla}^A \hat{\mathcal{G}}_{AB} = 0 + 0 + \hat{K}^{A}{}_{B}{}^{D} \hat{\mathcal{G}}_{AD} = -\hat{K}_{B}{}^{AD} \hat{\mathcal{G}}_{AD}$$
$$= -\hat{K}_{B}{}^{AD} \hat{\mathcal{G}}_{DA} = +\hat{K}_{B}{}^{DA} \hat{\mathcal{G}}_{DA} = +\hat{K}_{B}{}^{AD} \hat{\mathcal{G}}_{AD} = 0 \tag{9.3.6}$$

$$\hat{\nabla}^A (\hat{G}_{AB} + \hat{X}_{AB}) = 0 \tag{9.3.7}$$

And so there is a stress-energy conservation law with respect to the torsion connection also, at least in the completely antisymmetric case.

Further, still assuming complete antisymmetry of torsion, by definition of the Ricci tensor:

$$\hat{R}_{AB} = \hat{\mathcal{R}}_{AB} + \hat{K}_{DA}{}^{C} \hat{K}_{BC}{}^{D} - \partial_C \hat{K}_{BA}{}^{C} - \hat{K}_{BA}{}^{C} \hat{\Gamma}^{D}_{DC} + \hat{K}_{DA}{}^{C} \hat{\Gamma}^{D}_{DC} - \hat{K}_{DB}{}^{C} \hat{\Gamma}^{D}_{AC}$$
$$= \hat{\mathcal{R}}_{AB} - \hat{K}_{AD}{}^{C} \hat{K}_{BC}{}^{D} - \hat{\nabla}^C \hat{S}_{ABC} \tag{9.3.8}$$

$$\hat{G}_{[AB]} = \hat{R}_{[AB]} = -\hat{\nabla}^C \hat{S}_{ABC} = \hat{V}_{AB} \tag{9.3.9}$$

$\hat{V}_{AB}$ is the antisymmetric part of $\hat{G}_{AB}$ at this limit. And $\hat{X}_{AB}$ is a symmetric spin-torsion coupling adjustment - again only in the case of completely antisymmetric torsion.

In the general case (in the presence of symmetric torsion terms) such extensive identities and conservation laws are not present.

9.4 The Christoffel Symbols

Here we assume only the definitions of the Christoffel symbols and the new cylinder condition. (Without torsion terms shown, k set to 1)

$$2\hat{\Gamma}^{A}_{BC} = \textstyle\sum_d \hat{g}^{AD}(\partial_B \hat{g}_{CD} + \partial_C \hat{g}_{DB} - \partial_D \hat{g}_{BC})$$

$$= \textstyle\sum_d \hat{g}^{Ad}(\partial_B \hat{g}_{Cd} + \partial_C \hat{g}_{dB} - \partial_d \hat{g}_{BC})$$
$$+ \hat{g}^{A4}(\partial_B \hat{g}_{C4} + \partial_C \hat{g}_{4B} - \partial_4 \hat{g}_{BC})$$

$$2\hat{\Gamma}^{A}_{bc} = \textstyle\sum_d \hat{g}^{Ad}(\partial_b g_{cd} + \partial_c g_{db} - \partial_d g_{bc})$$
$$+ \textstyle\sum_d \hat{g}^{Ad}(\partial_b \phi^2 A_c A_d + \partial_c \phi^2 A_d A_b - \partial_d \phi^2 A_b A_c)$$
$$+ \hat{g}^{A4}(\partial_b \phi^2 A_c + \partial_c \phi^2 A_b - \partial_4 g_{bc} - \partial_4 \phi^2 A_b A_c)$$

$$2\hat{\Gamma}^{A}_{4c} = \sum_d \hat{g}^{Ad}(\partial_4 g_{cd} + \partial_4\phi^2 A_c A_d + \partial_c\phi^2 A_d - \partial_d\phi^2 A_c) + \hat{g}^{A4}\partial_c\phi^2$$

$$2\hat{\Gamma}^{A}_{44} = 2\sum_d \hat{g}^{Ad}\partial_4\phi^2 A_d - \sum_d \hat{g}^{Ad}\partial_d\phi^2 + \hat{g}^{A4}\partial_4\phi^2 \tag{9.4.1}$$

The Electromagnetic Limit $\boldsymbol{\phi^2 = 1}$

What is here called the electromagnetic limit is B1 in the text. For the purposes of full generality this is not assumed a priori.

$$\begin{aligned}
2\hat{\Gamma}^{A}_{bc} &= \sum_d \hat{g}^{Ad}(\partial_b g_{cd} + \partial_c g_{db} - \partial_d g_{bc}) + \sum_d \hat{g}^{Ad}(\partial_b A_c A_d + \partial_c A_d A_b - \partial_d A_b A_c) \\
&\quad + \hat{g}^{A4}(\partial_b A_c + \partial_c A_b - \partial_4 g_{bc} - \partial_4 A_b A_c) \\
2\hat{\Gamma}^{A}_{4c} &= \sum_d \hat{g}^{Ad}(\partial_4 g_{cd} + \partial_4 A_c A_d + \partial_c A_d - \partial_d A_c) \\
\hat{\Gamma}^{A}_{44} &= \sum_d \hat{g}^{Ad}\partial_4 A_d
\end{aligned} \tag{9.4.2}$$

Simplifying...

$$\begin{aligned}
2\hat{\Gamma}^{a}_{bc} &= 2\Gamma^{a}_{bc} + \sum_d g^{ad}(A_b F_{cd} + A_c F_{bd}) + A^a\partial_4 g_{bc} + A^a\partial_4 A_b A_c \\
2\hat{\Gamma}^{4}_{bc} &= -\sum_d A^d(\partial_b g_{cd} + \partial_c g_{db} - \partial_d g_{bc}) - \sum_d A^d(A_b F_{cd} + A_c F_{bd}) \\
&\quad - (1 + \sum_i A_i A^i)(\partial_4 g_{bc} + \partial_4 A_b A_c) + (\partial_b A_c + \partial_c A_b) \\
2\hat{\Gamma}^{a}_{4c} &= \sum_d g^{ad}(\partial_4 g_{cd} + \partial_4 A_c A_d) + \sum_d g^{ad}F_{cd} \\
2\hat{\Gamma}^{4}_{4c} &= -\sum_d A^d(\partial_4 g_{cd} + \partial_4 A_c A_d) - \sum_d A^d F_{cd} \\
\hat{\Gamma}^{a}_{44} &= \sum_d g^{ad}\partial_4 A_d \\
\hat{\Gamma}^{4}_{44} &= -\sum_d A^d\partial_4 A_d
\end{aligned} \tag{9.4.3}$$

The Scalar Limit $A_i = 0$

The scalar limit is not used in the main text as it would generally not be interesting when B1, the electromagnetic limit, was also imposed. The details are kept here for greater generality.

$$\begin{aligned}
2\hat{\Gamma}^{A}_{bc} &= \sum_d \hat{g}^{Ad}(\partial_b g_{cd} + \partial_c g_{ab} - \partial_d g_{bc}) - \hat{g}^{A4}\partial_4 g_{bc} \\
2\hat{\Gamma}^{A}_{4c} &= \sum_d \hat{g}^{Ad}\partial_4 g_{cd} + \hat{g}^{A4}\partial_c\phi^2 \\
2\hat{\Gamma}^{A}_{44} &= -\sum_d \hat{g}^{Ad}\partial_d\phi^2 + \hat{g}^{A4}\partial_4\phi^2
\end{aligned} \tag{9.4.4}$$

Simplifying...

$$\begin{aligned}
\hat{\Gamma}^{a}_{bc} &= \Gamma^{a}_{bc} \\
2\hat{\Gamma}^{4}_{bc} &= -\frac{1}{\phi^2}\partial_4 g_{bc} \\
2\hat{\Gamma}^{a}_{4c} &= \sum_d g^{ad}\partial_4 g_{cd} \\
2\hat{\Gamma}^{4}_{4c} &= \frac{1}{\phi^2}\partial_c\phi^2
\end{aligned}$$

$$
\begin{aligned}
&2\hat{\digamma}^{\,a}_{\,44} = -\textstyle\sum_d g^{ad}\partial_d\phi^2\\
&2\hat{\digamma}^{\,4}_{\,44} = \tfrac{1}{\phi^2}\partial_4\phi^2
\end{aligned}
\tag{9.4.5}
$$

The Electromagnetic Limit And Cylinder Condition

By applying equations (5.1.11) and the new cylinder condition in order to simplify terms of the electromagnetic limit with $k = 1$, at first omitting the torsion contribution (noting that these Christoffel symbols are symmetric in the lower indices):

$$
\begin{aligned}
&2\hat{\digamma}^{\,a}_{\,bc} = 2\digamma^{\,a}_{\,bc} + (A_bF_c{}^a + A_cF_b{}^a)\\
&2\hat{\digamma}^{\,4}_{\,bc} = -\textstyle\sum_d A^d(\partial_b g_{cd} + \partial_c g_{db} - \partial_d g_{bc}) + (\partial_b A_c + \partial_c A_b)\\
&\qquad = -2A_d\digamma^{\,d}_{\,bc} + (\partial_b A_c + \partial_c A_b)\\
&2\hat{\digamma}^{\,a}_{\,4c} = F_c{}^a\\
&\hat{\digamma}^{\,4}_{\,4c} = -\tfrac{1}{2}A^dF_{cd} \text{ and } \hat{\digamma}^{\,a}_{\,44} = \hat{\digamma}^{\,4}_{\,44} = 0
\end{aligned}
\tag{9.4.6}
$$

Now with torsion built into the Christoffel symbols, using equations (5.1.11) and others from that section, and noting that these Christoffel symbols are not necessarily symmetric in the lower indices:

$$
\begin{aligned}
&2\hat{\Gamma}^a_{bc} = g^{ad}(\partial_b g_{cd} + \partial_c g_{db} - \partial_d g_{bc}) + (A_bF_c{}^a + A_cF_b{}^a) - 2\hat{K}_{bc}{}^a\\
&\qquad = 2\digamma^{\,a}_{\,bc} + (A_bF_c{}^a + A_cF_b{}^a) - 2\hat{K}_{bc}{}^a\\
&2\hat{\Gamma}^4_{bc} = -\textstyle\sum_d A^d(\partial_b g_{cd} + \partial_c g_{db} - \partial_d g_{bc}) + (\partial_b A_c + \partial_c A_b)\\
&\qquad\quad - A^d(A_bF_{cd} + A_cF_{bd}) - 2\hat{K}_{bc}{}^4\\
&\qquad = -2A_d\digamma^{\,d}_{\,bc} + (\partial_b A_c + \partial_c A_b) - A^d(A_bF_{cd} + A_cF_{bd}) - 2\hat{K}_{bc}{}^4\\
&\hat{\Gamma}^a_{4c} = 0 \text{ and } \hat{\Gamma}^4_{4c} = \hat{\Gamma}^a_{44} = \hat{\Gamma}^4_{44} = 0\\
&\hat{\Gamma}^4_{c4} = -\tfrac{1}{2}A^dF_{cd}\\
&\hat{\Gamma}^a_{c4} = \tfrac{1}{2}F_c{}^a - \hat{K}_{c4}{}^a
\end{aligned}
\tag{9.4.7}
$$

9.5 Raised Levi-Civita Christoffel Symbols

Following the same procedure as with Christoffel symbols of the first and second kind a raised version of the Christoffel symbols can be derived, a third kind. It takes the value that would be guessed at (a guess because you can not raise across partial derivatives without caution) by inspecting the elements of the Christoffel symbol of the second kind and raising each element individually.

Starting from the covariant derivative of the raised metric tensor being zero:

$$\nabla^i g^{jk} = 0 = \partial^i g^{jk} + \Gamma^{jik} + \Gamma^{kij}$$

Cycling indices we have:

$$0 = \partial^k g^{ij} + \Gamma^{ikj} + \Gamma^{jki}$$
$$0 = \partial^j g^{ki} + \Gamma^{kji} + \Gamma^{ijk}$$

Adding the first two and subtracting the third:

$$\partial^i g^{jk} + \partial^k g^{ij} - \partial^j g^{ki} = 2\Gamma^{jik}$$

So, exactly as would be guessed:

$$\Gamma^{ijk} = \frac{1}{2}(\partial^j g^{ik} + \partial^k g^{ji} - \partial^i g^{jk}) \qquad (9.5.1)$$

9.6 Proper Kaluza Velocity As Space-Time Scalar

This section shows that the proper velocity **W** (written as a vector) with only one component in the Kaluza dimension is invariant under 4D space-time boosts orthogonal to it.

$W_4 = dx_4/d\tau$ proper velocity in a stationary space-time frame, but following the particle

$U_4 = \dfrac{W_4}{\sqrt{1+{W_4}^2}}$ coordinate velocity using proper velocity formula

Using orthogonal addition of coordinate velocities formula to boost space-time frame by orthogonal coordinate velocity **V**:

$$\mathbf{V} = (V, 0, 0, 0)$$
$$\mathbf{U} = (0, 0, 0, U_4)$$

Coordinate velocity vector in new frame, using the orthogonal velocity addition formula:

$$\overline{\mathbf{U}} = \mathbf{V} + \sqrt{1-V^2}\ \mathbf{U}$$

So,

$$\overline{U}_4 = \sqrt{1-V^2}\frac{W_4}{\sqrt{1+W_4^2}}$$

Define proper velocity in new frame: $\overline{\mathbf{W}}$, using proper velocity definition:

$$\overline{W}_4 = \frac{\overline{U}_4}{\sqrt{1-V^2-{\overline{U}_4}^2}}$$
$$= \frac{\sqrt{1-V^2}\frac{W_4}{\sqrt{1+W_4^2}}}{\sqrt{1-V^2-(\sqrt{1-V^2}\frac{W_4}{\sqrt{1+W_4^2}})^2}}$$

$$= \frac{W_4}{\sqrt{1+W_4^2}\sqrt{1-(\frac{W_4}{\sqrt{1+W_4^2}})^2}}$$

$$= \frac{W_4}{\sqrt{1+W_4^2-W_4^2}} = W_4 \tag{9.6.1}$$

$\overline{W_4} = W_4$ is the result required

The proper Kaluza velocity therefore is a scalar with respect to the Kaluza atlas.

10 Aknowledgements

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