Kaluza-Cartan Theory

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Abstract

Kaluza's 1921 theory of gravity and electromagnetism using a fifth wrapped-up spatial dimension is at the root of many modern attempts to develop new physical theories. Lacking important electromagnetic fields however, and having other problems, the theory is incomplete and generally considered untenable. An alternative approach is presented that includes torsion. Coulomb's law in the form of the Lorentz force law is investigated starting with a non-Maxwellian definition of charge, this is shown to be related to Maxwellian charge. It is concluded that Kaluza's 5D space and torsion should go together in what is here called a Kaluza-Cartan theory in order to form a unified theory of gravity and electromagnetism. A new cylinder condition is proposed that takes torsion into account and is fully covariant. Different formulations of the vacuum and matter models are called matter model regimes and are compared. Two connections are used, one with, and one without torsion. The concept of general covariance is investigated with respect to global properties that can be modelled via a non-maximal atlas. Differences from existing theories, general relativity being a limiting case, suggest experimental tests may be possible.

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1 Conventions

The following conventions are adopted unless otherwise specified:

Five dimensional metrics, tensors and pseudo-tensors are given the hat symbol. Five dimensional indices, subscripts and superscripts are given capital Roman letters. So for example the five dimensional Ricci flat 5-dimensional superspace-time of Kaluza theory is given as: $\hat{g}_{AB}$, all other tensors and indices are assumed to be 4 dimensional, if a general non-specified dimensional case is not being considered, for which either convention can be used. Index raising is referred to a metric $\hat{g}_{AB}$ if 5-dimensional, and to g_{ab} if 4-dimensional. The domain of partial derivatives carries to the end of a term without need for brackets, so for example we have $\partial_a g_{db} A_c + g_{db} g_{ac} = (\partial_a (g_{db} A_c)) + (g_{db} g_{ac})$. Terms that might repeat dummy variables or are otherwise in need of clarification use additional brackets. Square brackets can be used to make dummy variables local in scope.

Space-time is given signature $(-, +, +, +)$, Kaluza space $(-, +, +, +, +)$ in keeping with [6], except where stated and an alternative from [1] is referred to. Under the Wheeler et al [6] nomenclature, the sign conventions used here as a default are $[+, +, +]$. The first dimension (index 0) is always time and the 5^{th} dimension (index 4) is always the topologically closed Kaluza dimension. Universal constants defining physical units: c = 1, and $\mathbf{G}$ as a constant. The scalar field component is labelled ϕ^2 (in keeping with the literature) only as a reminder that it is associated with a spatial dimension, and to be taken as positive. The matrix of g_{cd} can be written as $|g_{cd}|$ when considered in a particular coordinate system to emphasize a component view. The Einstein summation convention may be used without special mention.

Some familiar defining equations consistent with [1] (using Roman lower-case for the general case only for ease of reference):

$$R_{ab} = \partial_c \Gamma^c_{ba} - \partial_b \Gamma^c_{ca} + \Gamma^c_{ba}\Gamma^d_{dc} - \Gamma^c_{da}\Gamma^d_{bc} \tag{1.0.1}$$

$$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab} = 8\pi \mathbf{G} T_{ab} \tag{1.0.2}$$

For convenience we will define $\alpha = \frac{1}{8\pi\mathbf{G}}$.

$$F_{ab} = \nabla_a A_b - \nabla_b A_a = \partial_a A_b - \partial_b A_a \text{ equally } F = dA \tag{1.0.3}$$

Any 5D exterior derivatives and differential forms could also be given a hat, thus: $\hat{d}\hat{B}$. However, the primary interest here will be 4D forms. $\Box$ represents the 4D D'Alembertian.

Torsion introduces non-obvious conventions in otherwise established definitions. The order of the indices in the Christoffel symbols comes to matter, and this includes in the Ricci tensor definition and the definition of the Christoffel symbols themselves:

$$\nabla_a w_b = \partial_a w_b - \Gamma^c_{ab} w_c \tag{1.0.4}$$

Christoffel symbols in general will take the usual form: Γ^c_{ab} or Γ^{abc} and so on. However, often we will need to distinguish a with-torsion Christoffel symbol from a without-torsion Christoffel symbol in some way. In the completely antisymmetric case, when the torsion tensor is completely anti-symmetric, we can write the without-torsion connection coefficients as: $\Gamma^c_{(ab)}$. But greater generality to specify the Levi-Civita connection at all times may call for the more explicit: $\{^{\,c}_{ab}\}$. More conveniently we may also refer to the Levi-Civita connection coefficients using: $\mathit{\Gamma}^c_{ab}$, and a covariant derivative operator: $\triangle_a$. In order to distinguish general G_{ab} and R_{ab} etc. from the case where the torsion has been explicitly excluded from the definition we use cursive: $\mathcal{G}_{ab}$ and $\mathcal{R}_{ab}$.

2 Introduction

Kaluza's 1921 theory of gravity and electromagnetism [2][3][4] using a fifth wrapped-up spatial dimension is at the heart of many modern attempts to develop new physical theories [1][5]. From supersymmetry to string theories topologically closed small extra dimensions are used to characterize the various forces of nature. It is therefore at the root of many modern attempts and developments in theoretical physics. However it has a number of foundational problems and is often considered untenable in itself. This paper looks at these problems from a purely classical perspective, without involving quantum theory. This perhaps runs against the grain for modern physics due to the great success of quantum mechanics, but is nevertheless worth doing as an independent enterprise.

The Metric

The theory assumes a (1,4)-Lorentzian Ricci flat manifold to be the underlying metric, split (analogously to the later ADM formalism) as follows:

$$\hat{g}_{AB} = \begin{bmatrix} g_{ab} + \phi^2 A_a A_b & \phi^2 A_a \\ \phi^2 A_b & \phi^2 \end{bmatrix} \tag{2.0.1}$$

Note that a common scaling factor has been set to $k = 1$ and so is not present, this will be reintroduced. By inverting this metric as a matrix (readily checked by multiplication) we get:

$$\hat{g}^{AB} = |\hat{g}_{AB}|^{-1} = \begin{bmatrix} g^{ab} & -A^a \\ -A^b & \frac{1}{\phi^2} + A_i A^i \end{bmatrix} \tag{2.0.2}$$

Maxwell's law are automatically satisfied: dF=0 follows from dd = 0. d*F= 4π*J can be set by construction. d*J=0, conservation of charge, follows also by dd=0 on most parts of the manifold.

However, in order to write the metric in this form there is a subtle assumption, that g_{ab}, which will be interpreted as the usual four dimensional space-time metric, is itself non-singular. However, this will always be the case for moderate or small values of A_x which will here be identified with the electromagnetic 4-vector potential. The raising and lowering of this 4-vector are defined in the obvious way in terms of g_{ab}. The 5D metric can be represented at every point on the Kaluza manifold in terms of this 4D metric g_{ab} (when it is non-singular), the vector potential A_x, and the scalar field ϕ^2. We have also assumed that topology is such as to allow the Hodge star to be defined. This means that near a point charge source the argument that leads to charge conservation potentially breaks down as the potential may cease to be well-defined. Whereas the Toth charge that will be defined in the sequel does not have this problem. So two different definitions of charge are to be given: the Maxwellian, and the Toth charge.

With values of ϕ^2 around 1 and relatively low 5-dimensional metric curvatures, we need not concern ourselves with this assumption beyond stating it on the basis that physically these parameters encompass tested theory. Given this proviso A_x is a vector and ϕ^2 is a scalar - with respect to the tensor system defined on any 4-dimensional submanifold that can take the induced metric g.

Kaluza's Cylinder Condition And The Original Field Equations

Kaluza's cylinder condition (KCC, or original KCC) is that all partial derivatives in the 5th dimension i.e. ∂_4 and $\partial_4\partial_4$ etc... of all metric components and of all tensors and their derivatives are zero. A perfect 'cylinder'. Here we extend it to torsion terms, and indeed all tensors and pseudo-tensors. This leads to constraints on g_{ab} given in [1] by three equations, the field equations of the original Kaluza theory, where the Einstein-Maxwell stress-energy tensor can be recognised embedded in the first equation:

$$G_{ab} = \frac{k^2\phi^2}{2}\left\{\frac{1}{4}g_{ab}F_{cd}F^{cd} - F^c_aF_{bc}\right\} - \frac{1}{\phi}\{\nabla_a(\partial_b\phi) - g_{ab}\Box\phi\} \tag{2.0.3}$$

$$\nabla^a F_{ab} = -3\frac{\partial^a\phi}{\phi}F_{ab} \tag{2.0.4}$$

$$\Box\phi = \frac{k^2\phi^3}{4}F_{ab}F^{ab} \tag{2.0.5}$$

Note that there is both a sign difference and a possible factor difference with respect to Wald [7] and Wheeler [6]. The sign difference appears to be due to the mixed use of metric sign conventions in [1] and can be ignored. A k factor is present and scaling will be investigated. These will be referred to as the first, second and third torsionless field equations, or original field equations, respectively. They are valid only in Kaluza vacuum, that is, outside of matter and charge models, and when there is no torsion. This requires Kaluza's original cylinder condition and the usual conception of matter models.

Definition 2.0.6: Perpendicular electromagnetic solutions.

Fields for which the following equation hold will be called perpendicular electromagnetic fields, and likewise those that do not satisfy this: non-perpendicular. Null solutions are perpendicular solutions with a further constraint. But perpendicular solutions will be of most interest here.

$$F_{ab}F^{ab} = 0 \tag{2.0.6}$$

By looking at the third field equation (2.0.5) it can be seen that if the scalar field does not vary then only a limited range of solutions result, that have perpendicular electric and magnetic fields. Eg null solutions. The second field equation (2.0.4) then also imposes no charge sources. Here the scalar term could be allowed to vary in order to allow for non-zero $F_{ab}F^{ab}$. This falls within Kaluza's original theory. This potentially allows for more electromagnetic solutions, but there are problems to overcome: the field equations cease being necessarily electrovacuum.

Foundational Problems

One inadequate and arbitrary fix is to set the scalar field term large, as is sometimes done to ensure that the second field equation (2.0.4) is identically zero despite scalar fluctuations. This approach will not be taken here. The stress-energy tensor under scalar field fluctuations is different from the Einstein-Maxwell tensor [6][7] and the accepted derivation of the Lorentz force law (for electrovacuums [6]) can not be assumed. A variable scalar field also implies non-conservation of Maxwell charge via the third field equation (2.0.5). Attempts to loosen constraints such as the KCC have also not been successful so far.

Another foundational issue of Kaluza theory is that even with a scalar field it does not have convincing sources of mass or charge built in. The second Field equation (2.0.4) has charge sources, but it's unlikely that realistic sources are represented by this equation. Matter and charge models in this work will be governed by different sets of equations in the Kaluza-Cartan space called 'matter model regimes', just as matter/energy is analogously assumed to be the conserved Einstein tensor in general relativity, and vacuum the state where the Ricci tensor is zero. We may propose alternatives. For example vacuum in the 5D Kaluza space of the original Kaluza theory yields electromagetic fields in the corresponding 4D space-time. There is room here for some experimentation with different notions.

As mentioned, charge will be given a possible alternative definition, Toth charge: as 5-dimensional momentum, following a known line of reasoning [8] within Kaluza theory. This will enable a derivation of Coulomb's law, via the Lorentz force law - leading to a genuine mathematical unification of electromagnetism and gravity. As momentum, the Toth charge is of necessity locally conserved, provided there are no irregularities in the topology of the Kaluza 5th dimension. Similarly the conservation of Maxwellian charge is normally guaranteed by the existence of the potential, except that this may not be valid in extreme curvatures where the values here associated with the 4-potential may cease to be a vector.

We will also assume global hyperbolicity in the sense of the existence of a Cauchy surface as is often done in general relativity to ensure 4 dimensional causality. Though this will not necessarily guarantee 5D causality in the event that either KCC is weakened or the theory presented here breaks down due to singularities. It also requires the usual energy conditions on the resulting 4D space-time and fields, or a 5D equivalent generalization.

One leading issue is that Kaluza theory offers limited electromagnetic solutions. Non-perpendicular electrovacuums more generally are not so easily supported as changes in the scalar field may force divergence of the field equations from those of the electrovacuum (see the first field equation (2.0.3)). This will lead to the potential failure of the Lorentz Force Law, in effect Coulomb's law. This is a fundamental problem to be overcome.

A Solution?

The Lorentz force law/Coulomb's law is to be derived from the theory (or theories, if we include different matter model regimes) independently of the electrovacuum solutions of general relativity. Note that in addition the derivation of the Lorentz force law within general relativity (from an assumed Einstein-Maxwell stress-energy tensor) is not without problems of principle [6]. But more importantly the stress-energy tensor that defines the electrovacuum geometry has to be assumed in classical electrodynamics within general relativity, whereas in this Kaluza-Cartan theory it is not. The Lorentz force law follows from different considerations. Thus, the program is to seek a link between the Toth charge and Maxwellian charge, so that they might be sufficiently interchangeable - as this can then be shown to lead to an alternative explanation for the Lorentz force law. The Lorentz force law, rather than electrovacuum solutions, will be looked for.

A Torsion Connection And A New Cylinder Condition

A further major change is the definition of a new cylinder condition based on the to be introduced torsion connection. Matter, charge and spin sources will also be investigated under the different matter model regimes. To obtain the sought for range of electromagnetic solutions a particular constraint will be weakened: that the torsion tensor is necessarily vanishing. Torsion will be allowed to vary allowing greater scope for solutions. Without this then the new cylinder condition (also to be defined) would be too strong.

The combination of torsion and the 5th spatial dimension justifies the label Kaluza-Cartan theory.

3 Overview Of Kaluza-Cartan Theory

The new theory (or theories) presented here, the Kaluza-Cartan theory (or Kaluza-Cartan theories), purports to resolve the foundational problems of the original Kaluza theory - even if presenting some new issues of its own.

3.1 The New Cylinder Condition

Throughout this work the limit of a new Kaluza Cylinder Condition (new KCC, or new cylinder condition) will be taken to be that the covariant derivative of all tensors in the Kaluza direction are to be zero, and that the covariant derivative depends on torsion. The 5D metric generally decomposes into 4D metric, vector potential and scalar field, at least when the embedded 4D metric is non-singular.

The new KCC by construction allows for an atlas of charts wherein also the partial derivatives are zero. This is true for a subatlas covering the 5D Lorentz manifold. But charts may exist in the maximal atlas for which these constraints are not possible. The atlas that is compliant is restricted. This means that the new KCC can be represented by a subatlas of the maximal atlas for the manifold. The set of local coordinate transformations that are compliant with this atlas (call it the Kaluza atlas) is in general non-maximal by design.

A further reduction in how the atlas might be interpreted is also implied by setting c=1, and constant **G**. The existence of a single unit for space and time can be assumed, and this must be scaled in unison for all dimensions. Consistently with cgs units we can choose either centimetres or seconds. This would leave velocities (and other geometrically unitless quantities) unchanged in absolute magnitude. This doesn't prevent reflection of an axis however, and indeed reflection of the Kaluza dimension will be equivalent to a charge inversion. However, given an orientation we can also remove this.

Space-time can not be an arbitrary 4D Lorentz submanifold as it must be one that is normal to a Kaluza axis and that satisfies certain constraints. This will generally have the interpretation best visualized as a cylinder with a longitudinal space-time and a perpendicular Kaluza dimension. However this will not be so simple when considering more than one connection.

We can further reduce the Kaluza atlas by removing boosts in the Kaluza dimension. Why? This requires the new Kaluza cylinder condition, as it significantly reduces the possible geometries. Space-time is taken to be a subframe within a 5D frame within the Kaluza subatlas wherein uncharged matter can be given a rest frame via a 4D Lorentz transformation. Boosting uncharged matter along the Kaluza axis will give it kinetic Toth charge (as described in the Introduction, and as detailed shortly). The Kaluza atlas represents the 4D view that charge is 4D covariant. Here we require that the Toth charge coincides with Maxwellian charge in some sense. The justification for this assertion will be clarified later. Rotations into the Kaluza axis can likewise be omitted. These result in additional constraints on the Christoffel symbols associated with charts of this subatlas, and enable certain geometrical objects to be more easily interpreted in space-time. The use of this subatlas does not prevent the theory being generally covariant, but simplifies the way in which we look at the Kaluza space through a 4D physical limit and worldview.

Preliminary Description 3.1.1: The Kaluza-Cartan space-time.

In the most general case a Kaluza-Cartan space-time will be a 5D Lorentz manifold with metric and metric torsion connection. As mentioned above the new Kaluza cylinder condition will be added (see postulate K7 later) and the topology of the Kaluza dimension will be closed and geometrically small. A global Kaluza direction will be defined as normal (relative to the new cylinder condition) to a 4D Lorentz submanifold. That submanifold, and all parallel submanifolds as a set, will constitute space-time. The torsionless Levi-Civita connection will also continue to be available. Charged particles will be those that are not restricted in movement to within space-time. Uncharged particles will be restricted to motion within each slice of the set of parallel space-time slices. The new cylinder condition will ensure that all the parallel space-times are equivalent. The rigidity of this is expressed via the definition of the Kaluza atlas.

A complete definition of the global Kaluza-Cartan space-time is given later in postulates K1-K8.

Definition 3.1.1: The Kaluza atlas.

(i) The Kaluza atlas is a subatlas of the maximal atlas of Kaluza-Cartan space-time. It can be given geometrized units when interpreted physically.

(ii) Boosts and rotations into the Kaluza dimension (as defined by the new cylinder condition) are explicitly omitted.

(iii) This represents the physical interpretation of charge as a covariant property of space-time even if it is not a covariant property of the 5D Kaluza-Cartan space.

(iv) Mathematically this is also an atlas of charts for which the partial derivatives of tensors and pseudo-tensors in the Kaluza dimension vanish.

(3.1.1)

3.2 Kinetic Toth Charge

Kinetic Toth charge is defined as the 5D momentum component in terms of the 5D Kaluza rest mass of a hypothesised particle: ie (i) its rest mass in the 5D Lorentz manifold (m_{k0}) and (ii) its proper Kaluza velocity ($dx_4/d\tau^*$) with respect to a frame in the maximal atlas that follows the particle. And equally it can be defined in terms of (i) the relativistic rest mass (m_0), relative to a projected frame where the particle is stationary in space-time, but where non-charged particles are stationary in the Kaluza dimension, and in terms of (ii) coordinate Kaluza velocity (dx_4/dt_0):

Definition 3.2.1: Toth charge (scalar).

$$Q^* = m_{k0}dx_4/d\tau^* = m_0 dx_4/dt_0 \tag{3.2.1}$$

This makes sense because mass can be written in fundamental units (i.e. in distance and time). And the velocities in question defined relative to particular frames. It is not a generally covariant definition but it is nevertheless mathematically meaningful. In the Appendix (9.8.1) it is shown that this kinetic Toth charge can be treated in 4D Space-Time, and the Kaluza atlas, as a scalar: the first equation above is covariant with respect to the Kaluza atlas. It can be generalized to a 4-vector as follows, and it is also conserved:

In general relativity at the special relativistic Minkowski limit the conservation of momentum-energy/stress-energy can be given in terms of the stress-energy tensor as follows [9]:

$$\frac{\partial \hat{T}^{00}}{\partial t} + \frac{\partial \hat{T}^{i0}}{\partial x_i} = 0 \tag{3.2.2}$$

Momentum in the j direction:

$$\frac{\partial \hat{T}^{0j}}{\partial t} + \frac{\partial \hat{T}^{ij}}{\partial x_i} = 0 \tag{3.2.3}$$

This is approximately true at a weak field limit and can be applied equally to Kaluza theory, in the absence of torsion. We have a description of conservation of momentum in the 5th dimension as follows:

$$\frac{\partial \hat{T}^{04}}{\partial t} + \frac{\partial \hat{T}^{i4}}{\partial x_i} = 0 \tag{3.2.4}$$

We also have i=4 vanishing by KCC. Thus the conservation of Toth charge becomes (when generalized to different space-time frames) the property of a 4-vector current, which we know to be conserved:

$$(\hat{T}^{04}, \hat{T}^{14}, \hat{T}^{24}, \hat{T}^{34}) \tag{3.2.5}$$

$$\partial_0 \hat{T}^{04} + \partial_1 \hat{T}^{14} + \partial_2 \hat{T}^{24} + \partial_3 \hat{T}^{34} = 0 \tag{3.2.6}$$

As in relativity this can be extended to a definition that is valid even when there is curvature. Nevertheless the original Toth charge definition (3.2.1) has meaning in all Kaluza atlas frames as a scalar.

Kinetic Toth charge current is the 4-vector, induced from 5D Kaluza-Cartan space as follows (using the Kaluza atlas to ensure it is well-defined as a 4-vector):

$$J^{*a} = -\alpha \hat{G}^{a4} \tag{3.2.7}$$

Noting that,

$$\hat{\nabla}_A \hat{G}^{AB} = \hat{\nabla}_a \hat{G}^{a4} = 0 \tag{3.2.8}$$

Using Wheeler et al [6] p.131, and selecting the correct space-time (or Kaluza atlas) frame, we have:

$$Q^* = J^*_a (1,0,0,0)^a \tag{3.2.9}$$

So we have a scalar, then a vector representation of relativisitic invariant charge current, and finally a 2-tensor unification with mass-energy.

When torsion is introduced this will not prevent the Levi-Civita connection having meaning and application on the same manifold contemporaneously. Kinetic Toth charge is defined in the same way even when torsion is present - via the Einstein tensor without torsion, and applying conservation of mass-energy relative to the torsionless connection. The new cylinder condition is defined, on the other hand, using the *torsion* connection. Distinction is therefore necessary between connections. So the 5D geoemtry depends on the torsion connection, but here a conservation still depends on the Levi-Civita connection.

The definition of kinetic charge and the conservation law of mass-energy-charge need to be written using the appropriate notation when torsion is used on the same manifold:

Definition 3.2.10: Toth charge current.

Toth charge current is defined to be the 4-vector $J^{*a} = -\alpha\hat{\mathcal{G}}^{a4}$, with respect to the Kaluza atlas, and noting:

$$\hat{\triangle}_A \hat{\mathcal{G}}^{AB} = 0 \tag{3.2.10}$$

3.3 Two Types Of Geometrized Charge

The metric components used in [1] as the 5D Kaluza metric, defined in terms of the original KCC follow. It will be equally used here in its new context, where the geometry of this space will depend on the new KCC defined in terms of a metric torsion tensor. It is called here the Kaluza-Cartan metric to remind us of this context. The vector potential and electromagnetic fields formed via the metric are sourced in Maxwell charge Q_M.

Definition 3.3.1: The 5D Kaluza-Cartan metric.

$$\hat{g}_{AB} = \begin{bmatrix} g_{ab} + k^2\phi^2 A_a A_b & k\phi^2 A_a \\ k\phi^2 A_b & \phi^2 \end{bmatrix} \tag{3.3.1}$$

This gives inverse as follows:

$$\hat{g}^{AB} = |\hat{g}_{AB}|^{-1} = \begin{bmatrix} g^{ab} & -kA^a \\ -kA^b & \frac{1}{\phi^2} + k^2 A_i A^i \end{bmatrix} \tag{3.3.2}$$

This gives (with respect to space-time) perpendicular solutions [1] under the original KCC, such that $G_{ab} = -\frac{k^2}{2} F_{ac}F_b^c$. Compare this with [7] where we have $G_{ab} = 2F_{ac}F_b^c$ in geometrized units we would need to have $k = 2$ or $k = -2$ for compatibility of results and formulas. Noting the sign change introduced by [1] - where it appears that the Einstein tensor was defined relative to (+, −, −, −), despite the 5D metric tensor being given in a form that can only be (−, +, +, +, +), which is confusing. Approximately the same result, but with consistent sign conventions, is achieved here in (5.2.4).

The geometrized units, Wald [7], give a relation for mass in terms of fundamental units. This leads to an expression for Toth charge in terms of Kaluza momentum when $k = 2$ and $\mathbf{G} = 1$. $\mathbf{G}$ and k are not independent however. If we fix one the other is fixed too, as a consequence of requiring the Lorentz force law written in familiar form. The relation between $\mathbf{G}$ and k is given in equation (6.5.5). Simple compatibility with Wald [7], where $k = 2$ and $\mathbf{G} = 1$, results however. The sign of k is also fixed by (6.1.4). The result as given in the Appendix, written in terms of the Toth charge Q^*, is:

$$Q^* = \frac{c}{\sqrt{\mathbf{G}}} P_4 \tag{3.3.3}$$

Generally speaking the approach here will be to do the calculations using $k = 1$ and then add in the general k term later, as and when needed, simply to ease calculation.

An important part of this theory is the nature of the relationship between these two types of charge: Q^* and Q_M - to be dealt with later.

3.4 Consistency With Special Relativity

Toth charge is identified with 5D momentum in a space-time rest frame. This is already known in the original Kaluza theory to obey a Lorentz-like force law, but will be extended here in scope.

That this is consistent with special relativity can be investigated. What this consistency means is that the relativistic mass created by momentum in the 5th dimension is kinematically identical to the relativistic rest mass.

The additions of velocities in special relativity is not obvious. Assume a flat 5D Kaluza space (i.e without geometric curvature or torsion, thus analogously to special relativity at a flat space-time limit, a 5D Minkowski limit). Space-time can be viewed as a 4D slice (or series of parallel slices) perpendicular to the 5th Kaluza dimension that minimizes the length of any loops that are perpendicular to it. Taking a particle and an inertial frame, the relativistic rest frame where the particle is stationary with respect to space-time but moving with velocity u in the 5th dimension, and a second frame where the charge is now moving in space-time at velocity v, but still with velocity u in the 5th dimension, then the total speed squared of the particle in the second frame is according to relativistic addition of orthogonal velocities:

$$s^2 = u^2 + v^2 - u^2 v^2 \tag{3.4.1}$$

The particle moving in the Kaluza dimension with velocity u, but stationary with respect to 4D space-time, will have a special relativistic 4D rest mass (m_0) normally greater than its 5D Kaluza rest mass (m_{k0}). We can see that the Kaluza rest mass definition (m_{k0}) is consistent with the orthogonal addition of velocities as follows:

$$m_0 = \frac{m_{k0}}{\sqrt{(1-u^2)}} \text{ where } u = \tanh[\sinh^{-1}(Q^*/(m_{k0}))] \quad (3.4.2)$$

$$m_{rel} = \frac{m_0}{\sqrt{(1-v^2)}} = \frac{m_{k0}}{\sqrt{(1-u^2)}} \times \frac{1}{\sqrt{(1-v^2)}} = \frac{m_{k0}}{\sqrt{(1-u^2-v^2+u^2v^2)}} \quad (3.4.3)$$

By putting $u = \tanh[\sinh^{-1}(Q^*/(m_{k0}))]$ (keeping the hyperbolics to recall the conversion between unidirectional proper and coordinate velocities) into the definition of relativistic rest mass in terms of Kaluza rest mass and solving, we get that charge, whether positive or negative, is related to the relativistic rest mass according to the following formula:

$$\begin{aligned} \cosh[\sinh^{-1}(Q^*/(m_{k0}))] &= m_0/m_{k0} = \frac{dt_0}{d\tau^*} \\ &= \sqrt{(Q^*/(m_{k0}))^2+1} \end{aligned} \quad (3.4.4)$$

Using $k = 2$ we also have, for a typical unit charge:

$$m_e = 9.1094 \times 10^{-28} g \quad (3.4.5)$$

$$Q^* = 4.8032 \times 10^{-10} statC = 4.8032 \times 3.87 \times 10^{-10+3} g = 1.859 \times 10^{-6} g \quad (3.4.6)$$

If we take these figures and equate $m_e = m_0$ then we end up with imaginary m_{k0} and imaginary proper Kaluza velocity. Obviously to detail this the Kaluza-Cartan space-time would have to be adapted further in some way. But on the other hand it causes no causality problems provided the net result is compliant with any energy conditions being applied. And what is important in this respect is that the figures we know to be physical in 4D remain so.

Further issues are pertinent.

Observed electrons have static charge, angular momentum, a magnetic moment, and a flavor. The only thing distinguishing the electron from the muon is the flavor. The mass difference between the muon and the electron is about 105 MeV, perhaps solely due to this difference in flavor. The issue of modeling particles within a classical theory is, not surprisingly, a difficult one! Thus at this stage the idealized hypothetical charges used here, and real particles, can only be tentatively correlated.

It is possible to proceed without concern for the foundational issues of such charge models or attempting to interpret this quandary, instead simply developing the mathematics as is and seeing where it leads without judging it a priori.

3.5 Matter And Charge Models, A Disclaimer

This theory assumes some sort of particle model of matter and charge is possible, that it can be added to the original theory without significantly changing the ambient space-time solution and thus its own path, which is as approximate as in general relativity, except as a limit. Secondly we might imagine that what has been described is a particle whizzing around the fifth dimension like a roller coaster on its spiralled tracks. The cylinder conditions can in fact be maintained if, instead of a 5D particle, the matter and charge sources were rather a 'solid' ring, locked into place around the 5th dimension, rotating at some predetermined proper Kaluza velocity (albeit imaginary). An exact solution could even involve changes in the size of the 5th dimension. None of that is investigated here, the aim was originally just to see whether non-perpendicular solutions can be found in a variant Kaluza theory, and what constraints are needed.

It is a proviso that a physically realistic matter and charge model has not been detailed, nor formally identified with a real charge source such as an electron. The assumption then that such a hypothetical model would necessarily follow (albeit approximately) some predetermined path such as geodesics is therefore an assumption - though not without analogs in other, experimentally valid classical theories. However, with the addition of torsion this becomes very much an assumption. Geodesics, or extremals, being followed by spinless particles in 4D Einstein-Cartan theory [11]. Other particles follow different paths when interaction with torsion is present.

An exact differential geometrical model of such a matter and charge source is presumed too difficult to produce here, even if possible, especially given the previous discussion about imaginary masses and velocities. In addition, the fact that real charge sources are quantum mechanical may also be discouraging, though a classical limit interpretation should be possible regardless. The philosophy here has not been to provide a Lagrangian for a hypothesised charge model either, but instead to simply delimit what might constitute such models, and to weaken such constraints as much as possible.

This work assumes limited concepts of particle charge models and attempts to find non-perpendicular electromagnetic fields and an explanation for the Lorentz force law. The theory is an attempt to replicate all the important features of classical physics, without predicting or imposing a particular model of charge as the correct one. The following assumption summarizes the preceding:

Geodesic Assumption: That any particle-like matter and charge model approximately follow extremals. (Auto-parallels and extremals being two analogs of geodesics used when torsion is present, but neither of which in the most general case determine the paths followed by particles).

Note that spinless particles according to [11] will follow extremals. Extremals coincide with auto-parallels when torsion is completely antisymmetric. Particles with spin may interact in other ways. So the assumption is that torsion-spin coupling does not significantly effect the path of the particle, at least to some approximation.

Further we shall introduce different sets of equations, constraints and definitions that exclusively define both matter regions and vacuum regions called 'matter model regimes'. Different matter model regimes will for the most part not be mutually compatible. That the vacuum consists of Ricci flat regions with respect to the Levi-Civita connection is the matter model regime of general relativity. This works well as the divergence of the Einstein tensor (without torsion) is zero, and the Einstein tensor (without torsion) is zero in vacuum (as defined within general relativity) regions. Thus we can say that the vacuum (within that conception of matter and vacuum) has some sort of 'integrity'. Alternatives are possible, and Kaluza's original theory suggests this in that 5D Ricci flatness implies 4D electromagnetic fields - it is interesting that a so-called 'vacuum' in one situation can be interpreted as having matter or energy in another. The definitions cease to follow the basic intuition of general relativity, and a formal regime, or the possibility to define different regimes, is needed. Generally matter and charge models will be the complement of the vacuum, which is different from general relativity.

3.6 Field Equations And Torsion

The detailed formulas for dealing with Christoffel symbols, torsion and other mathematical necessities are in the Appendix. Exploration of various new field equations is undertaken in the sequel given the new cylinder condition which is very tight and the extra degrees of freedom given by torsion.

Torsion is necessary to free up degrees of freedom after they have been reduced substantially by the new cylinder condition.

The field equations are dependent on the matter model regimes that are to be defined.

4 Postulates Of The Kaluza-Cartan Theory

In this section an axiomatization of the theory in terms of postulates, assumptions, notes and other definitions is presented. K1-K7, together with limit K8, constitute the core of the theory proper. The matter model regime then chosen constitutes a variation in the theory proper from one Kaluza-Cartan theory to another.

4.1 Kaluza-Cartan Space-Time Definition

A definition of Kaluza-Cartan Space-Time, or Kaluza-Cartan Space, follows. K7 is the new KCC:

Core Definitions and Postulates:

(K1) A Kaluza-Cartan manifold is a 5D smooth Lorentzian manifold.

(K2) One spatial dimension is topologically closed and small, the Kaluza dimension. There is a global unit vector that defines this direction and they form closed non-intersecting loops.

(K3) The other spatial dimensions and time dimension are large.

(K4) There is a connection that is a metric torsion connection with respect to the geometry, this is the torsion connection.

(K5) Kaluza-Cartan Space is assumed globally hyperbolic in the sense that there exists a 3D spatial cauchy surface plus time, extended in the obvious way via the new cylinder condition into 5D.

(K6) Kaluza-Cartan Space is oriented.

(K7) The covariant derivative (with respect to the torsion tensor) of all tensors and pseudotensors in the Kaluza direction are zero. This is the covariant derivative with lower index: $\hat{\nabla}_4$.

(K8) A relatively weak curvature limit will be assumed. The metric will be approximately flat. This will in effect define the classical limit.

The new KCC given here determines that local charts are possible with vanishing partial derivatives for all tensors and that a Kaluza atlas (3.1.1) is possible. That is, partial derivatives with lower index: ∂_4 are all zero. What we can do now is take a loop by following the Kaluza direction and forming a 1-dimensional submanifold for every point of space-time. By inspecting the bundles inherited by each loop submanifold we can observe that at every point they are necessarily static. As a result ∂^4 and $\hat{\nabla}^4$ must be vanishing for all tensors.

4.2 The Scalar Field, Torsion And Observation

In the sequel a justification (somewhat empirical - equation 6.4.1) is given for restricting the scalar field and accordingly the metric, maximal atlas and Kaluza atlas, as necessary:

(B1) $\phi^2 = 1$

(B2) The torsion tensor will be given the following well-behaved limit: It is to be weakly completely antisymmetric under B2 - A *weak antisymmetric limit* . This will be invoked only as required in the definition of certain matter model regimes. As with B1 its use will be delayed until necessary. It thus takes on an empirical quality. B2(a): Approaching this limit, symmetric parts of contorsion tensor (and their first derivatives) are treated as 'small' relative to typical electromagnetic terms, and any antisymmetric parts of the contorsion tensor. It is claimed that such a limit may be approached without loss of generality of the solutions from a physical perspective. In other words at the K8 limit (see S10 for related points), equation (5.1.9) is compatible with the weak antisymmetric limit, and poses no constraints due to the potential being treated as small as a metric component. As such we may approximately and with caution apply certain results associated with complete antisymmetry to this limit, noting that such results must be taken to be approached only approximately and not absolute. B2(b): We also require that the torsion term $\hat{K}_{ab}{}^{c}$ is not disproportionately large as it will later be multiplied by a small term, yet the resulting term is to remain of low significance.

4.3 Secondary Definitions And Points

Some useful observations, definitions and terminology:

(S1) The Kaluza-Cartan vacuum is a Ricci flat region of a Kaluza-Cartan manifold with respect to the torsion connection definition of the Ricci tensor. Similarly the Kaluza vacuum is a Ricci flat region with respect to the Levi-Civita connection. They are different: $\hat{R}_{ab} = 0$ and $\hat{\mathcal{R}}_{ab} = 0$ respectively. Here they are both defined in terms of the geometry implied by the new KCC. There is also the Kaluza space (and hence Kaluza vacuum) of the original Kaluza theory. So the definition of the vacuum, whichever may be being referenced, depends by this definition on the cylinder condition being used in two distinct ways. The Kaluza vacuum in the presence of torsion typically contains fields when the 4D metric is inspected, and will most often not be Ricci flat in 4D space-time. Likewise for the Kaluza vacuum under the original KCC, and for the Kaluza-Cartan vacuum under the new KCC. However, this is the case in three different ways. Other definitions may lead to similar complexities.

(S2) We will define matter model regimes that pick out the definition of vacuum we wish to use and that delimit matter and charge models. Note that the cosmological constant of general relativity could perhaps be included as part of an alternative regime not used here. Energy conditions may potentially also be included. There is scope for alternatives and further development.

(S3) A number of possible other global constraints can be experimented with and applied to matter models, the aim here is to keep the options as broad as possible. Assumptions pertaining to matter models are therefore defined separately within matter model regimes. This separation into different regimes allows for the maximum applicability of the work presented here, and to allow also for experimental input.

(S4) K7 can be used to decompose the entire 5D geometry into a 4D metric, a vector potential and a scalar field when curvatures are not so extreme as to lead to a singularity in the 4D metric. It defines how space-time is a parallel set of submanifolds.

(S5) Singularities resulting from break-down of the S4 decomposition may indicate regions where the theory as presented also breaks down. This could however also be useful in extending the theory to the quantum scale. This is not dealt with further. K8 is used to avoid this eventuality.

(S6) What is allowed as a physical solution needs to be delimited in some way, at the very least to avoid acausality as in classical physics. This can be done by using energy conditions as in relativity for the resultant 4D space-time, at least as long as the decomposition (S4, S5) doesn't break-down. The cylinder condition can then extrapolate that to the whole of the Kaluza-Cartan manifold. Here we will largely not worry about this problem and assume that it can ultimately be subsumed in an experimentally correct matter model regime at a later date.

(S7) Matter and charge models, and matter model regimes, must also be consistent with K7. A realistic elementary charge model would have imaginary Kaluza rest mass and Kaluza velocity, but this does not prevent it satisfying the cylinder condition.

(S8) K7 may well be a limit rather than a fundamental postulate. Similiarly the lack of a scalar field may be true only at a limit. This is not dealt with further. By definition of B2, it is necessary that complete antisymmetry is a limit and that it is not an absolute (due to 5.1.9). Even when it is assumed to be approximately true, it would most likely be just a limit (i.e strictly not always even approximately empiricially true), as a consequence of its crude definition in terms of a limit.

(S9) The new KCC is defined in terms of covariant derivatives instead of the usual frame dependent partial derivatives, and then a restricted Kaluza subatlas is used in which the partial derivatives are also zero. In particular it takes account of torsion.

(S10) A locally flat space limit will be invoked characterised by the vanishing of the first derivatives of metric components, S10(a), but not necessarily the vanishing of second derivatives. This will also correspond to a weak limit Maxwellian electromagnetic field by construction, cf definition (3.3.1). That is, at this limit, there are neither gravitational nor electromagnetic fields, but charge and mass sources remain. It is therefore a stronger notion of locality than is normally used in classical physics, because electromagnetism is formed here from the metric. Terms that would otherwise be significant in this respect, but that are multiplied by a vector potential, will be treated as tending to be 'small' relative to other significant terms via K8. Thus, in particular, in comparison to other more significant charge and 'mass' terms (i.e second derivatives of metric components). S10(b):

$$A_v \partial_w \partial_x g_{yz} \to 0 \qquad A_v \partial_w \partial_x g^{yz} \to 0$$

Similar considerations are used to obtain B2.

(S11) Note that in the general case with torsion, whilst every tensor and pseudo-tensor in sight has covariant derivative in the Kaluza direction of zero with respect to the torsion connection, and similarly partial derivatives of zero likewise, the covariant derivative with respect to the non-torsion derivative is not so constrained.

4.4 Matter And Charge Models

Some notes on matter and charge models:

(M1) Matter and charge models must be consistent with the preceding.

(M2) Simple examples could include black-hole singularities, though these are also not good examples in the general case.

(M3) Net positivity of mass and energy may be imposed by the usual energy conditions or similar extensions in 5D.

(M4) No comment is passed on the fact that the proportions of mass and charge of a realistic elementary charge model yield imaginary Kaluza rest mass and Kaluza proper velocity - this paper proceeds without judging this peculiar result.

(M5) There is a need to make an assumption about the paths of particles (ie small matter and charge models): the Geodesic Assumption, that light particle-like solutions are possible and follow 5D auto-parallels. I.e. geodesics when torsion is not present. This should in any case be the case for spinless particles, as seen in 4D Einstein-Cartan theory. As applied to particles with spin we are making an assumption about spin-torsion coupling that may well be incorrect, though likely correct at a limit.

(M6) Quantization of charge is not dealt with - no less than quantization of energy or momentum. The proper place for this is a quantum theory, or a theory that encompasses the quantum.

(M7) The exact definition of matter and charge in 5D, matter charge models, will be somewhat different from the related concept in general relativity, and, as with 5D vacuum, will depend on the matter model regime selected.

4.5 Matter Model Regimes

This sections defines the different matter model regimes used in this work. They are mutually exclusive in that with respect to any particular Kaluza-Cartan theory only one may be permissible at a time. It is a choice as to which one is being used for any particular theory. It should be noted that generally vacuum is here complementary to matter, unlike in general relativity where energy fields are meaningful which are neither matter fields nor the vacuum. This is possible as these energy fields represent Weyl curvature in the 5D vacuum in Kaluza's original theory, even if in 4D they also inherit Ricci curvature. However, there is some complexity here.

(R0) The basic matter model regime defines matter-charge models as any part of the Kaluza-Cartan manifold that are not Kaluza vacuums (according to definition S1) - analogously to Kaluza's original theory, but in the presence of torsion with respect to the new cylinder condition. R0(a): vacuum is defined as $\hat{\mathcal{R}}_{AB} = 0$. This implies $\hat{\mathcal{G}}_{AB} = 0$. This has the advantage of a ready-made conservation law for the Einstein tensor (without torsion). This is by definition free of Toth charge (3.2.1) and is shown to be free of local Maxwellian charge in (5.3.1).

(R1) The next most obvious definition of matter-charge models would be to define them as any regions that are not Kaluza-Cartan vacuums (according to definition S1). R1(a): vacuum is defined as $\hat{R}_{AB} = 0$. This implies in vacuum that $\hat{G}_{AB} = 0$. Matter models are then the complement of this. However, a conservation law is added in by hand. R1(b): $\hat{\nabla}^C \hat{R}_{AC} = \hat{\nabla}^C \hat{R}_{CB} = 0$. Not only must the vacuum have integrity, but matter models must conserve their mass in some sense. Equation (5.1.9) suggests that non-complete antisymmetry is required of torsion, meaning that a symmetrical component is to be allowed for torsion. This affects R1 and leads to R1(b). R1 will also apply the somewhat heavy-handed constraint R1(c): $\hat{K}_{a4}{}^{c} = 0$ on the entire manifold, to ensure the Kaluza-Cartan vacuum stays free of local Maxwell charge via (5.3.3).

(R2) This matter model regime proposes the paradoxical idea that we are mistaken in looking for matter or charge models, but that what is really required is a spin current model. The vacuum is defined as R2(a): $\hat{V}_{AB} = 0$, see (9.3.2). Matter-charge models, or now spin current models, are defined as the complement of this new concept of vacuum. Empirically expected behaviour is sought approaching the S10 plus B2 limit.

(R3) This matter model regime is simply R1 without R1(c). So we have R3(a): vacuum is defined as $\hat{R}_{AB} = 0$, and R3(b): $\hat{\nabla}^C \hat{R}_{AC} = \hat{\nabla}^C \hat{R}_{CB} = 0$, analogously. Matter is defined as the complement of this. Instead of R1(c) the S10 plus B2 limit is used to approach empirical behaviour, as with R2 above.

(R4) No constraints are applied. The Kaluza-Cartan vacuum is defined as R4(b): $\hat{R}_{AB} = 0$. Matter-charge models are defined as the complement of, contrarily, the Kaluza vacuum (S1). The S10 plus B2 limit is again used to approach empirical behaviour as a limit. The problems this entails, as matter models and certain fields overlap, will be analysed.

(R5) The following constraint will be imposed over the entire Kaluza-Cartan space for this matter model regime R4(a): $\hat{\triangle}_L \hat{\mathcal{R}}_{MB} = \hat{\triangle}_M \hat{\mathcal{R}}_{LB}$. Conceptually the conserved Bel hypothesis (see Appendix on Bel super-energy for motivation) has been used here. The Kaluza-Cartan vacuum is defined as R4(b): $\hat{R}_{AB} = 0$. Matter-charge models are defined as the complement of, contrarily, the Kaluza vacuum (S1). The S10 plus B2 limit is again used to approach empirical behaviour as a limit. This is R4 plus an extra constraint.

Clearly other alternatives are possible with varying degrees of potential validity. R1, R3, R4 and R5 are potentially viable matter model regimes for Kaluza-Cartan theory, R0 and R2 will be shown to be deficient in some way.

4.6 An Apology

Admittedly (M4) through (M7) and the consideration of different matter model regimes show that this theory is (or these theories are) in an early state of development. Many theories pass through such a state, and complete realization from the outset can not be expected. An objective in the foregoing has been to present the postulates that are needed for the theory to be well-defined, or have well-defined parameters, even if incomplete.

Further and better matter model regimes than those given here may well be possible and should be searched for.

Ultimately empirical observation determines validity.

5 The Field Equations

5.1 The Cylinder Condition And Scalar Field

Here we look at how K7 affects the Christoffel symbols of any coordinate system within the Kaluza atlas (using $k = 1$). Given that the Kaluza atlas is reduced in the following way: all partial derivatives of tensors in the Kaluza direction are set to zero. The Appendix (see section containing 9.4.1 and related) contains a reference for Christoffel symbol working both with and without the torsion component.

The following follow from the selection of coordinates that set the partial derivatives in the Kaluza dimension to zero; from K7, the new covariant version of the Kaluza cylinder condition used in this paper; and, from the relationship between these two and the Christoffel symbols given in Wald [7] p33 eqn (3.1.14) as applied to a number of test vectors. Note that there is no hint of symmetry of the (with torsion) Christoffel symbols suggested here. That is, these terms are forced zero by the fact that both the partial derivatives and the covariant derivatives in the Kaluza direction are zero. Cf equation (1.0.4) where the consequences of setting both the partial derivatives and the covariant derivative to zero can be seen on the Christoffel symbols. The new cylinder condition is quite strong and without torsion would lead immediately to insufficiently diverse geometry given the other postulates here.

$$0 = 2\hat{\Gamma}^A_{4c} = \sum_d \hat{g}^{Ad}(\partial_4 g_{cd} + \partial_4\phi^2 A_c A_d + \partial_c\phi^2 A_d - \partial_d\phi^2 A_c) + \hat{g}^{A4}\partial_c\phi^2 - 2\hat{K}_{4c}{}^A \tag{5.1.1}$$

$$0 = 2\hat{\Gamma}^A_{44} = 2\sum_d \hat{g}^{Ad}\partial_4\phi^2 A_d - \sum_d \hat{g}^{Ad}\partial_d\phi^2 + \hat{g}^{A4}\partial_4\phi^2 - 2\hat{K}_{44}{}^A \tag{5.1.2}$$

We have:

$$2\hat{K}_{4c}{}^A = \hat{g}^{Ad}(\partial_c\phi^2 A_d - \partial_d\phi^2 A_c) + \hat{g}^{A4}\partial_c\phi^2 \tag{5.1.3}$$

$$2\hat{K}_{44}{}^A = -\hat{g}^{Ad}\partial_d\phi^2 \tag{5.1.4}$$

Inspecting the first of these equations (5.1.1), and given that $K_{A(BC)} = 0$ (9.2.4), and further applying A=c without summing, we have a constraint on the scalar field in terms of the vector potential that further motivates postulate B1. Here, however, we make a priori use of postulate B1.

The immediate result of this is as follows (using $k=1$):

$$2\hat{K}_{4c}{}^{A} = \hat{g}^{Ad}(\partial_c A_d - \partial_d A_c) \tag{5.1.5}$$

$$2\hat{K}_{44}^{A} = 0 \tag{5.1.6}$$

Giving the contorsion a very clear interpretation here in terms of the electromagnetic field.

$$\hat{K}_{4c}{}^{a} = \frac{1}{2}F^a_c \tag{5.1.7}$$

$$\hat{K}_{4c}{}^{4} = -\frac{1}{2}A^d F_{cd} \tag{5.1.8}$$

In the case of complete antisymmetry of torsion/contorsion, again using (5.1.1), this specialises to:

$$\hat{K}_{4c}{}^{4} = \hat{K}_{c4}{}^{4} = \hat{\Gamma}^4_{c4} = \hat{\Gamma}^4_{4c} = A^d F_{cd} = 0 \tag{5.1.9}$$

This presents too tight a constraint on electromagnetism. For this reason non-completely antisymmetric torsion is allowed. In the general case:

$$\begin{aligned} \hat{K}_{[4c]}{}^{4} + \hat{K}_{(4c)}{}^{4} = \hat{K}_{4c}{}^{4} = -\frac{1}{2}A^d F_{cd} \\ \hat{K}_{c4}{}^{4} = 0 = -\hat{K}_{[4c]}{}^{4} + \hat{K}_{(c4)}{}^{4} \end{aligned} \tag{5.1.10}$$

$$\begin{aligned} \hat{K}_{[4c]}{}^{4} = \hat{K}_{(4c)}{}^{4} = \frac{1}{2}\hat{K}_{4c}{}^{4} = -\frac{1}{4}A^d F_{cd} \\ \hat{K}_{c4}^{4} = \hat{\Gamma}^4_{4c} = 0 \\ \hat{K}_{4c}^{4} = \hat{\Gamma}^4_{c4} = -\frac{1}{2}A^d F_{cd} \end{aligned} \tag{5.1.11}$$

5.2 The First Field Equation With Torsion, $k=1$

The first field equation in this theory is somewhat complicated (5.2.4), but an analysis here will show that Kaluza-Cartan theory and the original Kaluza theory share a limit for certain perpendicular solutions. This analysis also investigates whether we need to use an alternative matter model regime to R0, such as that of R1, in order to obtain a wider range of electromagnetic fields.

Looking at the Ricci tensor, but here with torsion (using equations 9.4.7 repeatedly, and the new KCC as required):

$$\hat{R}_{ab} = \partial_C \hat{\Gamma}^C_{ba} - \partial_b \hat{\Gamma}^C_{Ca} + \hat{\Gamma}^C_{ba}\hat{\Gamma}^D_{DC} - \hat{\Gamma}^C_{Da}\hat{\Gamma}^D_{bC}$$

$$\hat{R}_{ab} = \partial_c \hat{\Gamma}^c_{ba} - \partial_b \hat{\Gamma}^c_{ca} + \hat{\Gamma}^C_{ba} \hat{\Gamma}^D_{DC} - \hat{\Gamma}^C_{Da} \hat{\Gamma}^D_{bC}$$

$$\hat{R}_{ab} = \partial_c \hat{\Gamma}^c_{ba} - \partial_b \hat{\Gamma}^c_{ca} + \hat{\Gamma}^C_{ba} \hat{\Gamma}^d_{dC} - \hat{\Gamma}^C_{da} \hat{\Gamma}^d_{bC} \tag{5.2.1}$$

Doing the same for the without torsion definitions (using equations 9.4.6 repeatedly, and the new KCC as required):

$$\hat{\mathcal{R}}_{ab} = \partial_C \hat{\varGamma}^C_{ba} - \partial_b \hat{\varGamma}^C_{Ca} + \hat{\varGamma}^C_{ba} \hat{\varGamma}^D_{DC} - \hat{\varGamma}^C_{Da} \hat{\varGamma}^D_{bC}$$

$$\hat{\mathcal{R}}_{ab} = \partial_c \hat{\varGamma}^c_{ba} - \partial_b \hat{\varGamma}^c_{ca} + \frac{1}{2}\partial_b (A^d F_{ad}) + \hat{\varGamma}^c_{ba} \hat{\varGamma}^D_{Dc} - \hat{\varGamma}^C_{Da} \hat{\varGamma}^D_{bC} \tag{5.2.2}$$

In the original Kaluza theory the Ricci curvature of the 5D space is set to 0. The first field equation (2.0.3) comes from looking at the Ricci curvature of the space-time that results. Here there is a choice: whether to base the vacuum on a Kaluza vacuum or a Kaluza-Cartan vacuum. Taking as a lead the concept of matter-charge models according to R0 and conservation law (3.2.10), the Kaluza vacuum is investigated first. Setting $\hat{\mathcal{R}}_{ab} = 0$ (ie matter model regime R0) and using the local limit (S10):

$$\mathcal{R}_{ab} = \mathcal{R}_{ab} - \hat{\mathcal{R}}_{ab} = \partial_c \varGamma^c_{ba} - \partial_b \varGamma^c_{ca} - \partial_c \hat{\varGamma}^c_{ba} + \partial_b \hat{\varGamma}^c_{ca} - \frac{1}{2}\partial_b (A^d F_{ad})$$

$$= -\frac{1}{2}\partial_c (A_b F_a{}^c + A_a F_b{}^c) + \frac{1}{2}\partial_b (A_c F_a{}^c + A_a F_c{}^c) - \frac{1}{2}\partial_b (A^d F_{ad})$$

$$= -\frac{1}{2}\partial_c (A_b F_a{}^c + A_a F_b{}^c)$$

$$= -\frac{1}{2} A_b \partial_c F_a{}^c - \frac{1}{2} A_a \partial_c F_b{}^c - \frac{1}{2}(\partial_c A_b) F_a{}^c - \frac{1}{2}(\partial_c A_a) F_b{}^c \tag{5.2.3}$$

The local limit S10 reduces (5.2.3) further. Putting non-S10 local limit terms back in gives:

$$= -\frac{1}{2}\partial_c (A_b F_a{}^c + A_a F_b{}^c) + \varGamma^c_{ba} \varGamma^d_{dc} - \varGamma^c_{da} \varGamma^d_{bc} - \hat{\varGamma}^c_{ba} \hat{\varGamma}^D_{Dc} + \hat{\varGamma}^C_{Da} \hat{\varGamma}^D_{bC}$$

$$= -\frac{1}{2}\partial_c (A_b F_a{}^c + A_a F_b{}^c) + \varGamma^c_{ba} \varGamma^d_{dc} - \varGamma^c_{da} \varGamma^d_{bc}$$
$$- \hat{\varGamma}^c_{ba} \hat{\varGamma}^D_{Dc} + \hat{\varGamma}^c_{Da} \hat{\varGamma}^D_{bc} + \hat{\varGamma}^4_{Da} \hat{\varGamma}^D_{b4}$$

$$= -\frac{1}{2}\partial_c (A_b F_a{}^c + A_a F_b{}^c) + \varGamma^c_{ba} \varGamma^d_{dc} - \varGamma^c_{da} \varGamma^d_{bc}$$
$$-(\varGamma^c_{ba} + \frac{1}{2}(A_b F_a{}^c + A_a F_b{}^c))(\varGamma^d_{dc} + \frac{1}{2}(A_d F_c{}^d + A_c F_d{}^d)) - (\varGamma^c_{ba} + \frac{1}{2}(A_b F_a{}^c + A_a F_b{}^c))(-\frac{1}{2} A^d F_{cd})$$
$$+(\varGamma^c_{da} + \frac{1}{2}(A_d F_a{}^c + A_a F_d{}^c))(\varGamma^d_{bc} + \frac{1}{2}(A_b F_c{}^d + A_c F_b{}^d)) + (\frac{1}{2} F_a{}^c)(-A_d \varGamma^d_{bc} + \frac{1}{2}(\partial_b A_c + \partial_c A_b))$$
$$+(-A_c \varGamma^c_{da} + \frac{1}{2}(\partial_d A_a + \partial_a A_d))(\frac{1}{2} F_b{}^d) + (-\frac{1}{2} A^d F_{ad})(-\frac{1}{2} A^c F_{bc})$$

$$
\begin{aligned}
&= -\frac{1}{2}\partial_c(A_b F_a{}^c + A_a F_b{}^c) \\
&\quad -\frac{1}{2}(A_b F_a{}^c + A_a F_b{}^c)\Gamma^d_{dc} \\
&+\frac{1}{2}\Gamma^c_{da}(A_b F_c{}^d + A_c F_b{}^d) + \frac{1}{2}(A_d F_a{}^c + A_a F_b{}^c)\Gamma^d_{bc} + \frac{1}{2}(A_d F_a{}^c + A_a F_d{}^c)\frac{1}{2}(A_b F_c{}^d + A_c F_b{}^d) \\
&\quad +\frac{1}{2}F_a{}^c(-A_d\Gamma^d_{bc} + \frac{1}{2}(\partial_b A_c + \partial_c A_b)) \\
&\quad +(-A_c\Gamma^c_{da} + \frac{1}{2}(\partial_d A_a + \partial_a A_d))\frac{1}{2}F_b{}^d + \frac{1}{4}A^d F_{ad} A^c F_{bc} \\
&= -\frac{1}{2}A_b\partial_c F_a{}^c - \frac{1}{2}A_a\partial_c F_b{}^c - \frac{1}{2}(\partial_c A_b)F_a{}^c - \frac{1}{2}(\partial_c A_a)F_b{}^c - \frac{1}{2}(A_b F_a{}^c + A_a F_b{}^c)\Gamma^d_{dc} \\
&\quad +\frac{1}{2}\Gamma^c_{da}A_b F_c{}^d + \frac{1}{2}A_a F_b{}^c\Gamma^d_{bc} + \frac{1}{4}(A_d F_a{}^c + A_a F_d{}^c)(A_b F_c{}^d + A_c F_b{}^d) \\
&\quad +\frac{1}{4}F_a{}^c(\partial_b A_c + \partial_c A_b) + \frac{1}{4}(\partial_d A_a + \partial_a A_d)F_b{}^d + \frac{1}{4}A^d F_{ad} A^c F_{bc} \\
&= -\frac{1}{2}A_b\partial_c F_a{}^c - \frac{1}{2}A_a\partial_c F_b{}^c \\
&-\frac{1}{2}(\partial_c A_b)F_a{}^c - \frac{1}{2}(\partial_c A_a)F_b{}^c + \frac{1}{4}F_a{}^c(\partial_b A_c + \partial_c A_b) + \frac{1}{4}(\partial_d A_a + \partial_a A_d)F_b{}^d \\
&\quad -\frac{1}{2}(A_b F_a{}^c + A_a F_b{}^c)\Gamma^d_{dc} + \frac{1}{2}\Gamma^c_{da}A_b F_c{}^d + \frac{1}{2}A_a F_b{}^c\Gamma^d_{bc} \\
&\quad +\frac{1}{4}(A_d F_a{}^c + A_a F_d{}^c)(A_b F_c{}^d + A_c F_b{}^d) + \frac{1}{4}A^d F_{ad} A^c F_{bc} \\
&= -\frac{1}{2}A_b\partial_c F_a{}^c - \frac{1}{2}A_a\partial_c F_b{}^c + \frac{1}{2}F_{ac}F_b{}^c \\
&\quad -\frac{1}{2}(A_b F_a{}^c + A_a F_b{}^c)\Gamma^d_{dc} + \frac{1}{2}\Gamma^c_{da}A_b F_c{}^d + \frac{1}{2}A_a F_b{}^c\Gamma^d_{bc} \\
&\quad +\frac{1}{4}(A_d F_a{}^c + A_a F_d{}^c)(A_b F_c{}^d + A_c F_b{}^d) + \frac{1}{4}A^d F_{ad} A^c F_{bc}
\end{aligned}
\tag{5.2.4}
$$

The electrovacuum terms for a perpendicular electromagnetic field can be seen embedded in this equation as the third term, this shows that we are not producing a completely new theory from Kaluza's original theory. Kaluza-Cartan theory has a limit in common with Kaluza theory. However, if the charge terms are ignored then there is a lack in the above equation of likely significant terms to provide any other type of solution, non-perpendicular electromagnetic fields in particular.

For this reason we can try an alternative matter model regime such as R1 and the Kaluza-Cartan vacuum in order to obtain a fuller range of geometries.

Now, doing the same with respect to R1, and conservation law R1(b), so that we are defining vacuum as a Kaluza-Cartan vacuum (and matter models as strictly complement to vacuum) according to S1 ($\hat{R}_{AB} = 0$) - and using (5.2.1) - gives a still more complicated picture:

$$\mathcal{R}_{ab} = \mathcal{R}_{ab} - \hat{R}_{ab} = \partial_c \varGamma^{c}_{\ ba} - \partial_b \varGamma^{c}_{\ ca} + \varGamma^{c}_{\ ba}\varGamma^{d}_{\ dc} - \varGamma^{c}_{\ da}\varGamma^{d}_{\ bc}$$
$$-\partial_c \hat{\Gamma}^c_{ba} + \partial_b \hat{\Gamma}^c_{ca} - \hat{\Gamma}^C_{ba}\hat{\Gamma}^d_{dC} + \hat{\Gamma}^C_{da}\hat{\Gamma}^d_{bC} \tag{5.2.5}$$

Detailing each term here without a specific point to make is not profitable, is lengthy, and shall not be undertaken.

There is a limit in common between R0 and R1, towards the S10 local limit, when there is no appreciable $\hat{K}_{aB}{}^{C}$ (noting that perpendicular solutions under R0 need no such torsion). More generally allowing torsion terms of the form $\hat{K}_{ab}{}^{c}$ allows for non-perpendicular electromagnetic fields under R1.

With the exception of R0 all the matter model regimes are weaker than the constraint imposed by R0, so that as a constraint on non-perpendiclar solutions, this first field equation can be ignored. It's only R0 that creates a problem here, that restricts solutions to perpendicular solutions.

5.3 The Second Field Equation With Torsion

Rederivation of the second field equation under the new KCC:

$$\hat{\mathcal{R}}_{a4} = \partial_C \hat{\varGamma}^{C}_{\ 4a} - \partial_4 \hat{\varGamma}^{C}_{\ Ca} + \hat{\varGamma}^{C}_{\ 4a}\hat{\varGamma}^{D}_{\ DC} - \hat{\varGamma}^{C}_{\ Da}\hat{\varGamma}^{D}_{\ 4C}$$
$$= \partial_c \hat{\varGamma}^{c}_{\ 4a} + \hat{\varGamma}^{c}_{\ 4a}\hat{\varGamma}^{D}_{\ Dc} - \hat{\varGamma}^{c}_{\ Da}\hat{\varGamma}^{D}_{\ 4c} = \partial_c \hat{\varGamma}^{c}_{\ 4a} + \hat{\varGamma}^{c}_{\ 4a}\hat{\varGamma}^{d}_{\ dc} - \hat{\varGamma}^{c}_{\ da}\hat{\varGamma}^{d}_{\ 4c}$$
$$= \frac{1}{2}\partial_c F^c_a + \frac{1}{2}F^c_a \varGamma^{d}_{\ dc} + \frac{1}{4}F^c_a A^d F_{cd} - \frac{1}{2}(\varGamma^{c}_{\ da} + \frac{1}{2}(A_d F^c_a + A_a F^c_d))F^d_c$$

Looking at this locally (note S10) so that 1st derivatives are vanishing, but second derivatives remain (re-inserting general k):

$$\hat{\mathcal{R}}_{a4} = \frac{k}{2}\partial_c F^c_a \tag{5.3.1}$$

This couldn't be a clearer conception of Maxwell charge locally. Setting this to 0 in Kaluza vacuum would assure us that Maxwell charge is locally restricted to matter-charge models, however, matter model regime R0 has already been discounted as not providing sufficient solutions.

By definition (and the new KCC, and 9.4.7), we immediately get:

$$\hat{R}_{a4} = 0 \tag{5.3.2}$$

Whereas $\hat{R}_{4b}$ simplifies at the S10 local limit to:

$$\hat{R}_{4b} = \frac{1}{2}\partial_c F^c_b - \partial_c \hat{K}_{b4}{}^{c} + \partial_b \hat{K}_{c4}{}^{c} + \hat{K}_{b4}{}^{c}\hat{K}_{dc}{}^{d} - \hat{K}_{d4}{}^{c}\hat{K}_{bc}{}^{d} \tag{5.3.3}$$

This is also locally conserved Maxwell charge given no $\hat{K}_{a4}{}^{c}$, that is if R1(c) is satisfied under matter model regime R1. This limit is also approached when B2 is approached.

These calculations can equally be used for the other matter model regimes.

5.4 The Third Field Equation With Torsion, $k = 1$

This section shows how torsion releases the constraint of the third torsionless field equation (2.0.5), thus allowing non-perpendicular solutions. The constraint that the Ricci tensor be zero leads to no non-perpendicular solutions in the original Kaluza theory. This is caused by setting $\hat{R}_{44} = 0$ in that theory and observing the terms. The result is that (when the scalar field is constant) $0 = F_{cd}F^{cd}$ in the original Kaluza theory. The same issue arises here:

We have:

$$\begin{aligned}\hat{\mathcal{R}}_{44} &= \partial_C \hat{\Gamma}^{C}_{44} - \partial_4 \hat{\Gamma}^{C}_{C4} + \hat{\Gamma}^{C}_{44}\hat{\Gamma}^{D}_{DC} - \hat{\Gamma}^{C}_{D4}\hat{\Gamma}^{D}_{4C} \\ &= 0 - 0 + 0 - \hat{\Gamma}^{C}_{D4}\hat{\Gamma}^{D}_{4C} = -\hat{\Gamma}^{c}_{d4}\hat{\Gamma}^{d}_{4c} \\ &= -\frac{1}{4}F^{c}_{d}F^{d}_{c} \end{aligned} \tag{5.4.1}$$

The result is that whilst we can have non-perpendicular solutions, we can only have them outside of a strict Kaluza vacuum, contradicting R0. This suggests that the perpendicular solutions are those which correlate with a Kaluza vacuum, whilst the non-perpendicular solutions arise in Kaluza-Cartan vacuum.

By definition (and the new KCC, and 9.4.7), we immediately get:

$$\hat{R}_{44} = 0 \tag{5.4.2}$$

Thus no such limit is placed on matter model regime R1.

R1, therefore, is an example valid theory satisfying broadly the requirements demanded of it, provided a Lorentz force law is derived and the relationship between Toth charge and Maxwell charge clarified, as will be done in a later section.

The other mater models similarly leaves open the possibility for non-perpendicular electromagnetic fields here. There is no reason for (5.4.1) to be 0 in these cases, and so non-perpendicular solutions become possible.

6 The Lorentz Force Law

Toth [8] derives a Lorentz-like force law where there is a static scalar field and Kaluza's cylinder condition applies in the original Kaluza theory. The resulting 'charge' is the momentum term in the fifth dimension and it was not apparent how this related to the Maxwell current, except as Toth states via 'formal equivalence'. While this result was not new, Toth's calculation is extended here for the new theory and clarification of the issue is obtained. Here we make use of the Geodesic Assumption M5. First the identification of Toth charge and Maxwell charge is investigated.

6.1 Toth Charge

Now to investigate the relationship between Toth charge and Maxwell charge. For this we need the limit defined by both S10 and K8.

Using Appendix (9.7.3), at the defined limit S10 and K8:

$$\hat{\mathcal{G}}^{a4} = \hat{\mathcal{R}}^{a4} - \frac{1}{2}\hat{g}^{a4}\hat{\mathcal{R}} = \hat{\mathcal{R}}^{a4} - \frac{1}{2}(-A^a)\hat{\mathcal{R}} \to \hat{\mathcal{R}}^{a4}$$

$$\hat{\mathcal{R}}^{a4} = \partial_C \hat{\digamma}^{C4a} - \partial^4 \hat{\digamma}^{C\ a}_{\ C} + \hat{\digamma}^{Cba}\hat{\digamma}^{D}_{\ DC} - \hat{\digamma}^{C\ a}_{\ D}\hat{\digamma}^{Db}_{\ \ C}$$

$$\hat{\mathcal{G}}^{a4} \to \hat{\mathcal{R}}^{a4} = \partial_c \hat{\digamma}^{c4a} \tag{6.1.1}$$

Applying the local Maxwell flat space-time limit, and putting k back in, and by using Appendix equation (9.6.1) for the Christoffel symbol, we get:

$$\hat{\mathcal{R}}^{a4} \to \frac{1}{2}\partial_c k F^{ac} \tag{6.1.2}$$

And so by (3.2.10),

$$J^*_a \to -\frac{\alpha k}{2}\partial_c F_a^{\ c} \tag{6.1.3}$$

So Toth and Maxwell charges are related by a simple formula. The right hand side being Maxwell's charge current (see p.81 of [6]), and has the correct sign to identify a positive Toth charge Q^* with a positive Maxwell charge source $4\pi Q_M$, whenever $\alpha k > 0$. In the appropriate space-time frame, and Kaluza atlas frame, and at the appropriate limit, using (3.2.9):

$$4\pi Q_M = +\frac{2}{\alpha k}Q^* \tag{6.1.4}$$

6.2 A Lorentz-Like Force Law

The Christoffel symbols and the geodesic equation are the symmetric ones defined in the presence of totally antisymmetric torsion. We will here initially use $k = 1$, a general k can be added in later.

$\hat{\Gamma}^c_{(4b)} = \frac{1}{2}g^{cd}(\delta_4\hat{g}_{bd} + \delta_b\hat{g}_{4d} - \delta_d\hat{g}_{4b}) + \frac{1}{2}\hat{g}^{c4}(\delta_4\hat{g}_{b4} + \delta_b\hat{g}_{44} - \delta_4\hat{g}_{4b}) =$
$\frac{1}{2}g^{cd}[\delta_b(\phi^2 A_d) - \delta_d(\phi^2 A_b)] + \frac{1}{2}g^{cd}\delta_4\hat{g}_{bd} + \frac{1}{2}\hat{g}^{c4}\delta_b\hat{g}_{44} =$

$\frac{1}{2}\phi^2 g^{cd}[\delta_b A_d - \delta_d A_b] + \frac{1}{2}g^{cd}A_d\delta_b\phi^2 - \frac{1}{2}g^{cd}A_b\delta_d\phi^2 + \frac{1}{2}g^{cd}\delta_4\hat{g}_{bd} + \frac{1}{2}\hat{g}^{c4}\delta_b\phi^2 =$
$\frac{1}{2}\phi^2 F^c_b + \frac{1}{2}g^{cd}A_d\delta_b\phi^2 - \frac{1}{2}g^{cd}A_b\delta_d\phi^2 + \frac{1}{2}g^{cd}\delta_4\hat{g}_{bd} + \frac{1}{2}\hat{g}^{c4}\delta_b\phi^2 =$
$\frac{1}{2}\phi^2 F^c_b - \frac{1}{2}g^{cd}A_b\delta_d\phi^2 + \frac{1}{2}g^{cd}\delta_4\hat{g}_{bd} = \frac{1}{2}\phi^2 F^c_b \quad -\frac{1}{2}g^{cd}A_b\delta_d\phi^2$

(6.2.1)

$\hat{\Gamma}^c_{44} = \frac{1}{2}\hat{g}^{\mathrm{cD}}(\delta_4\hat{g}_{4D} + \delta_4\hat{g}_{4D} - \delta_D\hat{g}_{44}) = \text{-}\ \frac{1}{2}g^{\mathrm{cd}}\delta_d\phi^2$

(6.2.2)

We have:

$\hat{\Gamma}^c_{(ab)} = \frac{1}{2}g^{cd}(\delta_a g_{db} + \delta_b g_{da} - \delta_d g_{ab})$
$+\frac{1}{2}g^{cd}(\delta_a(\phi^2 A_d A_b) + \delta_b(\phi^2 A_a A_d) - \delta_d(\phi^2 A_a A_b)) + \frac{1}{2}\hat{g}^{c4}(\delta_a\hat{g}_{4b} + \delta_b\hat{g}_{4a} - \delta_4\hat{g}_{ab})$
$= \Gamma^c_{(ab)} \quad + \frac{1}{2}g^{cd}(\delta_a(\phi^2 A_d A_b) + \delta_b(\phi^2 A_a A_d) - \delta_d(\phi^2 A_a A_b))$
$-A^c(\delta_a\phi^2 A_b + \delta_b\phi^2 A_a)$

(6.2.3)

So, for any coordinate system within the maximal atlas:

$0 = \frac{d^2x^a}{d\tau^2} + \hat{\Gamma}^a_{(BC)}\frac{dx^B}{d\tau}\frac{dx^C}{d\tau}$
$= \frac{d^2x^a}{d\tau^2} + \hat{\Gamma}^a_{(bc)}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} + \hat{\Gamma}^a_{(4c)}\frac{dx^4}{d\tau}\frac{dx^c}{d\tau} + \hat{\Gamma}^a_{(b4)}\frac{dx^b}{d\tau}\frac{dx^4}{d\tau} + \hat{\Gamma}^a_{44}\frac{dx^4}{d\tau}\frac{dx^4}{d\tau}$
$= \frac{d^2x^a}{d\tau^2} + \hat{\Gamma}^a_{(bc)}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} + (\phi^2 F^a_b - g^{ad}A_b\delta_d\phi^2)\frac{dx^b}{d\tau}\frac{dx^4}{d\tau} - \frac{1}{2}g^{\mathrm{ad}}\delta_d\phi^2\frac{dx^4}{d\tau}\frac{dx^4}{d\tau}$

(6.2.4)

Taking $\phi^2 = 1$ and the charge-to-mass ratio to be:

$$Q'/m_{k0} = \frac{dx^4}{d\tau} \tag{6.2.5}$$

We derive a Lorentz-like force law:

$$\frac{d^2x^a}{d\tau^2} + \hat{\Gamma}^a_{(bc)}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} = -(Q'/m_{k0})F^a_b\frac{dx^b}{d\tau} \tag{6.2.6}$$

Putting arbitrary k and variable ϕ back in we have:

$$\frac{d^2x^a}{d\tau^2} + \hat{\Gamma}^a_{(bc)}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} = -k(Q'/m_{k0})(\phi^2 F^a_b - g^{ad}A_b\delta_d\phi^2)\frac{dx^b}{d\tau} - \frac{1}{2}g^{\mathrm{ad}}\delta_d\phi^2\frac{dx^4}{d\tau}\frac{dx^4}{d\tau}$$
(6.2.7)

6.3 Constant Toth Charge

Having derived a Lorentz-like force law we look also at the momentum of the charge in the Kaluza dimension. We look at this acceleration as with the Lorentz force law. We have, with torsion (and $k = 1$):

$$0 = \frac{d^2x^4}{d\tau^2} + \hat{\Gamma}^4_{(BC)}\frac{dx^B}{d\tau}\frac{dx^C}{d\tau}$$

$$= \frac{d^2x^4}{d\tau^2} + \hat{\Gamma}^4_{(bc)}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} + \hat{\Gamma}^4_{(4c)}\frac{dx^4}{d\tau}\frac{dx^c}{d\tau} + \hat{\Gamma}^4_{(b4)}\frac{dx^b}{d\tau}\frac{dx^4}{d\tau} + \hat{\Gamma}^4_{44}\frac{dx^4}{d\tau}\frac{dx^4}{d\tau}$$

$$= \frac{d^2x^4}{d\tau^2} + \hat{\Gamma}^4_{(bc)}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} + 2\hat{\Gamma}^4_{(4c)}\frac{dx^4}{d\tau}\frac{dx^c}{d\tau} + \frac{1}{2}A^d\delta_d\phi^2\frac{dx^4}{d\tau}\frac{dx^4}{d\tau} \tag{6.3.1}$$

6.4 Unitary Scalar Field And Torsion

Both equations above (6.2.7) and (6.3.1) have a term that wrecks havoc to any similarity with the Lorentz force law proper, the terms at the end. Both terms can however be eliminated by setting the scalar field to 1. This is postulate B1. This leads to torsion as a way to allow for non-perpendicular electromagnetic solutions.

The two equations under B1 become (for all k):

$$\frac{d^2x^a}{d\tau^2} + \hat{\Gamma}^a_{(bc)}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} = -k(Q'/m_{k0})F^a_b\frac{dx^b}{d\tau} \tag{6.4.1}$$

$$\frac{d^2x^4}{d\tau^2} + \hat{\Gamma}^4_{(bc)}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} = -k^2(Q'/m_{k0})A_cF^c_b\frac{dx^b}{d\tau} \tag{6.4.2}$$

This certainly looks more hopeful. The more extreme terms have disappeared, the general appearance is similar to the Lorentz force law proper. The right hand side of (6.4.2) is small, but in any case the well-behaved nature of charge follows from local momentum conservation and the required integrity of charge models - that they do not lose charge to the vacuum.

6.5 The Lorentz Force Law

It is necessary to confirm that equation (6.4.1) not only looks like the Lorentz force law formally, but is indeed the Lorentz force law. Multiplying both sides of (6.4.1) by $\frac{d\tau}{d\tau'}\frac{d\tau}{d\tau'}$, where τ' is an alternative affine coordinate frame, gives:

$$\frac{d^2x^a}{d\tau'^2} + \hat{\Gamma}^a_{(bc)}\frac{dx^b}{d\tau'}\frac{dx^c}{d\tau'} = -k\frac{d\tau}{d\tau'}(Q'/m_{k0})F^a_b\frac{dx^b}{d\tau'} \tag{6.5.1}$$

Given $Q^* = Q'\frac{d\tau}{d\tau^*}$ and therefore $\frac{m_{k0}}{m_0}Q^* = Q'\frac{d\tau}{dt_0}$ by definition, we can set the frame such that $\tau' = t_0$ via the projected 4D space-time frame of the charge. And the Lorentz force is derived:

$$\frac{d^2x^a}{d\tau'^2} + \hat{\Gamma}^a_{(bc)}\frac{dx^b}{d\tau'}\frac{dx^c}{d\tau'} = -k(Q^*/m_0)F^a_b\frac{dx^b}{d\tau'} \tag{6.5.2}$$

In order to ensure the correct Lorentz force law using the conventions of Wald [7] p69, this can be rewritten as follows, using the antisymmetry of $F^a_b = -F^a{}_b$:

$$= k(Q^*/m_0)F^a{}_b\frac{dx^b}{d\tau'} \tag{6.5.3}$$

Using (6.1.4) this can be rewritten again in terms of the Maxwell charge:

$$= k(\frac{\alpha k}{2}(4\pi Q_M)/m_0)F^a{}_b\frac{dx^b}{d\tau'} \tag{6.5.4}$$

The result is that we must relate $\mathbf{G}$ and k to obtain the Lorentz force law in acceptable terms:

$$\frac{d^2x^a}{d\tau'^2} + \hat{\Gamma}^a_{(bc)}\frac{dx^b}{d\tau'}\frac{dx^c}{d\tau'} = (Q_M/m_0)F^a_b\frac{dx^b}{d\tau'}$$

$$k = 2\sqrt{\mathbf{G}} \tag{6.5.5}$$

This shows that the Lorentz force law proper can be derived, but provides a constraint in so doing.

7 Analysis Of Matter Model Regimes

7.1 R0 and Integrity

R0 has the advantage of a ready-made conservation law for the Einstein tensor (without torsion) and the Kaluza vacuum has integrity in that it is a region without a matter or Toth charge source by definition, and this can also be inspected locally (5.3.3). However it does not provide sufficient variety of solutions as shown in relation to the first field equation (5.2.4).

What is meant by integrity here is that the definition is consistent and applicable in certain ways. It is necessary that a vacuum can not gain mass or charge without becoming itself a matter-charge model, othewise it is no vacuum. Similarly matter-charge models can have integrity by the same reasoning, if they do not lose their matter or charge to the vacuum - only to other matter charge model regions. Further, conservation laws need to apply if the words 'matter' and 'charge' are to mean anything. Integrity is a sufficient level of well-behaved properties so that the vacuum or matter model in question behaves as intended. This is necessarily defined and analysed on a case-by-case basis.

7.2 R1 - A First Solution

Maxwell charge is (locally, i.e with respect to S10) restricted to matter-charge models by (5.3.3) under R1, provided R1(c) is satisfied. The Kaluza-Cartan vacuum can not have local (under S10) Maxwell charge. R1(b) gives both the Kaluza-Cartan vacuum (S1) and associated matter models well-behaved properties in relation to matter. Under both R0 and R1 the charge terms $\hat{\mathcal{R}}_{a4}$ and $\hat{R}_{4a}$ are locally the same, the same local Maxwell charge. Both vacuums are well-behaved locally in that their respective vacuums exclude local Maxwell charge and matter by definition. And both have integrity for their respective matter models, as matter and charge is conserved, and both are restricted to presence within matter models. However, a problem with R1 is that the medicine seems worse than the disease. The complexity of the artificially added constraints, R1(b) and R1(c), seem even more convoluted than the original problem of simply having a stress energy tensor for electromagnetism added to general relativity. Nevertheless R1 is a demonstration of concept, that is, of the objective of this work in finding an alternative viable theory. Whereas R0 fails immediately. Search for a more elegant matter model regime is undertaken with the other matter model regimes.

7.3 R2 - Looking At Charge And Spin

The concept behind R2 is that it is a mistaken plan to give a definition of vacuum that has integrity in terms of the Ricci tensor, with or without torsion, since it may not be conserved if we no longer have R1(b). By defining the vacuum (and the spin current models - which replace the idea of matter charge models) in terms of the conserved spin current tensor V^{AB} the vacuum (and spin current models) can be given a different sort of integrity. The alternative conservation law is given in equation (9.3.5) and ensures that vacuum remains vacuum, and a matter charge model (or at least a 'spin' model) also has integrity in this sense.

Spin current can be identified with charge. This is done via B2 a *weak antisymmetric limit* (see Appendix for notes on how complete antisymmetry affects torsion equations). Not that any claim is made that complete antisymmetry holds in general - on the contrary problems would arise with respect to equation (5.1.9). The claim is that equation (5.1.1) et al place no constraint on the contorsion tensor that would prevent any *symmetric* parts being small relative to the electromagnetic field tensor. Two approximative conservation laws result: (9.3.7), (9.3.9). The S10 local limit will also be invoked, and K8 that defines weak fields. The result is the appearance of Maxwell charge as the significant term in (9.3.9) via (5.3.3) and (5.3.2) - approximately conserved (relative to the torsion connection) and with approximate integrity of both vacuum and charge models. The spin current being fully conserved and having full integrity with respect to both matter models and vacuum, by virtue of the unusual definition of matter models and vacuum assumed here. Maxwell charge will always tend to find equilibrium upon returning to the above limit (S10 and K8, plus B2) even in the event that the approximate limit is significantly violated as there is an equality binding an exact amount of Maxwell charge to an exact amount of spin current components via (9.3.9), (5.3.3) and (5.3.2), at that limit, and near that limit, and howsoever the limit arises. The history of the spin-charge does not change the end result back at the limit. It has in that sense a form of invariance. This makes physical sense as charge is not found without spin in reality, yet there is nothing here preventing spin occuring without charge - as also is the case in nature. This argument can be used with respect to other matter model regimes and need not be repeated.

Equation (5.3.3) can be written at this S10 plus B2 limit, using (5.1.7), as:

$$\hat{R}_{4b} = \frac{1}{2}\partial_c F^c_b - \partial_c(-\frac{1}{2}F^c_b)$$

So by (5.3.2) and (5.3.1), and still at this limit,

$$\hat{R}_{[4b]} = \frac{1}{2}\partial_c F^c_b = \hat{\mathcal{R}}_{b4} \tag{7.3.1}$$

As with charge we can try to do the same under R2 with matter. This might use (9.3.7), or rather (9.3.6), where at the weak antisymmetric limit a conservation law in terms of the torsion connection exists for the Einstein tensor (defined without torsion). Certainly by (9.3.9) spin current at the S10 and K8, plus B2 limit, requires Ricci curvature.

Note that at the relevant limit, S10 and K8, using (9.7.3):

$$\hat{\mathcal{G}}_{4A} = \hat{\mathcal{G}}_{A4} \to \hat{\mathcal{R}}_{A4} = \hat{\mathcal{R}}_{4A} \tag{7.3.2}$$

And this can be compared with (7.3.1).

This does not apply when derivatives of these terms are taken, as in conservation laws, where extra terms will be present (due to S10). However we may assume these extra terms to be small due to approximate consistency with respect to the Einstein tensor conservation law.

The torsionless Einstein tensor is conserved with respect to the Levi-Civita connection, and so mass-energy in this sense is conserved. The true, underlying matter-energy conservation law under R2 would be: (3.2.10), whilst the true, underlying charge and spin conservation law under R2 together would be: (9.3.5). R2 is still not quite right, however, as the 5D vacuum is defined such that chargeless/spinless matter (torsionless Ricci curvature terms) are generally present. Thus, while charge has integrity with respect to the vacuum, and while both charge and matter-energy are conserved when given the right interpretation, the matter models themselves do not have integrity.

This problem is sourced in the recurring consideration that the 5D vacuum needs to have non-Ricci flat curvature with respect to the torsionless definition in order to have the sought for non-perpendicular solutions (5.2.4) and this is most easily investigated by defining the Kaluza-Cartan vacuum as in R1, yet R2 does not resolve the problem either, yet at the same time we need that 5D matter can not leak into the Kaluza-Cartan vacuum, even if the Kaluza vacuum has energy in 4D, otherwise the matter models will lack internal consistency.

7.4 R3 - Making The Best Of R1 And R2

R3 is then simply R1 but with the above treatment of charge and spin replacing R1(c). The B2 limit is once again essential, but need not be true in general. And the integrity of the vacuum and matter models once again dealt with by the application of a somewhat artificial conservation law: R3(b). Resolving the recurring problem of integrity of matter models once again (as with R1) by force.

7.5 R4 - The Simplest Option

R4 attempts to remove this somewhat artificial constraint, R1(b) and R3(b) - by living with the consequences of its removal. The two fundamental conservation laws remain in force: (3.2.10) and (9.3.5). The B2 and S10 limits can be applied as above.

Spin current and charge are conserved as for R2, both being aligned approaching the relevant limit, and converging at that limit. Kaluza-Cartan vacuum can not contain spin current by definition. This is sufficient integrity for the Kaluza-Cartan vacuum and its complement with respect to charge and spin current.

Matter models do not here automatically have integrity with respect to spin and charge on account of the vacuum, as they are not here strictly the complement of the Kaluza-Cartan vacuum that has the integrity. Only at the relevant limit is there charge integrity via the equality of (5.3.1) and (5.3.3), and again using (9.3.9). The two fundamental conservation laws, however, ensure that upon returning to, or approaching, the relevant limit, that nothing has in fact been lost or gained.

The integrity of these matter models with respect to 5D (torsionless) mass-energy remains the problem. What is preventing a charged particle emitting its (torsionless) mass-energy to the Kaluza-Cartan vacuum and simply never getting it back? The Kaluza-Cartan vacuum has zero torsion-bearing Ricci tensor $\hat{R}_{AB}$, but does not generally have zero Ricci tensor $\hat{\mathcal{R}}_{AB}$ (without torsion), or 5D mass-energy (without torsion) $\hat{\mathcal{G}}_{AB}$. Recall (7.3.2). This is the core problem, as this is the relevant conserved quantity given the lack of R1(b) or R3(b).

In the case that the Kaluza vacuum and Kaluza-Cartan vacuum coincide, which is limited to certain perpendicular electromagnetic fields and pure gravitational fields, no problem arises. Perhaps the problem for when Kaluza-Cartan vacuums and Kaluza vacuums do not coincide actually represents some underlying physical reality? There is after all an actual and real energy transfer (and hence mass transfer) mediated via fields. It is only the relativistic rest mass that must not be leaked away, only the 4D rest mass that needs ‘integrity’. For charged particles this follows from Toth charge being defined as 5D (without torsion) momentum and the corresponding components of the (torsionless) Einstein tensor, and that by (6.1.4), at the S10 limit, Toth charge and Maxwell charge can be identified. The Kaluza momentum then must constitute the 4D rest mass. Therefore at the relevant limit (including B2) charge can further be identified with components of the conserved spin current. By M6 we are not concerned with quantization issues, and so for charged particles this is sufficient integrity. We could assume quantization if required however.

For a neutral particle the rest mass would be at least the net Kaluza kinetic energy of two (or more) opposing orientation charged particles, whose Kaluza momentums cancel out. In the R4 matter model regime whether the rest of the energy can dissipate across space is then an open question requiring further investigation or further definition by a specific matter model. The stability of neutral particles such as the neutron, or the stability of the mass difference between a proton and a positron would fall into this category. The possibility of dissipation would be exemplified in particle/anti-particle anihilations. In this way R4 treats any a priori integrity of matter as resulting from the properties of charge and spin, the rest to be dealt with by the matter models in question. The matter models also deal with any concerns (such as quantum mechanical issues) outside the scope of Kaluza-Cartan theory.

In summary, matter which is directly correlated with charge and/or spin has the sought for integrity under R4, and other 'energy' does not (and can be transmitted via fields). This is perhaps quite an unexpected result of the analysis. It limits the integrity of matter to matter directly associated with such charge and/or spin. This is not without analogs in quantum mechanics, but nevertheless delegates rather more responsibility for the matter model to the matter model in question than might be hoped for in a classical theory.

7.6 Alternatives

Further, we might consider adding to R4 constraints that ensure a degree of causality or other have other desirable/required characteristics, such as energy conditions. R5 adds a conservation of super-energy constraint that ensures a form of causality (see Appendix).

Other matter model regimes might be constructed to deal with the cosmological constant and other important empirical issues. Further analysis and or empirical testing may further differentiate the different matter model regimes.

8 Conclusion

Kaluza's 1921 theory of gravity and electromagnetism using a fifth wrapped-up spatial dimension is at the root of many modern attempts to develop new physical theories. However for a number of reasons it is sometimes considered untenable.

A new cylinder condition was imposed as with Kaluza's original theory, but one based on the covariant derivative and associated with a metric torsion connection. A generally covariant definition. A number of other constraints and definitions were provided. The result was the appearance of the missing non-perpendicular electromagnetic fields and a new definition of charge in terms of the 5D momentum. The new definition of kinetic charge and the Maxwellian charge coincide at an appropriate limit. In order to obtain non-perpendicular electromagnetic fields it was necessary to generalize matter-charge models. No single method for doing so was necessarily correct, so a number of matter model regimes were constructed.

Restrictions to the geometry and certain symmetries were handled by reducing the maximal atlas to a reduced Kaluza atlas that automatically handled the restrictions and symmetries without further deferment to general covariance. Physically this represents the idea that in 4D, charge is a generally covariant scalar, whereas in 5D, charge is entirely dependent on the frame. That this is meaningful stems from the global property of a small wrapped-up fifth spatial dimension with new cylinder condition. Mathematically the Kaluza atlas is a choice of subatlas for which the partial derivatives in the Kaluza direction are vanishing. This led to useful constraints on the Christoffel symbols for all coordinate systems in the Kaluza atlas.

Decomposition of the 5D metric into a 4D metric and a vector and scalar part is also possible.

One outstanding issue is that realistic charge models are not possible without involving imaginary numbers. However, that problem does not extend to 4D space-time and all the imaginary numbers actually disappear as soon as they are interpreted with respect to space-time. Thus no actual contradiction with experiment (or internal theoretical contradiction) need arise. The 4D construction can be investigated independently of the 5D model. Barring realistic charge models which pose peculiar challenges, the postulates actually required are straight forward. It is in this sense a simple theory. In effect all we have is a 5D manifold with a covariant cylinder condition on one spatial dimension defined with respect to an approximately completely antisymmetric metric torsion tensor limit, with weak fields and curvature.

Given certain assumptions about matter-charge models, the various matter model regimes, the entirety of classical electrodynamics is rederived. Gravity and electromagnetism are unified in a way not fully achieved by general relativity, Einstein-Cartan theory or Kaluza's original 5D theory.

The collocation of torsion with electromagnetism is different from other Einstein-Cartan theories where the torsion is limited to within matter models. Here certain specific components of torsion are an essential counterpart of electromagnetism, and other components of torsion can exist outside of matter models, depending on the matter model regime.

Due to the lack of realistic charge models, the open question regarding matter model regimes, and certain other considerations, this theory remains at an early stage of development, though the essential ingredients are present. Other issues that remain outstanding include the exact role of spin in the dynamics of particles, and a fuller exploration of the energy conditions and causality. Much work from Einstein-Cartan theory could probably be carried over or extended - similarly for Kaluza-Klein theories. An attempt to provide a new approach to causality was presented by using Bel super-energy.

The choice of a correct matter model regime has here been left an open question, though obviously the fourth or a variant of the fourth (such as the fifth) is preferred on the grounds of simplicity. It is hoped however that by such over-exemplification that feasibility of a solution to the problem posed with respect to Kaluza's original theory has been amply demonstrated. At the very least the possibility of further alternative approaches to variant gravitational or classical theories has been proven. The implied connection between spin current, the two different definitions of charge, and the Einstein (torsionless) tensor, and matter, that came out of the analysis of the last few matter model regimes, was particularly interesting.

The choice of matter model regime does not really change the postulates of the theory per se. The full theory is given by K1-8 (core postulates), plus B1 and B2 (empirically justified postulates), and then the matter model regime (for example R4) delimits the nature of empirical matter within the Kaluza theory framework, with further details coming from the particular matter model itself, not having been specified.

Why go to all this effort to unify electromagnetism and gravitation and to make electromagnetism fully geometric? Because experimental differences should be detectable given sufficient technology on the one hand, or, on the other, and equally, simply because such an attempt at unification might be right. This theory differs from both general relativity and Einstein-Cartan theory, and this may be empirically fruitful. Also the expected ω-consistency of Einstein-Cartan theory together with the derivation of a Lorentz force law via the Kaluza part of the theory gives a unique theoretical motivation, as does the fact that the other approaches beyond general relativity have not fulfilled their promise. Further, attempting to extend and unify classical theory prior to a full unification with quantum mechanics may even be a necessary step in a future unification, whether this turns out to be the right way or not. It may be that current attempts are more difficult than necessary as the question may not yet have been framed correctly.

It is often asserted that the true explanation for gravitational theory and space-time curvatures will most likely, by reductionist logic, emerge out of its constituent quantum phenomena. Such an approach has merit, but is overly optimistic. Before constituent quantum parts can be properly defined, the larger scale whole must have been present at first to then be so divided. On the contrary, a Euclidean cake may need to be sliced differently from a Riemannian cake if each slice is to be equal, and lumpiness has been induced by a gravitational field. The dividing and putting together of parts assumes a context, and a context assumes a whole, though in daily life we take our conception of space-time for granted, we take the whole as given. When comparing quantum mechanics and general relativity we should not. Something of the context is evidently missing from quantum mechanics on account of the difficulty in squaring the two worldviews, and therefore possibly from both. Thus reductionism assumes the whole, or contextual knowledge about that whole, before it is even applied. There is paradoxically an inherently non-reductionist assumption, or presumption, within that worldview. This is why the contrary approach of a purely classical context-seeking theory has been attempted.

Quantum mechanics has been hugely successful, so it is perhaps natural to try to find gravity by some extension of the quantum picture, a quantum gravity theory, that presumably would not need to unify gravity and electromagnetism, or would seek to do so within a quantum context. However success of a theory alone is not a fair test of the entire methodological outlook that led to that theory. Success in one domain may be the result of our bias and talents and opportunities, rather than nature's bias. But it is nature's bias that science seeks to uncover. In so doing, in the long run, our own bias may also be laid bare, and this can be doubly useful.

The bias of this work has perhaps been non-reductionist, classical, and non-quantum. In attempting to seek a geometrical unification of two forces that are considered well-understood by modern physics an old, and by now fairly unpopular program, has in effect been re-attempted.

9 Appendix

9.1 Geometrized Charge

The geometrized units, Wald [7], give a relation for mass in terms of fundamental units. This leads to an expression for Toth charge in terms of Kaluza momentum when $k = 2$ and $\mathbf{G} = 1$.

$$\mathbf{G} = 1 = 6.674 \times 10^{-8} cm^3 g^{-1} s^{-2} = 6.674 \times 10^{-8} cm^3 g^{-1} \times (3 \times 10^{10} cm)^{-2}$$
$$= 6.674 \times 10^{-8} cm g^{-1} \times (3 \times 10^{10})^{-2}$$

$$1g \approx 7.42 \times 10^{-29} cm \text{ for c=1, } \mathbf{G}\text{=1} \tag{9.1.1}$$

$$1g \approx G/c^2 cm \text{ for c=1, } \mathbf{G}\text{=1} \tag{9.1.2}$$

For $k = 2$, c=1, $\mathbf{G}$=1 we have:

$$1statC = 1cm^{3/2} s^{-1} g^{1/2} = cm^{1/2} \times (7.42 \times 10^{-29} cm)^{1/2} / (3.00 \times 10^{10})$$

$$= 8.61 \times 10^{-15} cm/(3 \times 10^{10}) \approx 2.87 \times 10^{-25} cm \approx 3.87 \times 10^3 g \tag{9.1.3}$$

Using cgs (Gaussian) units and the cgs versions of $\mathbf{G}$ and c, ie $\mathbf{G} = 6.67 \times 10^{-7} cm^3 g^{-1} s^{-1}$ and $c = 3 \times 10^{10} cms^{-1}$, the charge can be written in terms of 5D proper momentum P_4 as follows:

$$1statC = 1cm^{3/2} s^{-1} g^{1/2} = 1(cm/s) cm^{1/2} g^{1/2} = \frac{c}{\sqrt{\mathbf{G}}} g.cm/s$$

$$Q^* = \frac{c}{\sqrt{\mathbf{G}}} P_4 \tag{9.1.4}$$

9.2 Introducing The Geometry Of Torsion

5D Cartan torsion will be admitted. This will provide extra and required degrees of freedom since the new cylinder condition (K7) would be too tight to yield interesting geometry otherwise. It is noted that Einstein-Cartan theory, that adds torsion to the dynamics of relativity theory is most probably a minimal ω-consistent extension of general relativity [13][14] and therefore the use of torsion is not only natural, but arguably a necessity on philosophical and physical grounds. That argument can also be applied here. What we have defined by this addition can be called Kaluza-Cartan theory as it takes Kaluza's theory and adds torsion. We assume that the torsion connection is metric.

For both 5D and 4D manifolds (i.e. dropping the hats and indices notation for a moment), torsion will be introduced into the Christoffel symbols as follows, using the notation of Hehl [11]. Metricity of the torsion tensor will be assumed [19], the reasonableness of which (in the context of general relativity with torsion) is argued for in [20] and [21]:

$$\frac{1}{2}(\Gamma^k_{ij} - \Gamma^k_{ji}) = S_{ij}{}^k \tag{9.2.1}$$

This relates to the notation of Kobayashi and Nomizu [12] and Wald [7] as follows:

$$T^i{}_{jk} = 2S_{jk}{}^i \equiv \Gamma^i_{jk} - \Gamma^i_{kj} \tag{9.2.2}$$

We have the contorsion tensor $K_{ij}^{\;\;k}$ [11] as follows, and a number of relations [11]:

$$\Gamma^k_{ij} = \frac{1}{2}g^{kd}(\partial_i g_{dj} + \partial_j g_{di} - \partial_d g_{ij}) - K_{ij}{}^k = \{^{\,k}_{ij}\} - K_{ij}{}^k \tag{9.2.3}$$

$$K_{ij}{}^k = -S_{ij}{}^k + S_j{}^k{}_i - S^k{}_{ij} = -K_i{}^k{}_j \tag{9.2.4}$$

Notice how the contorsion is antisymmetric in the last two indices.

With torsion included, the auto-parallel equation becomes [11]:

$$\frac{d^2x^k}{ds^2} + \Gamma^k_{(ij)}\frac{dx^i}{ds}\frac{dx^j}{ds} = 0 \tag{9.2.5}$$

$$\Gamma^k_{(ij)} = \{^{\,k}_{ij}\} + S^k{}_{(ij)} - S_{(j}{}^k{}_{i)} = \{^{\,k}_{ij}\} + 2S^k{}_{(ij)} \tag{9.2.6}$$

Only when torsion is completely antisymmetric is this the same as the extremals [11] which give the path of spinless particles and photons in Einstein-Cartan theory: extremals are none other than geodesics with respect to the Levi-Civita connection.

$$\frac{d^2x^k}{ds^2} + \{^{\,k}_{ij}\}\frac{dx^i}{ds}\frac{dx^j}{ds} = 0 \tag{9.2.7}$$

When completely antisymmetric we have many simplifications such as:

$$K_{ij}^{\ \ k} = -S_{ij}^{\ \ k} \tag{9.2.8}$$

9.3 Stress-Energy Tensors And Conservation Laws

The stress-energy tensor for a torsion bearing non-symmetric connection in Einstein-Cartan theory is usually labelled $\kappa\hat{P}_{AB}$, it need not be symmetric. In the literature the constant κ is included analogously to the 8π in general relativity. Here we will use the Einstein tensor $\hat{G}_{AB}$, taking a purely geometrical view.

The Belinfante-Rosenfeld [15] stress-energy tensor $\hat{B}$ is a symmetric adjustment of $\kappa\hat{P}$ that adjusts for spin currents as sources for Riemann-Cartan spaces. It can be defined equally for the 5D case. It is divergence free. It is the torsion equivalent according to Belinfante and Rosenfeld of the original Einstein tensor $\hat{G}$ [12] in some sense: the Einstein-Hilbert action. However the usual definition in terms of stress-energies and Noether currents, rather than the Einstein tensor, is not appropriate here. In effect repeating the Belinfante-Rosenfeld procedure, by defining the torsionless Einstein tensor in terms of torsion bearing components, yields what can be interpreted as extra spin-spin coupling term $\hat{X}_{AB}$:

$$\hat{\mathcal{G}}_{AB} = \hat{G}_{AB} + \hat{V}_{AB} + \hat{X}_{AB} \tag{9.3.1}$$

$$\hat{V}_{AB} = \frac{1}{2}\hat{\nabla}^C(\hat{\sigma}_{ABC} + \hat{\sigma}_{BAC} + \hat{\sigma}_{CBA}) \tag{9.3.2}$$

Where σ is defined as the spin tensor in Einstein-Cartan theory. However, here we do not start with spin (and some particle Lagrangians), but with the torsion tensor. So instead the spin tensor is defined in terms of the torsion tensor using the Einstein-Cartan equations. Here spin is explicitly defined in terms of torsion:

$$\hat{\sigma}_{ABC} = 2\hat{S}_{ABC} + 2\hat{g}_{AC}\hat{S}_{BD}^{\ \ \ D} - 2\hat{g}_{BC}\hat{S}_{AD}^{\ \ \ D} \tag{9.3.3}$$

This simplifies definition (9.3.2):

$$\hat{V}_{AB} = \frac{1}{2}\hat{\nabla}^C(\hat{\sigma}_{CBA}) = \hat{\nabla}^C(\hat{S}_{CBA} + \hat{g}_{CA}\hat{S}_{BD}^{\ \ \ D} - \hat{g}_{BA}\hat{S}_{CD}^{\ \ \ D}) \tag{9.3.4}$$

By considering symmetries and antisymmetries we get a conservation law:

$$\hat{\nabla}_B\hat{V}^{AB} = 0 \tag{9.3.5}$$

The Case Of Complete Anytisymmetry

Note that the mass-energy-charge conservation law for the torsionless Einstein tensor is in terms of the torsionless connection, but the spin source conservation law here is in terms of the torsion-bearing connection. However, for completely antisymmetric torsion we have:

$$\hat{\nabla}_C \hat{\mathcal{G}}_{AB} = \hat{\triangle}_C \hat{\mathcal{G}}_{AB} + \hat{K}_{CA}{}^{D} \hat{\mathcal{G}}_{DB} + \hat{K}_{CB}{}^{D} \hat{\mathcal{G}}_{AD}$$

So,

$$\hat{\nabla}^A \hat{\mathcal{G}}_{AB} = 0 + 0 + \hat{K}^{A}{}_{B}{}^{D} \hat{\mathcal{G}}_{AD} = -\hat{K}_{B}{}^{AD} \hat{\mathcal{G}}_{AD}$$
$$= -\hat{K}_{B}{}^{AD} \hat{\mathcal{G}}_{DA} = +\hat{K}_{B}{}^{DA} \hat{\mathcal{G}}_{DA} = +\hat{K}_{B}{}^{AD} \hat{\mathcal{G}}_{AD} = 0 \qquad (9.3.6)$$

$$\hat{\nabla}^A (\hat{G}_{AB} + \hat{X}_{AB}) = 0 \qquad (9.3.7)$$

And so there is a stress-energy conservation law with respect to the torsion connection also, at least in the completely antisymmetric case.

Further, still assuming complete antisymmetry of torsion, by definition of the Ricci tensor:

$$\hat{R}_{AB} = \hat{\mathcal{R}}_{AB} + \hat{K}_{DA}{}^{C} \hat{K}_{BC}{}^{D} - \partial_C \hat{K}_{BA}{}^{C} - \hat{K}_{BA}{}^{C} \hat{\digamma}^{D}{}_{DC} + \hat{K}_{DA}{}^{C} \hat{\digamma}^{D}{}_{DC} - \hat{K}_{DB}{}^{C} \hat{\digamma}^{D}{}_{AC}$$
$$= \hat{\mathcal{R}}_{AB} - \hat{K}_{AD}{}^{C} \hat{K}_{BC}{}^{D} - \hat{\nabla}^C \hat{S}_{ABC} \qquad (9.3.8)$$

$$\hat{G}_{[AB]} = \hat{R}_{[AB]} = -\hat{\nabla}^C \hat{S}_{ABC} = \hat{V}_{AB} \qquad (9.3.9)$$

$\hat{V}_{AB}$ is the antisymmetric part of $-\hat{G}_{AB}$. And $\hat{X}_{AB}$ is a symmetric spin-torsion coupling adjustment - again only in the case of completely antisymmetric torsion.

In the general case such identities and conservation laws are less easily found.

9.4 The Christoffel Symbols

Here we assume only the definitions of the Christoffel symbols and the new KCC. (Without torsion terms shown, k set to 1)

$$2\hat{\digamma}^{A}{}_{BC} = \sum_d \hat{g}^{AD} (\partial_B \hat{g}_{CD} + \partial_C \hat{g}_{DB} - \partial_D \hat{g}_{BC})$$

$$= \sum_d \hat{g}^{Ad} (\partial_B \hat{g}_{Cd} + \partial_C \hat{g}_{dB} - \partial_d \hat{g}_{BC})$$
$$+ \hat{g}^{A4} (\partial_B \hat{g}_{C4} + \partial_C \hat{g}_{4B} - \partial_4 \hat{g}_{BC})$$

$$2\hat{\digamma}^{A}{}_{bc} = \sum_d \hat{g}^{Ad} (\partial_b g_{cd} + \partial_c g_{db} - \partial_d g_{bc})$$

$$+\sum_d \hat{g}^{Ad}(\partial_b\phi^2 A_c A_d + \partial_c\phi^2 A_d A_b - \partial_d\phi^2 A_b A_c)$$
$$+\hat{g}^{A4}(\partial_b\phi^2 A_c + \partial_c\phi^2 A_b - \partial_4 g_{bc} - \partial_4\phi^2 A_b A_c)$$

$$2\hat{\Gamma}^A_{4c} = \sum_d \hat{g}^{Ad}(\partial_4 g_{cd} + \partial_4\phi^2 A_c A_d + \partial_c\phi^2 A_d - \partial_d\phi^2 A_c) + \hat{g}^{A4}\partial_c\phi^2$$

$$2\hat{\Gamma}^A_{44} = 2\sum_d \hat{g}^{Ad}\partial_4\phi^2 A_d - \sum_d \hat{g}^{Ad}\partial_d\phi^2 + \hat{g}^{A4}\partial_4\phi^2$$

(9.4.1)

The Electromagnetic Limit $\phi^2 = 1$

$$2\hat{\Gamma}^A_{bc} = \sum_d \hat{g}^{Ad}(\partial_b g_{cd} + \partial_c g_{db} - \partial_d g_{bc}) + \sum_d \hat{g}^{Ad}(\partial_b A_c A_d + \partial_c A_d A_b - \partial_d A_b A_c)$$
$$+\hat{g}^{A4}(\partial_b A_c + \partial_c A_b - \partial_4 g_{bc} - \partial_4 A_b A_c)$$
$$2\hat{\Gamma}^A_{4c} = \sum_d \hat{g}^{Ad}(\partial_4 g_{cd} + \partial_4 A_c A_d + \partial_c A_d - \partial_d A_c)$$
$$\hat{\Gamma}^A_{44} = \sum_d \hat{g}^{Ad}\partial_4 A_d$$

(9.4.2)

Simplifying...

$$2\hat{\Gamma}^a_{bc} = 2\Gamma^a_{bc} + \sum_d g^{ad}(A_b F_{cd} + A_c F_{bd}) + A^a\partial_4 g_{bc} + A^a\partial_4 A_b A_c$$
$$2\hat{\Gamma}^4_{bc} = -\sum_d A^d(\partial_b g_{cd} + \partial_c g_{db} - \partial_d g_{bc}) - \sum_d A^d(A_b F_{cd} + A_c F_{bd})$$
$$-(1 + \sum_i A_i A^i)(\partial_4 g_{bc} + \partial_4 A_b A_c) + (\partial_b A_c + \partial_c A_b)$$
$$2\hat{\Gamma}^a_{4c} = \sum_d g^{ad}(\partial_4 g_{cd} + \partial_4 A_c A_d) + \sum_d g^{ad}F_{cd}$$
$$2\hat{\Gamma}^4_{4c} = -\sum_d A^d(\partial_4 g_{cd} + \partial_4 A_c A_d) - \sum_d A^d F_{cd}$$
$$\hat{\Gamma}^a_{44} = \sum_d g^{ad}\partial_4 A_d$$
$$\hat{\Gamma}^4_{44} = -\sum_d A^d\partial_4 A_d$$

(9.4.3)

The Scalar Limit $A_i = 0$

$$2\hat{\Gamma}^A_{bc} = \sum_d \hat{g}^{Ad}(\partial_b g_{cd} + \partial_c g_{ab} - \partial_d g_{bc}) - \hat{g}^{A4}\partial_4 g_{bc}$$
$$2\hat{\Gamma}^A_{4c} = \sum_d \hat{g}^{Ad}\partial_4 g_{cd} + \hat{g}^{A4}\partial_c\phi^2$$
$$2\hat{\Gamma}^A_{44} = -\sum_d \hat{g}^{Ad}\partial_d\phi^2 + \hat{g}^{A4}\partial_4\phi^2$$

(9.4.4)

Simplifying...

$$\hat{\Gamma}^a_{bc} = \Gamma^a_{bc}$$
$$2\hat{\Gamma}^4_{bc} = -\frac{1}{\phi^2}\partial_4 g_{bc}$$
$$2\hat{\Gamma}^a_{4c} = \sum_d g^{ad}\partial_4 g_{cd}$$
$$2\hat{\Gamma}^4_{4c} = \frac{1}{\phi^2}\partial_c\phi^2$$
$$2\hat{\Gamma}^a_{44} = -\sum_d g^{ad}\partial_d\phi^2$$
$$2\hat{\Gamma}^4_{44} = \frac{1}{\phi^2}\partial_4\phi^2$$

$$(9.4.5)$$

The Electromagnetic Limit And Cylinder Condition

By applying equations (5.1.11) and the new KCC in order to simplify terms of the electromagnetic limit with $k = 1$, at first omitting the torsion contribution (noting that these Christoffel symbols are symmetric in the lower indices):

$$\begin{aligned}
&2\hat{\mathit{\Gamma}}^{a}_{bc} = 2\mathit{\Gamma}^{a}_{bc} + (A_b F_c{}^a + A_c F_b{}^a)\\
&2\hat{\mathit{\Gamma}}^{4}_{bc} = -\textstyle\sum_d A^d(\partial_b g_{cd} + \partial_c g_{db} - \partial_d g_{bc}) + (\partial_b A_c + \partial_c A_b)\\
&\qquad = -2A_d \mathit{\Gamma}^{d}_{bc} + (\partial_b A_c + \partial_c A_b)\\
&2\hat{\mathit{\Gamma}}^{a}_{4c} = F_c{}^a\\
&\hat{\mathit{\Gamma}}^{4}_{4c} = -\tfrac{1}{2}A^d F_{cd} \text{ and } \hat{\mathit{\Gamma}}^{a}_{44} = \hat{\mathit{\Gamma}}^{4}_{44} = 0
\end{aligned} \tag{9.4.6}$$

Now with torsion built into the Christoffel symbols, using equations (5.1.11) and others from that section, and noting that these Christoffel symbols are not necessarily symmetric in the lower indices:

$$\begin{aligned}
&2\hat{\Gamma}^{a}_{bc} = g^{ad}(\partial_b g_{cd} + \partial_c g_{db} - \partial_d g_{bc}) + (A_b F_c{}^a + A_c F_b{}^a) - 2\hat{K}_{bc}{}^a\\
&\qquad = 2\mathit{\Gamma}^{a}_{bc} + (A_b F_c{}^a + A_c F_b{}^a) - 2\hat{K}_{bc}{}^a\\
&2\hat{\Gamma}^{4}_{bc} = -\textstyle\sum_d A^d(\partial_b g_{cd} + \partial_c g_{db} - \partial_d g_{bc}) + (\partial_b A_c + \partial_c A_b)\\
&\qquad - A^d(A_b F_{cd} + A_c F_{bd}) - 2\hat{K}_{bc}{}^4\\
&\qquad = -2A_d \mathit{\Gamma}^{d}_{bc} + (\partial_b A_c + \partial_c A_b) - A^d(A_b F_{cd} + A_c F_{bd}) - 2\hat{K}_{bc}{}^4\\
&\hat{\Gamma}^{a}_{4c} = 0 \text{ and } \hat{\Gamma}^{4}_{4c} = \hat{\Gamma}^{a}_{44} = \hat{\Gamma}^{4}_{44} = 0\\
&\hat{\Gamma}^{4}_{c4} = -\tfrac{1}{2}A^d F_{cd}\\
&\hat{\Gamma}^{a}_{c4} = \tfrac{1}{2}F_c{}^a - \hat{K}_{c4}{}^a
\end{aligned} \tag{9.4.7}$$

9.5 The Generalized Bel Super-Energy Tensor

The Generalized Bel tensor for a Lorentz manifold (or simply Bel tensor) is the super-energy tensor associated with the Riemannian curvature [17]. The definition of super-energy tensor does not require that torsion be vanishing in either the connection or any of the defining tensors [17], and the important dominant super-energy property [17] follows in all cases. This leads to the causality of the Rieman tensor [16] under specific conditions without deference to energy conditions. The super-energy tensor definition depends on the antisymmetries of the Riemannian tensor definition, that is [17], that it is a double symmetric (2,2)-form. The definition of the super-energy tensor with respect to basic properties such as it being a 4-tensor are dependent on the admissibility of the interpretation of the Riemann tensor as a (2,2)-form.

Now the Riemann tensor can be written as [12]:

$$R^i_{jkl} = \partial_k \Gamma^i_{lj} - \partial_l \Gamma^i_{kj} + \Gamma^m_{lj}\Gamma^i_{km} - \Gamma^m_{kj}\Gamma^i_{lm} \tag{9.5.1}$$

It is a (2,2)-form if its antisymmetries are as follows: $R_{[ij][kl]}$. This is clearly the case for [k,l]. For [i,j] it is a known result provided that the torsion-bearing connection is metric. The argument requires the torsion analog of Wald's equation (3.2.12) [7] and then follows for the same reasons as given there for the torsionless case.

In [16] the derivation of the causality of the fields underlying any particular super-energy tensor is given in terms of the divergence of the field's super-energy tensor. A divergence condition is given that ensures causality of the underlying field associated with any such super-energy tensor. The divergence of the generalized Bel tensor would therefore need to be bounded by this condition if the Riemannian curvature were to remain causal. This condition is theorem 4.2 in [17].

A sufficient case would be if the divergence of the superenergy tensor were zero (and assuming global hyperbolicity). The important details are on page 4 of [16]. The argument does not require that the connection be torsion free. Thus the vanishing divergence of the generalized Bel tensor would yield causal Riemannian curvature, assuming the Riemann tensor remained a (2,2)-form (as indicated above), with no deferment to energy conditions in both the case when torsion is used to define the Bel tensor and when it isn't

On p24 of [17] we have a calculation of this divergence in the torsion-free case, and it can be seen that when the Ricci curvature is zero that the divergence of super-energy is also zero. This however references symmetry properties (in addition to antisymmetry properties) and thus further consideration of the case with torsion would be required to extend or generalize this theorem. Theorem 6.1 on p25 of [17] may well not apply in the case that the tensors and connection are defined in terms of torsion. Nevertheless it nicely characterizes an important property of the Kaluza vacuum, that it can not be a source of Bel super-energy with respect to 5D definitions.

The Conserved Bel Hypothesis will be that the divergence of the Generalized Bel superenergy tensor be vanishing (when defined with respect to the torsionless connection and torsion free tensors) over all of the Kaluza-Cartan space-time.

We might also have *the nearly Conserved Bel Hypothesis* such that the divergence of the Generalized Bel superenergy tensor be bounded (when defined with respect to the torsionless connection and torsion free tensors) over all of the Kaluza-Cartan space-time. Thus including matter/charge models and torsion. The bound here being defined by theorem 4.2 of [17].

These 2 tentative suggestions would have explanatory power if shown to be correct as a way of rationalizing the various energy conditions used in general relativity, and in explaining some aspects of classical scale causality. It could also be carried forward to genuinely 5D extended theories, that is theories that require the weakening of the new cylinder condition. Work would need to be done on these hypotheses to clarify them, and that is not the program here.

It can be noted that in 4D and 5D in particular p29 of [17] the generalized Bel tensor (torsion not considered) has the nice property of being completely symmetric. It is curious that it should be completely symmetric precisely in the 4D and 5D cases.

Theorem 6.1 of [17] though not proven for its torsion analog would link divergence of super-energy with what Senovilla et al [17] call the matter current.

The divergence of the Bel super-energy is given as follows [17] (not using the hat or index notation to indicate dimensionality, and not considering torsion):

$$\nabla_a B^{ablm} = R^{b\ l}_{\ r\ s} J^{msr} + R^{b\ m}_{\ r\ s} J^{lsr} - \frac{1}{2} g^{lm} R^{b}_{\ rsy} J^{syr} \tag{9.5.2}$$

$$J_{lmb} = -J_{mlb} \equiv \nabla_l R_{mb} - \nabla_m R_{lb} \tag{9.5.3}$$

Given $J_{lmb} = 0$, which implies conservation of Bel super-energy, we then have (remembering that this has not been proven when torsion is used in the definitions):

$$\nabla_4 R_{mb} = \nabla_m R_{4b} \tag{9.5.4}$$

So, using the full hat notation again as used throughout this text (ie terms without torsion), we have, when the so-called matter current is zero:

$$\hat{\triangle}_L \hat{\mathcal{R}}_{MB} = \hat{\triangle}_M \hat{\mathcal{R}}_{LB} \tag{9.5.5}$$

Further, by applying the new cylinder condition, when the torsion is explicitly vanishing:

$$\hat{\nabla}_4 \hat{\mathcal{R}}_{AB} = \hat{\triangle}_A \hat{\mathcal{R}}_{4B} = 0 \tag{9.5.6}$$

However we are really interested in the case where torsion defines the new cylinder condition in which case in general (see note S11):

$$\hat{\nabla}_4 \hat{\mathcal{R}}_{AB} \neq \hat{\triangle}_4 \hat{\mathcal{R}}_{AB} = \hat{\triangle}_A \hat{\mathcal{R}}_{4B} \neq 0 \tag{9.5.7}$$

This is quite counter-intuitive, but as such potentially leads to unexpected considerations.

9.6 Raised Levi-Civita Christoffel Symbols

Following the same procedure as with Christoffel symbols of the first and second kind a raised version of the Christoffel symbols can be derived, a third kind. It takes the value that would be guessed at (a guess because you can not raise across partial derivatives without caution) by inspecting the elements of the Christoffel symbol of the second kind and raising each element individually.

Starting from the covariant derivative of the raised metric tensor being zero:

$$\nabla^i g^{jk} = 0 = \partial^i g^{jk} + \Gamma^{jik} + \Gamma^{kij}$$

Cycling indices we have:

$$0 = \partial^k g^{ij} + \Gamma^{ikj} + \Gamma^{jki}$$
$$0 = \partial^j g^{ki} + \Gamma^{kji} + \Gamma^{ijk}$$

Adding the first two and subtracting the third:

$$\partial^i g^{jk} + \partial^k g^{ij} - \partial^j g^{ki} = 2\Gamma^{jik}$$

So, exactly as would be guessed:

$$\Gamma^{ijk} = \frac{1}{2}(\partial^j g^{ik} + \partial^k g^{ji} - \partial^i g^{jk}) \tag{9.6.1}$$

9.7 Local Ricci Scalar Curvature, $k = 1$

Here we calculate the Ricci scalar curvature for $k = 1$ at the local limit defined by vanishing first derivatives of metric components (see S10), with Ricci curvature defined relative to the torsionless connection. Using (5.2.2), (5.3.1) and (5.4.1):

$$|\hat{\mathcal{R}}_{AB}| \to \begin{bmatrix} \partial_c \hat{\Gamma}^c_{\ ba} - \partial_b \hat{\Gamma}^c_{\ ca} & \frac{1}{2}\partial_c F_a^{\ c} \\ \frac{1}{2}\partial_c F_a^{\ c} & 0 \end{bmatrix} \tag{9.7.1}$$

The Ricci scalar can now be calculated as follows, the apparent asymmetry being caused by the limit terms:

$$|\hat{g}^{DA}||\hat{\mathcal{R}}_{AB}| = |\hat{\mathcal{R}}^D_B|$$

$$\to \begin{bmatrix} g^{da}(\partial_c \hat{\Gamma}^c_{\ ba} - \partial_b \hat{\Gamma}^c_{\ ca}) - A^d \frac{1}{2}\partial_c F_b^{\ c} & \frac{1}{2}\partial_c F^{dc} \\ -A^a(\partial_c \hat{\Gamma}^c_{\ ba} - \partial_b \hat{\Gamma}^c_{\ ca}) + \frac{1}{2}\partial_c F_b^{\ c}(\frac{1}{\phi^2} + A_i A^i) & -A^a \frac{1}{2}\partial_c F_a^{\ c} \end{bmatrix} \tag{9.7.2}$$

$$\hat{\mathcal{R}} \to g^{da}(\partial_c \hat{\Gamma}^c_{\ ba} - \partial_b \hat{\Gamma}^c_{\ ca}) - A^d \frac{1}{2}\partial_c F_d^{\ c} - A^a \frac{1}{2}\partial_c F_a^{\ c} = g^{da}(\partial_c \hat{\Gamma}^c_{\ ba} - \partial_b \hat{\Gamma}^c_{\ ca}) - A^d \partial_c F_d^{\ c}$$

$$\hat{\mathcal{R}} \to g^{da}(\partial_c \hat{\Gamma}^c_{\ ba} - \partial_b \hat{\Gamma}^c_{\ ca}) - A^d \partial_c F_d^{\ c} \tag{9.7.3}$$

9.8 Proper Kaluza Velocity As Space-Time Scalar

This section shows that the proper velocity $\mathbf{W}$ (written as a vector) with only one component in the Kaluza dimension is invariant under 4D space-time boosts orthogonal to it.

$W_4 = dx_4/d\tau$ proper velocity in a stationary space-time frame, but following the particle

$$U_4 = \frac{W_4}{\sqrt{1+{W_4}^2}}$$ coordinate velocity using proper velocity formula

Using orthogonal addition of coordinate velocities formula to boost space-time frame by orthogonal coordinate velocity $\mathbf{V}$:

$$\mathbf{V} = (V, 0, 0, 0)$$
$$\mathbf{U} = (0, 0, 0, U_4)$$

Coordinate velocity vector in new frame, using the orthogonal velocity addition formula:

$$\overline{\mathbf{U}} = \mathbf{V} + \sqrt{1 - V^2}\ \mathbf{U}$$

So,

$$\overline{U}_4 = \sqrt{1 - V^2}\frac{W_4}{\sqrt{1+W_4^2}}$$

Define proper velocity in new frame: $\overline{\mathbf{W}}$, using proper velocity definition:

$$\begin{aligned}
\overline{W}_4 &= \frac{\overline{U}_4}{\sqrt{1 - V^2 - {\overline{U}_4}^2}} \\
&= \frac{\sqrt{1-V^2}\frac{W_4}{\sqrt{1+W_4^2}}}{\sqrt{1 - V^2 - (\sqrt{1-V^2}\frac{W_4}{\sqrt{1+W_4^2}})^2}} \\
&= \frac{W_4}{\sqrt{1+W_4^2}\sqrt{1 - (\frac{W_4}{\sqrt{1+W_4^2}})^2}} \\
&= \frac{W_4}{\sqrt{1+W_4^2 - W_4^2}} = W_4
\end{aligned} \tag{9.8.1}$$

$\overline{W_4} = W_4$ is the result required

The proper Kaluza velocity therefore is a scalar with respect to the Kaluza atlas.

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11 References

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