On the Growth of Meromorphic Solutions of a type of Systems of Complex Algebraic Differential Equations*

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Abstract This paper is concerned with the growth of meromorphic solutions of a class of systems of complex algebraic differential equations. A general estimate the growth order of solutions of the systems of differential equation is obtained by Zalacman Lemma. We also take an example to show that the result is right.
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1 Introduction and Main Results

We use the standard notation of the Nevanlinna theory of meromorphic functions (see .e.g.[1], [2]).

In 1998, W. Bergweiler [3] considered the order of the solutions of complex differential equation $(f')^n = P[f]$, where $P[f](z) = \sum_{r\in I} a_r(z,f)(f')^{r_1}\cdots(f^{(n)})^{r_n}$, $a_r(z,f)$ is a rational function in z and f,I is a finite index set.

He proved the following result

Theorem A[2] Let $w(z)$ be any meromorphic solution of algebraic differential equation (2), $n > u$, then the growth order $\sigma(w)$ of $w(z)$ are finite.

In 2008, Su X.F. and Gao L.Y. [4] investigated the order of the solutions of a type of the systems of higher-order complex algebraic differential equations as follows:

$$\begin{cases} (w_2^{(n)})^{m_1} = a(w_1 + c(z))^p, \\ (w_1^{(n)})^{m_2} = \frac{\Omega(w_2)}{\Omega_k(w_2)}, \end{cases} \tag{1.1}$$

where $\Omega_j(w_2) = \sum_{j=0}^{q} b_j(z)(w_2)^{j_0}(w_2')^{j_1}(w_2^n)^{j_n}$ is a differential polynomial, $b_j(z)$ $(j = 0, 1, 2, \cdots, n)$ is a polynomial, $\Omega_k(w_2) = (w_2')^{k_1}\cdots(w_2^{(n)})^{k_n}$ is a differential monomial, m_1, m_2, p and q are nonnegative integers.

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Definition 1.1 *For (1.1), write* $u_j = j_1 + 2j_2 + \cdots + nj_n,\ j \in I,\ u = max_{j\in I}\{u_j\},\ v = k_1 + 2k_2 + \cdots + nk_n$.

Definition 1.2 *Let* $w = (w_1, w_2)$ *be a solution of (1.1), the order* ρ_w *of the solution a system of higher-order complex algebraic differential equations* $w = (w_1, w_2)$ *is defined by* $\rho_w = max_{i=1,2}\{\rho_{w_i}\}$, ρ_{w_i} *where denote the order of* $w_i (i = 1, 2)$.

Let $\mathcal{F}$ be a family of meromorphic functions defined on D, $\mathcal{F}$ is said to normal on D, if every sequence $f_n \in \mathcal{F}$, there exists a subsequence $\{f_{n_j}\}$, such that $\{f_{n_j}\}$ uniformly converges in every point on D, conversely, $\mathcal{F}$ is not normal on D.

They obtained

Theorem B[4] Let $w = (w_1, w_2)$ be a non-polynomial meromorphic solution of (1.1). If $c^{(n)}(z) = 0, nm_1m_2 + vp > n^2m_2p + up$, then $\rho_w < \infty$.

A recent paper Yuan et al[4] established a general estimate of growth order of $w(z)$, the result may be stated as follows:

Theorem C[5]Let $w(z)$ be meromorphic in complex plane, $n \in N$, $\Omega[w]$ be a differential polynomial with the form (2), $n > u$. If $w(z)$ satisfies the differential equation $[w'(z)]^n = \Omega[w]$, then the growth order $\sigma(w)$ of $w(z)$ satisfies

$$\sigma(w) \le 2 + \frac{2deg_{z,\infty}a}{n-u}.$$

It is natural to ask weather get more the precise estimate of growth of solutions of the system of differential equations (1.1)? we get following theorem:

Theorem 1.1 *Let* $w = (w_1, w_2)$ *be a non-polynomial meromorphic solution of (1.1). If* $c^{(n)}(z) = 0, nm_1m_2 + vp > n^2m_2p + up$, *then* $\rho_w \le 2\alpha + 2$,
where

$$\alpha = \frac{pq}{(m_1 - np)nm_2 + vp - up}$$

.

First we quote the following Lemma.

Lemma 1.1 *(Zalcman* [6]*) Let* F *be a family of meromophic functions in unit disc* $\triangle$*, then* F *is not normal in* D *if and only if there exist*

1). a real number $0 < r < 1$,

2). a sequence of complex number $\{z_n\}, |z_n| < r$,

3). a sequence of functions $f_n \in F$ *and*

4). a sequence of positive numbers $\rho_n \to 0^+$,

such that $g_n(\xi) = f_n(z_n + \rho_n\xi)$ *converges locally uniformly (with respect to the spherical metric) to a non-constant meromorphic function* $g_n(\xi)$ *for any compact subset on* C*, where* $\rho_k = \frac{1}{f^\sharp(c_k)}$ *and* $\rho^\sharp$ *denote the spherical derivative of* f .

Lemma 1.2 [5] *Let* f *be a meromorphic function in complex plane,* $\sigma := \sigma(f)$*. then for each* $0 < \rho < \frac{\sigma-2}{2}$*, there exist points* $a_n \to \infty (n \to \infty)$*, such that*

$$lim_{n\to\infty}\frac{f_n^\sharp(a_n)}{|a_n|^\rho} = +\infty \tag{1.2}$$

2 Proof of Theorem 1.1

For the systems of complex differential equations (1.1), differentiating the first of equation, We get

$$\frac{(w_2^{(n)})^{m_1}}{a}(\frac{m_1 w_2^{(n+1)}}{w_2^{(n)}} - \frac{a'}{a})^p = p^p(w_1' + c'(z))^p,$$

that is

$$(w_2^{(n)})^{m_1-p}(\frac{m_1 a w_2^{(n+1)} - a' w_2^{(n)}}{a})^p = p^p(w_1' + c'(z))^p,$$

In general, we have

$$(w_2^{(n)})^{m_1-np}(\frac{Q_n(z,w_2)}{a^n})^p = p^{np}(w_1^{(n)} + c^{(n)}(z))^p, \tag{2.1}$$

where

$$Q_1(z,w_2) = m_1 a w_2^{(n+1)} - a' w_2^{(n)},$$
$$Q_{n+1}(z,w_2) = (m_1 - np)a w_2^{(n+1)} - (np+1)a' w_2^{(n)} Q_n(z,w_2) + apQ_n'(z,w_2),$$

$Q_n(z,w_2)$ is a polynomials of $w_2^{(n)}, w_2^{(n+1)}, \cdots, w_2^{(2n)}$ and $a, a', \cdots, a^{(n)}$ homogenous of degree n with respect to $w_2^{(n)}, w_2^{(n+1)}, \cdots, w_2^{(2n)}$ and $a, a', \cdots, a^{(n)}$ respectively.

By (2.1) and the second equation of the systems (1.1), we obtain

$$(\frac{(w_2^{(n)})^{m_1-np}(Q_n(z,w_2))^p}{a^{np+1}p^{np}})^{m_2} = (\frac{\Omega(w_2)}{\Omega_k(w_2)})^p, \tag{2.2}$$

We suppose the growth $\alpha < \frac{\rho_{w_2}}{2} - 1$, by lemma 1.2, we have a sequence $\{z_k\}$, $z_k \to \infty$,

$$\frac{w_2^\sharp(z_k)}{z_k^\alpha} \to \infty (n \to \infty).$$

Where $w_2^\sharp$ denote the spherical derivative of w_2. It show that functional family is not normal at $z = 0$. By Lemma 1, we have both sequence $\{c_k\}$ and $\{\rho_k\}$, they satisfy $|c_k - z_k| < 1, \rho_k \to 0$, meanwhile $h_k(z) = w_2(c_k + \rho_k z)$ is local convergence to nontrivial meromorphic function h. By the proof of Lemma 1, we can suppose $\rho_k = \frac{1}{w_2^\sharp(c_k)}$ and $w_2^\sharp(c_k) \le w_2^\sharp(z_k)$, such that $c_k^d \rho_k \to 0 (k \to \infty)$ for any constant d.

When $c_k + \rho_k z$ replace z in (2.2), we obtain

$$(\frac{(w_2^{(n)}(c_k+\rho_k z))^{m_1-np} Q_n^p(c_k+\rho_k z, w_2(c_k+\rho_k z))}{a^{np+1}p^{np}})^{m_2} = (\frac{\Omega(c_k+\rho_k z, w_2(c_k+\rho_k z))}{\Omega_k(c_k+\rho_k z, w_2(c_k+\rho_k z))})^p,$$

that is

$$(\frac{(w_2^{(n)}(c_k+\rho_k z))^{m_1-np} Q_n^p(c_k+\rho_k z, h_k(z))}{a^{np+1}p^{np}})^{m_2} = (\frac{\Omega(c_k+\rho_k z, h_k(z))}{\Omega_k(c_k+\rho_k z, h_k(z))})^p. \tag{2.3}$$

Meanwhile, we have

$$(w_2^{(n)})(c_k + \rho_k z) = \rho_k^{-n} h_k^{(n)}(z). \tag{2.4}$$

Using (2.3) and (2.4), we obtain

$$((h_k^{(n)}(z))^{m_1-np}\rho_k^{-n(m_1-np)} Q_n^p(c_k+\rho_k z, h_k(z)))^{m_2}$$
$$= a^{m_2(np+1)}p^{m_2 np}(\frac{\sum_{j=0}^q b_j(c_k+\rho_k z)(h_k(z))^{j_0} P_j(h_k(z))\rho_k^{-u_j}}{\Omega_k(h_k(z))\rho^{-v}})^p.$$

$$|(h_k^{(n)}(z)|^{(m_1-np)m_2} \leq |a|^{m_2(np+1)}p^{m_2np}\cdot\frac{\sum_{j=0}^{q}|b_j(c_k+\rho_k z)(h_k(z))^{j_0}P_j(h_k(z))|^p\rho_k^{(m_1-np)nm_2+vp-u_jp}}{|\Omega_k(h_k(z))|^p|Q_n^p(h_k(z))|^{m_2}}$$

$$=|a|^{m_2(np+1)}p^{m_2np}\frac{\sum_{j=0}^{q}|b_j(c_k+\rho_k z)(h_k(z))^{j_0}c_k^{-q}P_j(h_k(z))|^p|c_k|^{pq}\rho_k^{(m_1-np)nm_2+vp-u_jp}}{|\Omega_k(h_k(z))|^p|Q_n^p(h_k(z))|^{m_2}},$$

where $u_j=j_1+2j_2+\cdots+nj_n, P_j(h_k(z))=(h_k'(z))^{j_1}\cdots(h_k^{(n)}(z))^{j_n}$.

For every fixed $|z|$, $|b_j(c_k+\rho_k z)h_k^{j_0}(z)c_k^{-q}|^p$ is bound, there maybe outside a set of finite measure as $k\to\infty$.

Because of $nm_1m_2+vp>n^2m_2p+u_jp$, we have $|c_k|^{pq}\rho_k^{(m_1-np)nm_2+vp-u_jp}\to 0$ i.e. $|c_k|^{\frac{pq}{(m_1-np)nm_2+vp-u_jp}}\rho_k\to 0$.

Because $\frac{pq}{(m_1-np)nm_2+vp-u_jp}\leq\alpha=\frac{pq}{(m_1-np)nm_2+vp-up}<\frac{\rho_{w_2}}{2}-1$, we have

$$h_k^{(n)}(z)=0 \tag{2.5}$$

$h(z)$ is a polynomial with respect to z by (2.5), it is a contradiction to condition of theorem 1.1. Therefore $\rho_{w_2}\leq 2\alpha+2$.

From the first equation of (1.1), we have $\rho_{w_1}\leq 2\alpha+2$. Hence $\rho_w\leq 2\alpha+2$.

The proof of Theorem 1.1 is complete.

3 Example

Example 3.1 *For solutions $w_1(z)=e^{z^2}, w_2(z)=e^{z^2}$ satisfies the following system of algebraic differential equations*

$$\begin{cases}(w_2'(z))^2=4z^2w_1^2(z),\\ (w_1'(z))^3=\frac{8z^2w_2^3(z)w_2'(z)+32z^5w_2^4(z)}{w_2''(z)},\end{cases} \tag{3.1}$$

where $n=p=u=q=1, vp=2, m_1=2, m_2=3, a(z)=4z^2, c(z)=0$, we have $\alpha=\frac{1}{4}$. It is easy to get $\rho(w)=2<2+\frac{1}{2}$, which show that our result is right.

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