A series of constants in the first three iterations of the logistic map

S. Halayka *

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Abstract

A rough analysis of the first three iterations of the logistic map $x' = rx(1-x)$ produces a series of special constants. The three constants are 1, the inverse of the golden ratio, and Catalan's constant.

1 The first iteration of the logistic map

The paper "Fibonacci order in the period-doubling cascade to chaos" by Linage *et al* [1] describes how the golden ratio

$$\phi = \frac{\sqrt{5}+1}{2} \simeq 1.61803. \tag{1}$$

can be found indirectly by analyzing the bifurcation diagram that is generated by continually iterating the logistic map

$$x' = rx(1-x) = -rx^2 + rx. \tag{2}$$

This paper will describe how the golden ratio can be found directly by analyzing the polynomials that correspond to the first three iterations of the logistic map.

Figure 1 is a plot of the solutions x' to the polynomial given in Eq. (2) for the interval $r = [0,4]$, $x = [0,1]$, where the magnitudes of the solutions are represented by brightness. Essentially, the plot shows the emergence of a continually widening trunk that effectively ends at $r = 4$. Note that the trunk is not quite fully mirror symmetric with regard to the line $x = 0.5$, insomuch that the trunk is hanging slightly low. Figure 2 is an alternative version of Figure 1 with extremely high contrast. The pure white region in this second figure is the superlevel set of this polynomial where brightness ≥ 0.5. This high contrast plot makes the emergence of the trunk at $r = 2$ all the more obvious.

Where $x = 0.5$, the trunk starts at $r = 2$, effectively ends at $r = 4$, and has a length of $\ell_{trunk} = 2$. At $r = 4$, the trunk starts at roughly $x \simeq 0.85187$, ends at roughly $x \simeq 0.14714$, and has a width of $w_{trunk} \simeq 0.70473$. Rough approximations to the Feigenbaum constants

$$\delta \simeq 4.669202, \tag{3}$$
$$\alpha \simeq 2.502908, \tag{4}$$

can be found by forming ratios out of the trunk's length and width. For instance, the ratio of the width of the trunk to the width of the gap below the trunk is $w_{trunk}/0.14714 \simeq 4.78952 \simeq 1.03 \times \delta$, which is a rough approximation of the first Feigenbaum constant. The ratio of the width of the trunk to the width of the gap above the trunk is $w_{trunk}/(1-0.85187) \simeq 4.75751 \simeq 1.02 \times \delta$, which is also a rough approximation of the first Feigenbaum constant. Finally, the ratio of the length of the trunk to the width of the trunk is $\ell_{trunk}/w_{trunk} \simeq 2.83797 \simeq 1.13 \times \alpha$, which is a rough approximation of the second Feigenbaum constant.

Setting $x = 0.5$ as a constant, we get a cross-section along the middle of the trunk in the form of the expression

$$Brightness = \frac{r}{4}. \tag{5}$$

*Little Red River Park, SK Canada – email: shalayka@gmail.com

The indefinite integral of this expression, without the constant of integration, is

$$\int \left(\frac{r}{4}\right) dr = \frac{r^2}{8}. \tag{6}$$

Note that the area under the expression's solutions between $r = 2$, $r = 4$ is precisely

$$A_0 = \frac{4^2}{8} - \frac{2^2}{8} = 1.5 = \ln(4.48169). \tag{7}$$

Also, the area under the expression's solutions between $r = 0$, $r = 4$ is precisely

$$A_1 = \frac{4^2}{8} - \frac{0^2}{8} = 2 = \ln(e^2). \tag{8}$$

Finally, the area under the line $x = 0.5$ between $r = 2$, $r = 4$ is precisely

$$A_2 = 0.5 \times (4 - 2) = 1 = \ln(e). \tag{9}$$

2 The second iteration of the logistic map

The golden ratio can be found directly by analyzing the second iteration of the logistic map

$$\begin{aligned} x'' = rx'(1-x') = & -r^3x^4 + 2r^3x^3 - r^3x^2 \\ & -r^2x^2 + r^2x. \end{aligned} \tag{10}$$

Note that the signs of the terms in the expanded polynomial are biased. Altogether there are 5 terms, of which 3 are negative, and 2 are positive.

Figure 3 is a plot of the solutions x'' to the polynomial given in Eq. (10) for the interval $r = [0,4]$, $x = [0,1]$, where the magnitudes of the solutions are represented by brightness. Essentially, the plot shows the emergence of a continually widening trunk that splits at roughly $r \simeq \sqrt{5}+1$ into two branches that effectively end at $r = 4$. Figure 4 is an alternative version of Figure 3 with extremely high contrast. This high contrast plot goes to show how the trunk emerges at $r = 2$, and then splits at $r \simeq \sqrt{5}+1$. Of course, $2/(\sqrt{5}+1) = 1/\phi$ and $(\sqrt{5}+1)/2 = \phi$. Doing some arithmetic to get the lengths involved

$$\ell_0 = 2 - 0 \quad = \quad 2, \tag{11}$$

$$\ell_1 = (\sqrt{5}+1) - 2 = \sqrt{5} - 1 \quad \simeq \quad 1.23607, \tag{12}$$

$$\ell_2 = 4 - (\sqrt{5}+1) \quad \simeq \quad 0.763932, \tag{13}$$

we find that both ℓ_0/ℓ_1 and ℓ_1/ℓ_2 are equal to the golden ratio ϕ. Note that the branches are not quite fully mirror symmetric with regard to the line $x = 0.5$, insomuch that the top branch is hanging slightly low, *etc*.

Setting $x = 0.5$ as a constant, we get a cross-section along the middle of the trunk in the form of the expression

$$Brightness = -\frac{r^3}{16} + \frac{r^2}{4}. \tag{14}$$

Figure 5 is a plot of this cross-section.

The indefinite integral of the expression given in Eq. (14) is

$$\int \left(-\frac{r^3}{16} + \frac{r^2}{4}\right) dr = -\frac{r^4}{64} + \frac{r^3}{12}. \tag{15}$$

The area under the curve between $r = 2$, $r = \sqrt{5}+1$ is

$$A_0 \simeq 0.693853 = \ln(2.00141). \tag{16}$$

The area under the curve between $r = 0$, $r = \sqrt{5}+1$ is

$$A_1 \simeq 1.11052 = \ln(3.03594). \tag{17}$$

These two areas are shown in Figures 6, 7 respectively. Note that the area under the line $x = 0.5$ between $r = 2$, $r = \sqrt{5}+1$ is precisely

$$A_2 = 0.5 \times \left((\sqrt{5}+1) - 2\right) = 1/\phi = \phi - 1 = \ln(1.85528). \tag{18}$$

3 The third iteration of the logistic map

Better approximations of the two Feigenbaum constants can be found by analyzing the third iteration of the logistic map.

$$
\begin{aligned}
x''' = rx''(1-x'') = & -r^7x^8 + 4r^7x^7 - 6r^7x^6 + 4r^7x^5 - r^7x^4 \\
& -2r^6x^6 + 6r^6x^5 - 6r^6x^4 + 2r^6x^3 \\
& -r^5x^4 + 2r^5x^3 - r^5x^2 \\
& -r^4x^4 + 2r^4x^3 - r^4x^2 \\
& -r^3x^2 + r^3x.
\end{aligned} \tag{19}
$$

Note that the signs of the terms in the third iteration of the map are biased. Altogether there are 17 terms, of which 10 are negative, and 7 are positive.

Figure 8 is a plot of the solutions x''' to the polynomial given in Eq. (19) for the interval $r = [0,4]$, $x = [0,1]$. Essentially, the plot shows the emergence of a continually widening trunk that splits into three branches at $r = \sqrt{5}+1$, and then splits again at roughly $r \simeq 3.83187$ into four branches that effectively end at $r = 4$. Figure 9 is an alternative version of Figure 8 with extremely high contrast.

Consider just the two branches above the line $x = 0.5$ in Figure 9. The top branch starts at $r = \sqrt{5}+1$, effectively ends at $r = 4$, and has a length of $\ell_{\text{top}} \simeq 0.763932$. The bottom branch starts at roughly $r \simeq 3.83187$, effectively ends at $r = 4$, and has a length of $\ell_{\text{bottom}} \simeq 0.16813$. The ratio of these lengths $\ell_{\text{top}}/\ell_{\text{bottom}} \simeq 4.543698 \simeq 0.97 \times \delta$ is a rough approximation of the first Feigenbaum constant. At $r = 4$, the top branch starts at roughly $x \simeq 0.990393$, ends at roughly $x \simeq 0.915735$, and has a width of $w_{\text{top}} \simeq 0.074658$. At $r = 4$, the bottom branch starts at roughly $x \simeq 0.777785$, ends at roughly $x \simeq 0.597545$, and has a width of $w_{\text{bottom}} \simeq 0.18024$. The ratio of these widths $w_{\text{bottom}}/w_{\text{top}} \simeq 2.414209 \simeq 0.96 \times \alpha$ is a rough approximation of the second Feigenbaum constant.

Finally, setting $x = 0.5$ as a constant, the indefinite integral of the cross-section along the middle of the trunk is

$$
\int \left(-\frac{r^7}{256} + \frac{r^6}{32} - \frac{r^5}{16} - \frac{r^4}{16} + \frac{r^3}{4} \right) dr = -\frac{r^8}{2048} + \frac{r^7}{224} - \frac{r^6}{96} - \frac{r^5}{80} + \frac{r^4}{16}. \tag{20}
$$

The area under the curve between $r = 2$, $r \simeq 3.83187$ is

$$
A_0 \simeq 1.2499 \simeq \ln(3.4899). \tag{21}
$$

The area under the curve between $r = 0$, $r \simeq 3.83187$ is

$$
A_1 \simeq 1.62966 = \ln(5.10214). \tag{22}
$$

Note that the area under the line $x = 0.5$ between $r = 2$, $r \simeq 3.83187$ is

$$
A_2 = 0.5 \times (3.83187 - 2) \simeq 0.915935 = \ln(2.49911), \tag{23}
$$

which is a rough approximation of Catalan's constant

$$
K = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \simeq 0.915966. \tag{24}
$$

4 Discussion

The area A_0 may or may not have anything to do with entropy, although it is interesting to note that the act of making a which-way choice is bound to occur during the act of moving along a branching object. Even if the area A_0 is not related to entropy, it is interesting to note that A_0 is the natural logarithm of something very close to either an integer or an integer + 1/2.

Likewise, the areas A_1 and A_2 may or may not have anything to do with entropy.

Note that finding the area A_2 for interations four and above is not as simple as treating the region as a rectangle. This is because the solutions to the polynomial rise above and dip below the threshold value of $x = 0.5$ many times before the last dip (ie. the last branch) occurs, each taking a chunk out of the otherwise ideal rectangle.

If a polynomial's signs – either fully expanded, or with combined like terms as shown here – are used to create a random Fibonacci sequence, the end result would definitely not be exactly Viniswath's constant, since there is a sign bias the polynomial. For instance, see Figures 10 and 11 for plots of the solutions x'''''' to the polynomial corresponding to the sixth iteration of the logistic map.

References

[1] Linage G, Montoya F, Sarmiento A, Showalter K, Parmananda P. Fibonacci order in the period-doubling cascade to chaos. (2006) Physics Letters A 359

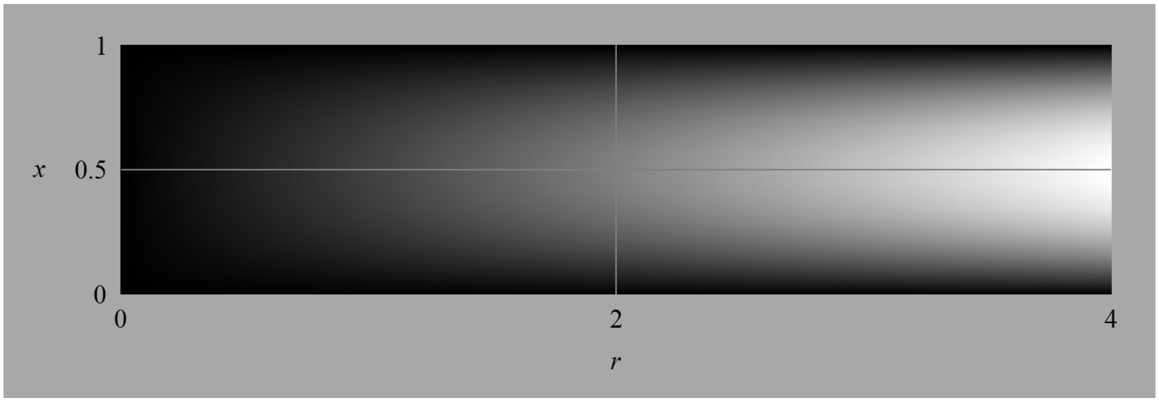

Figure 1: The solutions x' to the polynomial given in Eq. (2) for the interval $r = [0,4]$, $x = [0,1]$.

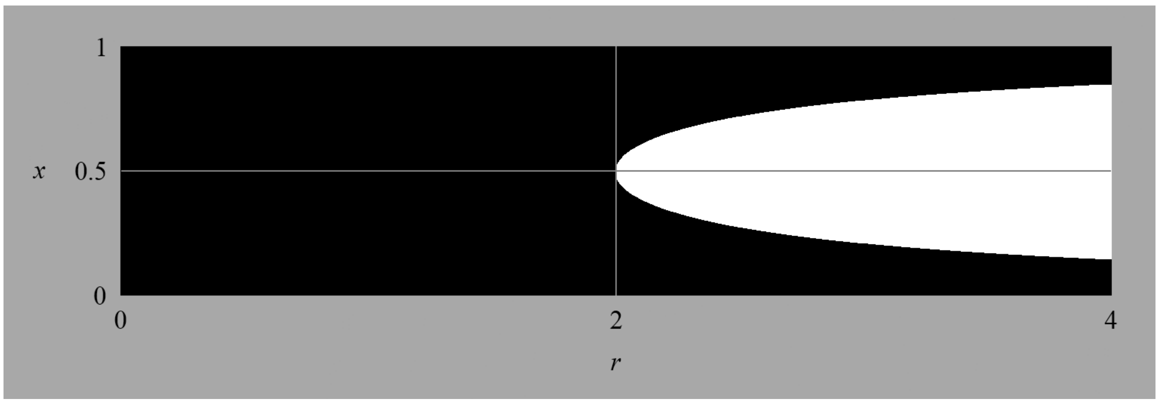

Figure 2: A high contrast version of Figure 1. The pure white region is the superlevel set of the polynomial where brightness ≥ 0.5. This goes to show how the trunk emerges at $r = 2$.

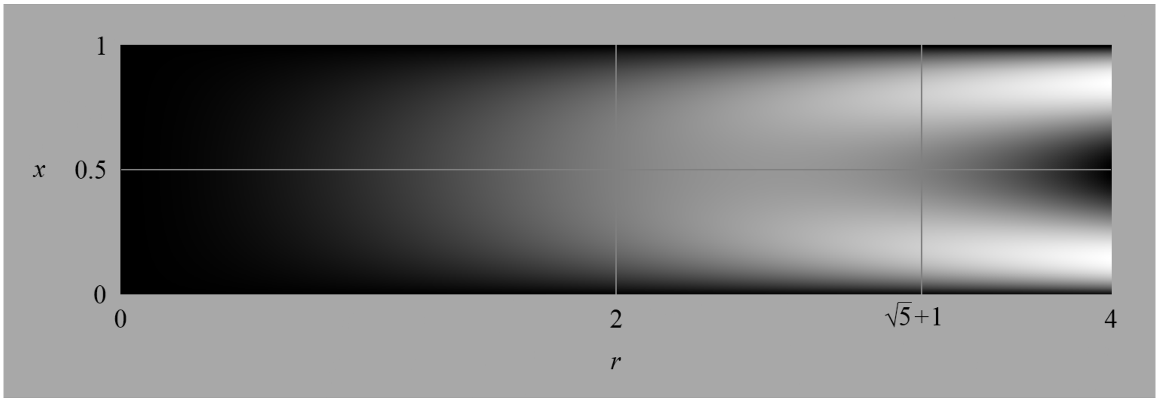

Figure 3: The solutions x'' to the polynomial given in Eq. (10) for the interval $r = [0,4]$, $x = [0,1]$.

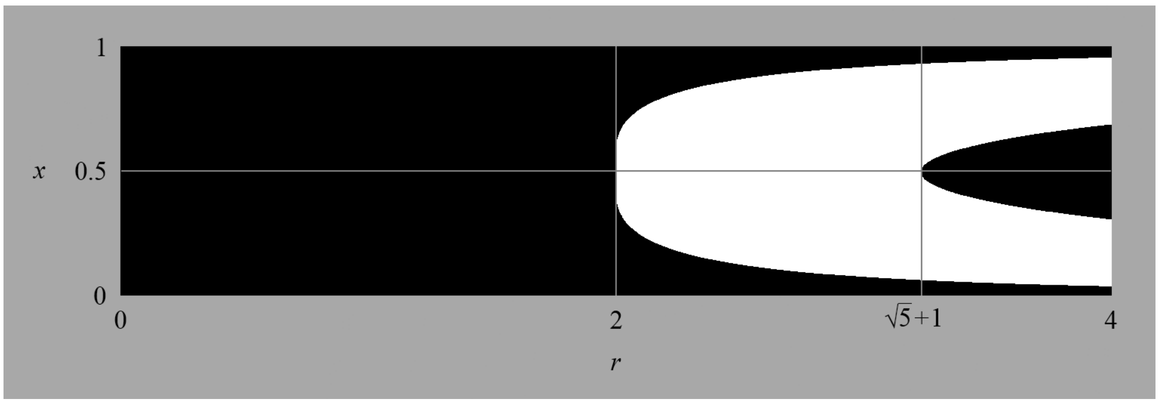

Figure 4: A high contrast version of Figure 3. The pure white region is the superlevel set of the polynomial where brightness ≥ 0.5. This goes to show how the trunk emerges at $r = 2$, and splits into two branches at $r = \sqrt{5}+1$.

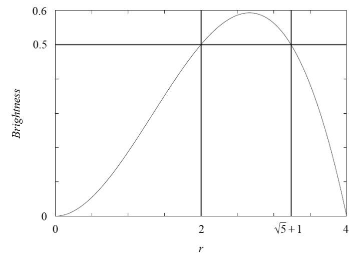

Figure 5: Cross-section of constant $x = 0.5$.

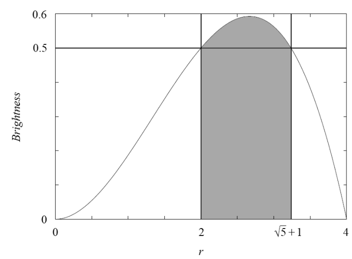

Figure 6: Cross-section of constant $x = 0.5$. The gray region under the curve has an area of 0.693853. Also, the gray subregion under the line $x = 0.5$ has an area of $1/\phi = \phi - 1 \simeq 0.618034$.

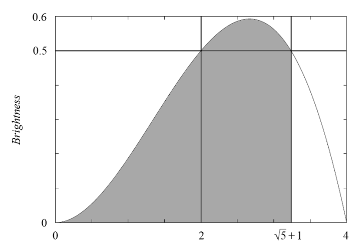

Figure 7: Cross-section of constant $x = 0.5$. The gray region under the curve has an area of 1.11052.

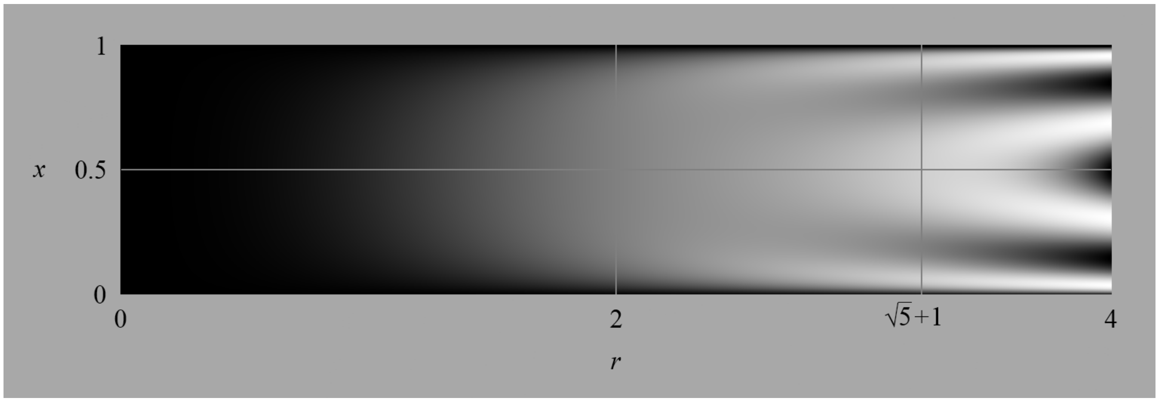

Figure 8: The solutions x''' to the polynomial given in Eq. (19) for the interval $r = [0,4]$, $x = [0,1]$.

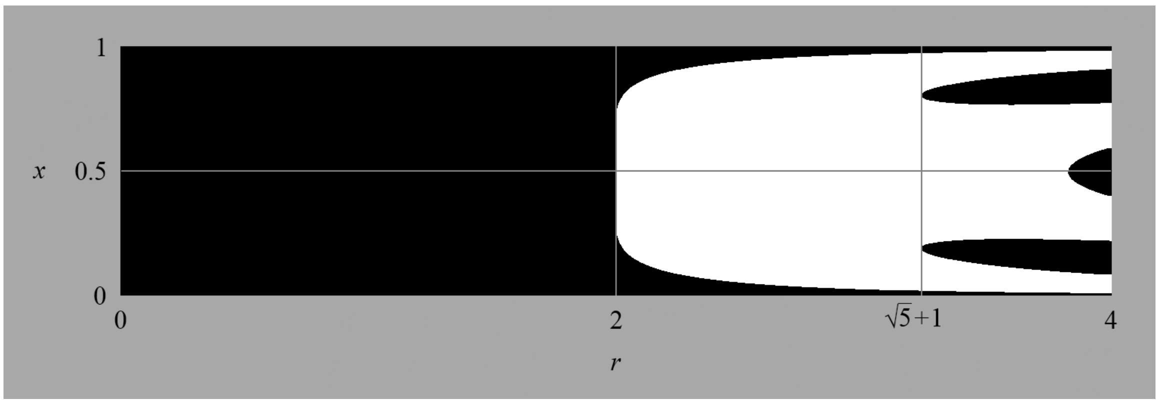

Figure 9: A high contrast version of Figure 8. The pure white region is the superlevel set of the polynomial where brightness ≥ 0.5. This goes to show how the trunk emerges at $r = 2$, splits into three branches at $r \simeq \sqrt{5}+1$, and then splits again into four branches at roughly $r \simeq 3.83$.

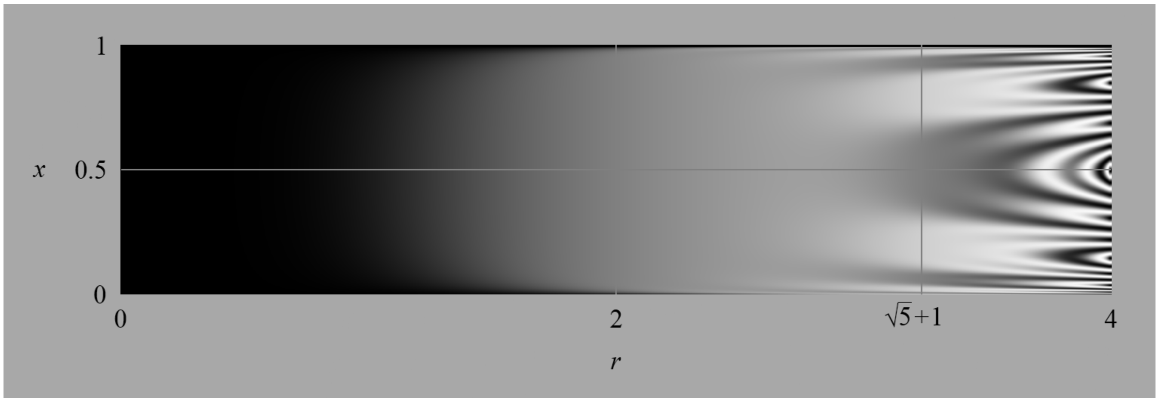

Figure 10: The solutions x'''''' to the polynomial corresponding to the sixth iteration of the logistic map for the interval $r = [0,4]$, $x = [0,1]$.

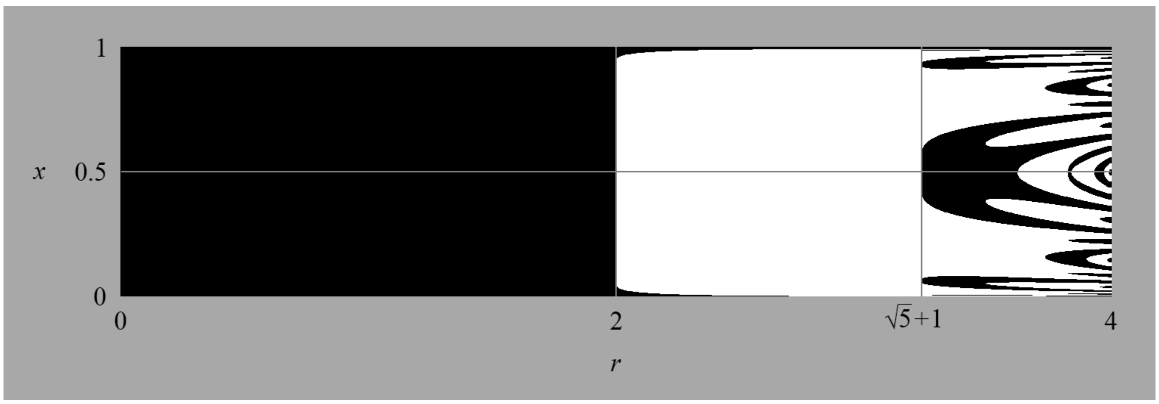

Figure 11: A high contrast version of Figure 10. The pure white region is the superlevel set of the polynomial where brightness ≥ 0.5.