Germán Andrés Paz
(2000) Rosario, Santa Fe, Argentina
Solution to One of Landau's Problems
Solución a Uno de los Problemas de Landau

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Introduction

Landau's Problems are four problems in Number Theory concerning prime numbers:

- **Goldbach's Conjecture:** This conjecture states that every positive even integer greater than 2 can be expressed as the sum of two (not necessarily different) prime numbers.
- **Twin Prime Conjecture:** Are there infinitely many prime numbers *p* such that $p+2$ is also a prime number? This problem was solved by **Carlos Giraldo Ospina (Lic. Matemáticas, USC, Cali, Colombia)**, who proved that if *k* is any positive even integer, then there are infinitely many prime numbers *p* such that $p+k$ is also a prime number.
- **Legendre's Conjecture:** Is there always at least one prime number between n^2 and $(n+1)^2$ for every positive integer *n*?
- **Primes of the form n^2+1:** Are there infinitely many prime numbers of the form n^2+1 (where *n* is a positive integer)?

Please see the article ***Primos Gemelos, Demostración Kmelliza***, where C. G. Ospina shows the method he used to solve the Twin Prime Conjecture.

Primes of the form $n^2 + 1$

Abstract:
In this document we are going to prove that there are infinitely many prime numbers of the form n^2+1. In order to achieve our goal, we are going to use the same method that C. G. Ospina used in his paper.

Note: In this document, whenever we say that a number *b* is **between** a number *a* and a number *c*, it will mean that $a < b < c$, which means that *b* will never be equal to *a* or *c* (the same rule will be applied to intervals). Moreover, the number *n* that we will use in this document will always be a **positive integer**.

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Theorems 1, 2, 3 and 4

Let us suppose that n is a positive integer. We need to know what value n needs to have so that there is always a perfect square a^2 such that $n < a^2 < \frac{3n}{2}$.

Note: We say a number is a 'perfect square' if it is the square of an integer. In other words, a number x is a perfect square if $\sqrt{x}$ is an integer. Perfect squares are also called 'square numbers'.

In general, if m is any positive integer, we need to know what value n needs to have so that there is always a positive integer a such that $n < a^m < \frac{3n}{2}$.

We have

$$n < a^m < \frac{3n}{2}$$

This means that

$$n < a^m \quad \text{and} \quad a^m < \frac{3n}{2}$$

$$\sqrt[m]{n} < a \quad \text{and} \quad a < \sqrt[m]{\frac{3n}{2}}$$

To sum up,

$$\sqrt[m]{n} < a < \sqrt[m]{\frac{3n}{2}}$$

As we said before, the number a is a positive integer. Now, the integer immediately following the number $\sqrt[m]{n}$ will be called $\sqrt[m]{n} + d$. In other words, $\sqrt[m]{n} + d$ is the smallest integer that is greater than $\sqrt[m]{n}$:

- If $\sqrt[m]{n}$ is an integer, then $\sqrt[m]{n} + d = \sqrt[m]{n} + 1$, because in this case we have $d = 1$.

- If $\sqrt[m]{n}$ is not an integer, then in the expression $\sqrt[m]{n} + d$ we have $0 < d < 1$, but we do not have any way of knowing the exact value of d if we do not know the value of $\sqrt[m]{n}$ first.

Now, let us make the calculation:

$$\sqrt[m]{n}+d \quad < \quad \sqrt[m]{\frac{3n}{2}}$$

We need to take the largest possible value of *d*, which is $d=1$ (if $\sqrt[m]{n}+1<\sqrt[m]{\frac{3n}{2}}$, then $\sqrt[m]{n}+d<\sqrt[m]{\frac{3n}{2}}$ for all *d* such that $0<d\leq 1$):

$$\sqrt[m]{n}+1 \quad < \quad \sqrt[m]{\frac{3n}{2}}$$

$$1 \quad < \quad \sqrt[m]{\frac{3n}{2}}-\sqrt[m]{n}$$

$$1 \quad < \quad \sqrt[m]{\frac{3}{2}n}-\sqrt[m]{n}$$

$$1 \quad < \quad \sqrt[m]{1.5n}-\sqrt[m]{n}$$

$$1 \quad < \quad \sqrt[m]{1.5}\sqrt[m]{n}-\sqrt[m]{n}$$

$$1 \quad < \quad \sqrt[m]{n}\left(\sqrt[m]{1.5}-1\right)$$

$$\frac{1}{\sqrt[m]{1.5}-1} \quad < \quad \sqrt[m]{n}$$

$$\left(\frac{1}{\sqrt[m]{1.5}-1}\right)^m \quad < \quad n$$

$$n \quad > \quad \left(\frac{1}{\sqrt[m]{1.5}-1}\right)^m$$

$$n \quad > \quad \frac{1^m}{\left(\sqrt[m]{1.5}-1\right)^m}$$

$$n \quad > \quad \frac{1}{\left(\sqrt[m]{1.5}-1\right)^{m}}$$

This means that if m is a positive integer, then for every positive integer $n > \frac{1}{\left(\sqrt[m]{1.5}-1\right)^{m}}$ there is at least one positive integer a such that $n < a^m < \frac{3n}{2}$. Now, if $n > \frac{14.4}{\left(\sqrt[m]{1.5}-1\right)^{m}}$ then $n > \frac{1}{\left(\sqrt[m]{1.5}-1\right)^{m}}$.

- Consequently, if m is a positive integer, then for every positive integer $n > \frac{14.4}{\left(\sqrt[m]{1.5}-1\right)^{m}}$ there is at least one positive integer a such that $n < a^m < \frac{3n}{2}$. This true statement will be called ***Theorem 1***.

Now we are going to prove that if m is any positive integer, then $\frac{1}{\left(\sqrt[m]{1.5}-1\right)^{m}} > 1$.

Proof:

$$\frac{1}{\left(\sqrt[m]{1.5}-1\right)^{m}} \quad > \quad 1$$

$$1 \quad > \quad 1\left(\sqrt[m]{1.5}-1\right)^{m}$$

$$1 \quad > \quad \left(\sqrt[m]{1.5}-1\right)^{m}$$

$$\sqrt[m]{1} \quad > \quad \sqrt[m]{1.5}-1$$

$$1 \quad > \quad \sqrt[m]{1.5}-1$$

$$1+1 \quad > \quad \sqrt[m]{1.5}$$

$$2 \quad > \quad \sqrt[m]{1.5}$$

$$2^m > 1.5$$

It is very easy to verify that $2^m > 1.5$ for every positive integer *m*. Consequently, if *m* is any positive integer, then $\frac{1}{\left(\sqrt[m]{1.5}-1\right)^m} > 1$. If $\frac{1}{\left(\sqrt[m]{1.5}-1\right)^m} > 1$, then $\frac{14.4}{\left(\sqrt[m]{1.5}-1\right)^m} > 14.4$. This means that $\frac{14.4}{\left(\sqrt[m]{1.5}-1\right)^m} > 14.4$ for every positive integer *m*.

Note: *In general, to prove that an inequality is correct, we can solve that inequality step by step. If we get a result which is obviously correct, then we can start with that correct result, 'work backwards from there' and prove that the initial statement is true.*

- As a consequence, if *m* is any positive integer and $n > \frac{14.4}{\left(\sqrt[m]{1.5}-1\right)^m}$, then $n > 14.4$.

  This true statement will be called ***Theorem 2***.

- In the document ***Infinitely Many Prime Numbers of the Form ap±b*** it was proved that for every positive integer $n > 14.4$ there exist prime numbers *r* and *s* such that $n < r < \frac{3n}{2} < s < 2n$ (please see that document for a proof). This true statement will be called ***Theorem 3***.

- According to Theorems 1, 2 and 3, if *m* is a positive integer, then for every positive integer $n > \frac{14.4}{\left(\sqrt[m]{1.5}-1\right)^m}$ there exist a positive integer *a* and a prime number *s* such that $n < a^m < \frac{3n}{2} < s < 2n$. This true statement will be called ***Theorem 4***.

Theorems 5, 6 and 7

Now, if *m* is a positive integer, let us calculate what value *n* needs to have so that there is always a positive integer *a* such that $\frac{3n}{2} < a^m < 2n$:

We have

$$\frac{3n}{2} < a^m < 2n$$

This means that

$$\frac{3n}{2} < a^m \quad \text{and} \quad a^m < 2n$$

$$\sqrt[m]{\frac{3n}{2}} < a \quad \text{and} \quad a < \sqrt[m]{2n}$$

To sum up,

$$\sqrt[m]{\frac{3n}{2}} < a < \sqrt[m]{2n}$$

The number *a* is a positive integer. Now, the integer immediately following the number $\sqrt[m]{\frac{3n}{2}}$ will be called $\sqrt[m]{\frac{3n}{2}} + d$. In other words, $\sqrt[m]{\frac{3n}{2}} + d$ is the smallest integer that is greater than $\sqrt[m]{\frac{3n}{2}}$:

- If $\sqrt[m]{\frac{3n}{2}}$ is an integer, then $\sqrt[m]{\frac{3n}{2}} + d = \sqrt[m]{\frac{3n}{2}} + 1$, because in this case we have $d = 1$.

- If $\sqrt[m]{\frac{3n}{2}}$ is not an integer, then in the expression $\sqrt[m]{\frac{3n}{2}} + d$ we have $0 < d < 1$, but we do not have any way of knowing the exact value of *d* if we do not know the value of $\sqrt[m]{\frac{3n}{2}}$ first.

Let us make the calculation:

$$\sqrt[m]{\frac{3n}{2}} + d \quad < \quad \sqrt[m]{2n}$$

We need to take the largest possible value of *d*, which is *d* =1 (if $\sqrt[m]{\frac{3n}{2}} + 1 < \sqrt[m]{2n}$, then $\sqrt[m]{\frac{3n}{2}} + d < \sqrt[m]{2n}$ for all *d* such that $0 < d \leq 1$):

$$\sqrt[m]{\frac{3n}{2}}+1 < \sqrt[m]{2n}$$

$$1 < \sqrt[m]{2n}-\sqrt[m]{\frac{3n}{2}}$$

$$1 < \sqrt[m]{2n}-\sqrt[m]{\frac{3}{2}n}$$

$$1 < \sqrt[m]{2n}-\sqrt[m]{1.5n}$$

$$1 < \sqrt[m]{2}\sqrt[m]{n}-\sqrt[m]{1.5}\sqrt[m]{n}$$

$$1 < \sqrt[m]{n}\left(\sqrt[m]{2}-\sqrt[m]{1.5}\right)$$

$$\frac{1}{\sqrt[m]{2}-\sqrt[m]{1.5}} < \sqrt[m]{n}$$

$$\left(\frac{1}{\sqrt[m]{2}-\sqrt[m]{1.5}}\right)^m < n$$

$$\frac{1^m}{\left(\sqrt[m]{2}-\sqrt[m]{1.5}\right)^m} < n$$

$$\frac{1}{\left(\sqrt[m]{2}-\sqrt[m]{1.5}\right)^m} < n$$

$$n > \frac{1}{\left(\sqrt[m]{2}-\sqrt[m]{1.5}\right)^m}$$

This means that if m is a positive integer, then for every positive integer $n>\frac{1}{\left(\sqrt[m]{2}-\sqrt[m]{1.5}\right)^m}$ there is at least one positive integer a such that $\frac{3n}{2}<a^m<2n$. Now, if $n>\frac{14.4}{\left(\sqrt[m]{2}-\sqrt[m]{1.5}\right)^m}$ then $n>\frac{1}{\left(\sqrt[m]{2}-\sqrt[m]{1.5}\right)^m}$.

- Consequently, if *m* is a positive integer, then for every positive integer $n > \frac{14.4}{\left(\sqrt[m]{2}-\sqrt[m]{1.5}\right)^m}$ there is at least one positive integer *a* such that $\frac{3n}{2} < a^m < 2n$.
  This true statement will be called ***Theorem 5***.

Now we are going to prove that if *m* is any positive integer, then $\frac{1}{\left(\sqrt[m]{2}-\sqrt[m]{1.5}\right)^m} > 1$.

Proof:

$$\frac{1}{\left(\sqrt[m]{2}-\sqrt[m]{1.5}\right)^m} > 1$$

$$1 > 1\left(\sqrt[m]{2}-\sqrt[m]{1.5}\right)^m$$

$$1 > \left(\sqrt[m]{2}-\sqrt[m]{1.5}\right)^m$$

$$\sqrt[m]{1} > \sqrt[m]{2}-\sqrt[m]{1.5}$$

$$1 > \sqrt[m]{2}-\sqrt[m]{1.5}$$

$$1+\sqrt[m]{1.5} > \sqrt[m]{2}$$

$\sqrt[m]{1.5} > 1$ because $1.5 > 1^m$, that is to say, $1.5 > 1$.
$\sqrt[m]{2} \leq 2$ because $2 \leq 2^m$.

This means that

$$1+\sqrt[m]{1.5} > 2 \geq \sqrt[m]{2}$$

which proves that

$$1+\sqrt[m]{1.5} > \sqrt[m]{2}$$

Therefore, if *m* is any positive integer, then $\frac{1}{\left(\sqrt[m]{2}-\sqrt[m]{1.5}\right)^m} > 1$. If $\frac{1}{\left(\sqrt[m]{2}-\sqrt[m]{1.5}\right)^m} > 1$, then $\frac{14.4}{\left(\sqrt[m]{2}-\sqrt[m]{1.5}\right)^m} > 14.4$. This means that $\frac{14.4}{\left(\sqrt[m]{2}-\sqrt[m]{1.5}\right)^m} > 14.4$ for every positive integer *m*.

- As a consequence, if *m* is any positive integer and $n > \frac{14.4}{\left(\sqrt[m]{2}-\sqrt[m]{1.5}\right)^m}$, then $n > 14.4$.

  This true statement will be called ***Theorem 6***.

- According to Theorems 3, 5 and 6, if *m* is a positive integer, then for every positive integer $n > \frac{14.4}{\left(\sqrt[m]{2}-\sqrt[m]{1.5}\right)^m}$ there exist a prime number *r* and a positive integer *a* such that $n < r < \frac{3n}{2} < a^m < 2n$. This true statement will be called ***Theorem 7***.

Infinitely many prime numbers of the form $n^2 + 1$

According to ***Theorem 4***, for every positive integer $n > \frac{14.4}{\left(\sqrt{1.5}-1\right)^2}$ there exist a positive integer *a* and a prime number *s* such that $n < a^2 < \frac{3n}{2} < s < 2n$.

The numbers a^2 and *s* form what we will call *'pair (perfect square, prime) of order k'*. This is because:

- We have a pair of numbers: a perfect square a^2 and a prime number *s*.
- The perfect square a^2 is followed by the prime numbers *s*. In other words, $a^2 < s$.
- We say that $a^2 + k = s$. In other words, we say the pair (perfect square, prime) is 'of order *k*' because the difference between the numbers forming this pair is *k*.

Now we need to define some other concepts:

- The set made up of all positive integers *z* such that $z > \frac{14.4}{\left(\sqrt[m]{1.5}-1\right)^m}$ will be called **Set A(*m*)**. Examples:

    - Set A(2) is the set of all positive integers *z* such that $z > \frac{14.4}{\left(\sqrt{1.5}-1\right)^2}$. In other words, Set A(2) is made up of all positive integers *z* such that $z > 285.09\ldots$.

    - Set A(3) is the set of all positive integers *z* such that $z > \frac{14.4}{\left(\sqrt[3]{1.5}-1\right)^3}$. In other words, Set A(3) is made up of all positive integers $z > 4751.47\ldots$.

- The set made up of all positive integers *z* such that $z > \frac{14.4}{\left(\sqrt[m]{2}-\sqrt[m]{1.5}\right)^m}$ will be called **Set B(*m*)**.

Let us prove that there are infinitely many prime numbers of the form n^2+1. In order to achieve the goal, we are going to use the same method that C. G. Ospina used in his article ***Primos Gemelos, Demostración Kmelliza***.

1. Let us suppose that in Set A(2) there are no pairs (perfect square, prime) of order $k < u$ starting from $n = u$.

   The numbers *n* and *u* are positive integers which belong to Set A(2).

2. Between *u* and 2*u*, $n = u$, there is at least one pair (perfect square, prime), according to ***Theorem 4***.

3. The difference between two integers located between *u* and 2*u* is $k < u$.

4. Between *u* and 2*u*, $n = u$, there is at least one pair (perfect square, prime) of order $k < u$, according to statements 2. and 3.

5. In Set A(2), starting from $n = u$ there is at least one pair (perfect square, prime) of order $k < u$, according to statement 4.

6. Statement 5. contradicts statement 1.

7. Therefore, no kind of pair (perfect square, prime) of any order *k* can be finite, according to statement 6.

Note: It is already known that for every positive integer *k* the polynomial n^2+k is irreducible over $\mathbb{R}$ and thus irreducible over $\mathbb{Z}$, since every second-degree polynomial whose discriminant is a negative number is irreducible over $\mathbb{R}$.

According to statement 7., for every positive integer *k* there are infinitely many pairs (perfect square, prime) of order *k*. This means that pairs (perfect square, prime) of order 1 can not be finite. In other words, prime numbers of the form n^2+1 can not be finite.

All this proves that there are infinitely many prime numbers of the form n^2+1.

In general, if we use **Set A(*m*)**, **Set B(*m*)** and Theorems 4 and 7 and we use the same method, we could prove that there are infinitely many prime numbers of the form n^m+k and infinitely many prime numbers of the form n^m-k for certain values of *m* and *k* (we only have to take into account the cases where the polynomials n^m+k and n^m-k are irreducible over $\mathbb{Z}$).

Conclusion

We will restate the most important theorem that was proved in this document:

There are infinitely many prime numbers of the form n^2+1, where *n* is a positive integer.

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New conjecture

If n is any positive integer and we take n consecutive integers located between n^2 and $(n+1)^2$, then among those n integers there is at least one prime number.

In other words, if a_1, a_2, a_3, a_4, … , a_n are n consecutive integers such that $n^2 < a_1 < a_2 < a_3 < a_4 < ... < a_n < (n+1)^2$, then at least one of those n integers is a prime number. This conjecture will be called ***Conjecture C***.

❖ Legendre's Conjecture

It is very easy to verify that the amount of integers located between n^2 and $(n+1)^2$ is equal to $2n$.

Proof:

$$(n+1)^2 - n^2 = 2n+1$$

$$n^2 + 2n + 1 - n^2 = 2n+1$$

$$2n+1 = 2n+1$$

We need to exclude the number $(n+1)^2$ because we are taking into consideration the integers that are greater than n^2 and smaller than $(n+1)^2$:

$$2n+1-1 = 2n$$

According to this, between n^2 and $(n+1)^2$ there are two groups of n consecutive integers each that do not have any integer in common. Example for $n=3$:

$$(3)^2 \quad \underbrace{\underbrace{10 \quad 11 \quad 12}_{\substack{\textbf{Group A} \\ (n \textbf{ consecutive integers})}} \quad \underbrace{13 \quad 14 \quad 15}_{\substack{\textbf{Group B} \\ (n \textbf{ consecutive integers})}}}_{2n \textbf{ consecutive integers}} \quad (3+1)^2$$

Group A and **Group B** do not have any integer in common. According to **Conjecture C**, Group A contains at least one prime number and Group B also contains at least one prime number, which means that between 3^2 and $(3+1)^2$ there are at least **two** prime numbers. This is true because the numbers 11 and 13 are both prime numbers.

All this means that if Conjecture C is true, then there are at least two prime numbers between n^2 and $(n+1)^2$ for every positive integer *n*. As a result, if Conjecture C is true, then Legendre's Conjecture is also true.

❖ Brocard's Conjecture

This conjecture states that if p_n and p_{n+1} are consecutive prime numbers greater than 2, then between $(p_n)^2$ and $(p_{n+1})^2$ there are at least four prime numbers.

Since $2 < p_n < p_{n+1}$, we have $p_{n+1} - p_n \geq 2$. This means that there is at least one positive integer a such that $p_n < a < p_{n+1}$. As a result, there is at least one positive integer *a* such that $(p_n)^2 < a^2 < (p_{n+1})^2$.

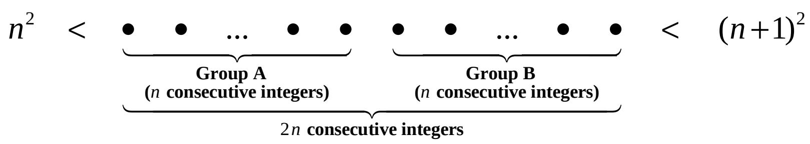
Conjecture C states that between $(p_n)^2$ and a^2 there are at least two prime numbers and that between a^2 and $(p_{n+1})^2$ there are also at least two prime numbers. In other words, if Conjecture C is true then there are at least four prime numbers between $(p_n)^2$ and $(p_{n+1})^2$. As a consequence, if Conjecture C is true then Brocard's Conjecture is also true.

❖ Andrica's Conjecture

This conjecture states that $\sqrt{p_{n+1}} - \sqrt{p_n} < 1$ for every pair of consecutive prime numbers p_n and p_{n+1} (of course $p_n < p_{n+1}$).

Obviously, every prime number is located between two consecutive perfect squares. If we take any prime number p_n, which is obviously located between n^2 and $(n+1)^2$, two things may happen:

- Case 1: The number p_n is located among the first *n* consecutive integers that are located between n^2 and $(n+1)^2$. These *n* integers form what we call *Group A*, and the following *n* integers form what we call *Group B*, as shown below:



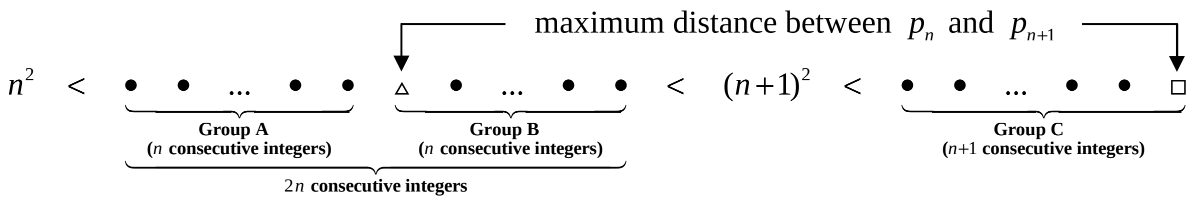
If p_n is located in Group A and Conjecture C is true, then p_{n+1} is either located in Group A or in Group B. In both cases we have $\sqrt{p_{n+1}} - \sqrt{p_n} < 1$, because $\sqrt{(n+1)^2} - \sqrt{n^2} = 1$ and the numbers $\sqrt{p_{n+1}}$ and $\sqrt{p_n}$ are closer to each other than $\sqrt{(n+1)^2}$ in relation to $\sqrt{n^2}$.

- Case 2: The prime number p_n is located in Group B.

If p_n is located in Group B and Conjecture C is true, it may happen that p_{n+1} is also located in Group B. In this case it is very easy to verify that $\sqrt{p_{n+1}} - \sqrt{p_n} < 1$, as explained before.

Otherwise, if p_{n+1} is not located in Group B, then p_{n+1} is located in Group C (see the graphic below). In this case the largest value p_{n+1} can have is $p_{n+1} = (n+1)^2 + n + 1 = n^2 + 2n + 1 + n + 1 = n^2 + 3n + 2$ and the smallest value p_n can have is $p_n = n^2 + n + 1$ (in order to make the process easier, we are not taking into account that in this case the numbers p_n and p_{n+1} have different parity, so they can not be both prime at the same time).

This means that the largest possible difference between $\sqrt{p_{n+1}}$ and $\sqrt{p_n}$ is $\sqrt{p_{n+1}} - \sqrt{p_n} = \sqrt{n^2 + 3n + 2} - \sqrt{n^2 + n + 1}$. Let us look at the graphic below:



$\triangle \;=\; n^2+n+1 \;=\; p_n$

$\square \;=\; n^2+3n+2 \;=\; p_{n+1}$

It is easy to prove that $\sqrt{n^2+3n+2}-\sqrt{n^2+n+1}<1$.

Proof:

$$\sqrt{n^2+3n+2}-\sqrt{n^2+n+1} \;<\; 1$$

$$\sqrt{n^2+3n+2} \;<\; 1+\sqrt{n^2+n+1}$$

$$n^2+3n+2 \;<\; \left(1+\sqrt{n^2+n+1}\right)^2$$

$$n^2+3n+2 \;<\; 1+2\sqrt{n^2+n+1}+n^2+n+1$$

$$n^2+3n+2-n^2-n-1 \;<\; 1+2\sqrt{n^2+n+1}$$

$$2n+1 \;<\; 1+2\sqrt{n^2+n+1}$$

$$2n \;<\; 2\sqrt{n^2+n+1}$$

$$n \;<\; \frac{2\sqrt{n^2+n+1}}{2}$$

$$n \;<\; \sqrt{n^2+n+1}$$

$$n^2 \;<\; n^2+n+1$$

which is true for every positive integer *n*.

We can see that even when the difference between p_{n+1} and p_n is the largest possible difference, we have $\sqrt{p_{n+1}}-\sqrt{p_n}<1$. If the difference between p_{n+1} and p_n were smaller, then of course it would also happen that $\sqrt{p_{n+1}}-\sqrt{p_n}<1$.

According to Cases 1 and 2, if Conjecture C is true then Andrica's Conjecture is also true.

To conclude, if Conjecture C is true, then Legendre's Conjecture, Brocard's Conjecture and Andrica's Conjecture are all true.

❖ Possible new interval

It is easy to verify that if Conjecture C is true, then in the interval $\left[n^2+n+1,\ n^2+3n+2\right]$ (see the graphics on previous pages) there are at least two prime numbers for every positive integer *n*.

The number n^2+n+1 will always be an odd integer.

Proof:

- If *n* is even, then n^2 is also even. Then we have

$$(even\ integer + even\ integer)+1 = even\ integer + odd\ integer = odd\ integer$$

- If *n* is odd, then n^2 is also odd. Then we have

$$(odd\ integer + odd\ integer)+1 = even\ integer + odd\ integer = odd\ integer$$

Since the number n^2+n+1 will always be an odd integer, then it may be prime or not.

Now, the number n^2+3n+2 can never be prime because this number will always be an even integer (and it will be greater than 2).

Proof:

- If $n=1$ (smallest value *n* can have), then $n^2+3n+2=1+3+2=6$.
- If *n* is even, then n^2 and $3n$ are both even integers. The number 2 is also an even integer, and we know that

$$even\ integer + even\ integer + even\ integer = even\ integer$$

- If *n* is odd, then n^2 and $3n$ are both odd integers, and we know that

$$(odd\ integer + odd\ integer) + even\ integer = even\ integer + even\ integer = even\ integer$$

From all this we deduce that if Conjecture C is true, then the maximum distance between two consecutive prime numbers is the one from the number n^2+n+1 to the number $n^2+3n+2-1 = n^2+3n+1$, which means that in the interval $[n^2+n+1,\ n^2+3n+1]$ there are at least two prime numbers. In other words, in the interval $[n^2+n+1,\ n^2+3n]$ there is at least one prime number.

The difference between the numbers n^2+n+1 and n^2+3n is $n^2+3n-(n^2+n+1)=n^2+3n-n^2-n-1=2n-1$. In addition to this, $\left\lfloor \sqrt{n^2+n+1} \right\rfloor = n$. This means that in the interval $\left[n^2+n+1,\ \ n^2+n+1+2\left\lfloor \sqrt{n^2+n+1} \right\rfloor -1\right]$ there is at least one prime number. In other words, if $a=n^2+n+1$ then the interval $\left[a,\ a+2\left\lfloor \sqrt{a} \right\rfloor -1\right]$ contains at least one prime number.

Note: The symbol $\lfloor\ \rfloor$ represents the *floor function*. The floor function of a given number is the largest integer that is not greater than that number. For example, $\lfloor x \rfloor$ is the largest integer that is not greater than *x*.

Now, if Conjecture C is true, then the following statements are all true:

1. If a is a perfect square, then in the interval $\left[a,\ a+\left\lfloor \sqrt{a} \right\rfloor\right]$ there is at least one prime number.

2. If a is an integer such that $n^2 < a \le n^2+n+1 < (n+1)^2$, then in the interval $\left[a,\ a+\left\lfloor \sqrt{a} \right\rfloor -1\right]$ there is at least one prime number.

3. If a is an integer such that $n^2 < n^2+n+2 \le a < (n+1)^2$, then in the interval $\left[a,\ a+2\left\lfloor \sqrt{a} \right\rfloor -1\right]$ there is at least one prime number.

We know that $a+2\left\lfloor \sqrt{a} \right\rfloor -1 \ge a+\left\lfloor \sqrt{a} \right\rfloor$.

Proof:

$$a+2\left\lfloor \sqrt{a} \right\rfloor -1 \ge a+\left\lfloor \sqrt{a} \right\rfloor \iff 2\left\lfloor \sqrt{a} \right\rfloor -1 \ge \left\lfloor \sqrt{a} \right\rfloor \iff 2\left\lfloor \sqrt{a} \right\rfloor \ge \left\lfloor \sqrt{a} \right\rfloor +1 \iff$$

$\iff \left\lfloor \sqrt{a} \right\rfloor + \left\lfloor \sqrt{a} \right\rfloor \ge \left\lfloor \sqrt{a} \right\rfloor +1 \iff \left\lfloor \sqrt{a} \right\rfloor \ge 1$, which is true for every positive integer *a*.

And we also know that $a+2\lfloor\sqrt{a}\rfloor-1>a+\lfloor\sqrt{a}\rfloor-1$.

Proof:

$a+2\lfloor\sqrt{a}\rfloor-1>a+\lfloor\sqrt{a}\rfloor-1 \Leftrightarrow 2\lfloor\sqrt{a}\rfloor>\lfloor\sqrt{a}\rfloor$, which is obviously true for every positive integer a.

All this means that the interval $\left[a,\ a+2\lfloor\sqrt{a}\rfloor-1\right]$ can be applied to the number a from statement 1., to the number a from statement 2. and to the number a from statement 3.

Therefore, if n is any positive integer and Conjecture C is true, then in the interval $\left[n,\ n+2\lfloor\sqrt{n}\rfloor-1\right]$ there is at least one prime number (we change letter a for letter n). According to this, we can also say that if Conjecture C is true then in the interval $\left[n,\ n+2\sqrt{n}-1\right]$ there is always a prime number for every positive integer n.

Now… how can we prove Conjecture C?

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Email: german_a_paz@hotmail.com

See also:

- *Números primos, fórmula precisa*. Original paper: *Cálculo de la cantidad de números primos que hay por debajo de un número dado* (How to calculate the amount of prime numbers that are less than a given number)
- *Números Primos de Sophie Germain, Demostración de su Infinitud* (There Are Infinitely Many Sophie Germain Prime Numbers)
- *Infinitely Many Prime Numbers of the Form ap±b Infinitos Números Primos de la Forma ap±b*

Papers by Carlos Giraldo Ospina I recommend (these papers are available at http://numerosprimos.8m.com/Documentos.htm):

- *Números Primos* (Prime Numbers)
- *Primos, Dispersión Parabólica* (Primes, Parabolic Dispersion)
- *Primos de Mersenne, Dispersión Parabólica* (Mersenne Primes, Parabolic Dispersion)
- *Proof of Legendre's and Brocard's Conjectures*