Model of superluminal oscillating neutrinos

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We present a simple quantum relativistic model of neutrino oscillations and propagation in space. Matrix elements of the neutrino Hamiltonian depend on momentum and this dependence is responsible for the observed neutrino speed. On average neutrino travels with the speed of light, but instantaneous speed oscillates around c in a pattern synchronized with flavor oscillations. Due to low masses of the familiar neutrino species ν_e, ν_μ, ν_τ the predicted effect is extremely small. However, if one assumes the existence of a fourth supermassive ($m > 0.3\ GeV/c^2$) neutrino flavor, then this theory can explain the superluminal propagation of ν_μ seen in the MINOS and OPERA experiments. Based on this assumption we provide specific predictions for future neutrino velocity measurements. The consistence of our approach with fundamental principles of relativity and causality is discussed as well.

I. INTRODUCTION

A recent preprint [1] published by the OPERA collaboration claims observation of a superluminal effect in neutrino propagation. A similar result was obtained in an earlier MINOS experiment [2], although the accuracy of this measurement was not sufficient to insist on the presence of superluminality. The great importance of these observations is obvious as they challenge the fundamentals of Einstein's special relativity, which forbids any kind of superluminal propagation of particles and asserts that any such effect would mean a violation of the principle of causality. Dozens of attempts at theoretical explanation of the OPERA effect have already appeared in the literature – some of them were reviewed in [3, 4] – but no one gained a universal acceptance. So, there is a great deal of scepticism in the scientific community regarding these remarkable observations, and the prevailing attitude seems to be to blame a yet undiscovered systematic experimental error. In this paper we will offer a possible explanation for the neutrino superluminality, which, on one hand, is fully within mainstream quantum relativistic physics, and on the other hand, challenges the traditional interpretation of Einstein's relativity theory.

In section II we discuss the present experimental situation around the issue of neutrino superluminality. In sections III - V we suggest a simple but realistic model of neutrino propagation in space. Our model is formulated in one spatial dimension, but its generalization for the real 3D world is not expected to bring about any significant changes. The model is fully relativistic, meaning that commutation relations of the Poincaré Lie algebra are explicitly satisfied by operators of the total momentum, total energy and boost. According to this model, the velocity of high energy neutrinos oscillates around the average value c, so that in certain conditions a superluminal propagation speed can be detected. The oscillating time dependence of the neutrino speed is directly related to well-known flavor oscillations.

The simplest versions of our model, which involve two or three neutrino flavors, agree with observations qualitatively but they have a serious defect that is explained in detail in section VI: Small neutrino masses and the observed persistence of the superluminal effect across a wide energy range [1] imply that the magnitude of superluminality should be very small. The most natural resolution of this contradiction can be found if one postulates existence of a fourth neutrino flavor having extremely large mass exceeding 0.3 GeV/c^2. In section VII we will see that this idea allows us to reproduce all existing experimental data and to make specific predictions for future experiments. In section VIII we will discuss how these results may affect our interpretation of the principles of relativity and causality.

II. EXPERIMENTAL DATA

There were three major experiments [1, 2, 5] measuring μ-neutrino propagation speed. Their essential parameters are listed in Table I. All these experiments shared the same basic design: An energetic proton beam from accelerator collided with a target thus producing charged π and K mesons which decayed in-flight into muon and a μ-neutrino. The neutrino beam was captured by a distant detector. Then, knowing the time-of-flight t and the propagation length L one could determine the apparent propagation speed as $v_\mu \equiv L/t$. It is convenient to express experimental results in terms of the parameter

$$\delta v \equiv \frac{v_\mu - c}{c} \quad (1)$$

Positive values of δv correspond to superluminal propagation.

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TABLE I. Experiments measuring neutrino velocity.

Property	Fermilab [5, 6]	MINOS [2]	OPERA [1]	SN1987A [7–9]
Neutrino flavor	ν_μ	ν_μ	ν_μ	$\overline{\nu}_e$
Average energy E (GeV)	32 - 195	3	17	0.0075 - 0.040
Base L (km)	0.55 - 0.895	734	730	15×10^{12}
$\beta = L/E$ (km/GeV)	$0.003 - 0.028$	245	43	$(0.38 - 2.0) \times 10^{15}$
δv exp.	$(0 \pm 4) \times 10^{-5}$	$(5.1 \pm 2.9) \times 10^{-5}$	$(2.37 \pm 0.43) \times 10^{-5}$	$(0 \pm 2) \times 10^{-9}$
δv theor.	2.37×10^{-5}	3.0×10^{-5}	2.37×10^{-5} (fitted)	≈ 0

In the early experiment at Fermilab [5, 6] this scheme was not followed as the propagation time t was not actually measured. Instead, the experimentalists have noticed that neutrinos arrived in the detector almost simultaneously with muons originated from the same meson decay. Since energetic muons are known to travel with the speed of light, it was concluded that neutrino speed does not exceed c as well. The experimental limit on the parameter δv was found to be $|\delta v| < 4 \times 10^{-5}$ for a number of neutrino energies E ranging from 32 GeV to 195 GeV.

A more recent MINOS experiment [2] used a lower neutrino energy of about 3 GeV and performed direct time of flight measurements on a long baseline of 734 km. A significant superluminal effect $\delta v = 5.1 \times 10^{-5}$ was observed, but experimental uncertainties were too high to definitively claim the discovery.

A similar design was used in the recent OPERA experiment [1]: Muon-type neutrinos with energies of about 17 GeV were produced at the CERN accelerator site and registered by the OPERA detector 730 kilometers away. Due to improved time and distance measurement techniques the superluminality $\delta v = 2.37 \times 10^{-5}$ was confirmed with an impressive 6σ significance. For our discussion below it is important that neutrino energies were in a broad interval 13.8 GeV - 40.7 GeV, and no significant energy dependence of δv was found.

Relevant data from a different kind of observation are presented in the last column of the Table. In this case electron antineutrinos emitted by the SN1987A supernova were detected on Earth [7, 8]. So, the propagation length was $L = 160000$ light years. It was concluded that parameter δv was essentially zero with an extremely low uncertainty of 2×10^{-9} [9].

For our discussion in this work we will also need numerical values of essential neutrino properties, such as their masses and mixing angles shown in Table II. These numbers are useful not only for comparison of our results with experiments in section VII, but also for justifying approximations in various formulas throughout this work. Neutrino masses are not well established: neither their free (non-interacting) values $m_{e,\mu,\tau}$ nor eigenvalues $m_{1,2,3}$ of the interacting mass operator. The present consensus is that these masses are rather low – on the order of 1 eV/c^2. It is well established that in the course of propagation neutrinos of one flavor experience partial conversion into other flavors due to the effect of *neutrino oscillations* [10, 11]. Experimental studies of neutrino oscillation frequencies [12, 13] provide rather precise values of differences of squared mass eigenvalues shown in the Table. Observed oscillation amplitudes are related to mixing angles. Only the θ_{23} angle is relevant for this work. Note that the mixing coefficient $\sin^2 2\theta_{23}$ is only known to be higher than 0.9 [12]. In our calculations we used the value of 0.97 for illustration purposes. The considered energy range from 3 GeV to 40 GeV covers values characteristic for two experiments (OPERA and MINOS) in which the superluminal effect in neutrino propagation has been observed with some certainty.

TABLE II. Neutrino properties used in this work.

Property	Value
Masses $m_{1,2,3}$	≈ 1 eV/c^2
$\|m_3^2 - m_2^2\|$	2.43×10^{-3} eV^2/c^4 [12]
$m_2^2 - m_1^2$	8.0×10^{-5} eV^2/c^4 [13]
Mixing coefficient $\sin^2 2\theta_{23}$	0.97
Energy E	3 - 40 GeV [1, 2]

III. NON-INTERACTING NEUTRINOS

We would like to describe a free neutrino system oscillating between two states: μ-neutrino and τ-neutrino. For simplicity, at this stage we will ignore the possible effect of the third (electron) neutrino species. Then the Hilbert space can be constructed as a direct sum of two one-particle subspaces

$$\mathcal{H} = \mathcal{H}_\mu \oplus \mathcal{H}_\tau \tag{2}$$

This Hilbert space will be used for both non-interacting and interacting neutrino systems. This introductory section will cover the case in which the oscillation-causing interaction is turned off.

A. Representation of the Poincaré group

Both $\mathcal{H}_\mu$ and $\mathcal{H}_\tau$ are Hilbert spaces carrying unitary irreducible representations of the Poincaré group characterized by (non-observable) free neutrino masses m_μ and m_τ, respectively. In our 1-dimensional model neutrinos are also spinless. The noninteracting representation of the Poincaré group acting in the Hilbert space $\mathcal{H}$ can be built as a direct sum of these two irreducible representations. To write explicit formulas we will choose a convenient basis set in (2). For each momentum p we select two orthonormal basis states of definite flavor. Then each normalized state vector $|\psi\rangle$ can be represented as a 2-component momentum-dependent vector in this (flavor) basis

$$|\psi\rangle \equiv \begin{bmatrix} \Phi_\mu(p) \\ \Phi_\tau(p) \end{bmatrix}$$

where $\Phi_{\mu,\tau}(p)$ are complex wave functions satisfying the normalization condition

$$\int dp \left(|\Phi_\mu(p)|^2 + |\Phi_\tau(p)|^2\right) = 1$$

Projection operators on the particle subspaces $\mathcal{H}_\mu$ and $\mathcal{H}_\tau$ are

$$\Pi_\mu = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad (3)$$

$$\Pi_\tau = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad (4)$$

respectively.

In this paper we adopt Schrödinger representation: Any inertial change of the observer is reflected in a change of system's state vector or wave function. Different observers use the same Hermitian operator to describe a given observable. Finite transformations from the Poincaré group (space translations, time translations and boosts) are represented in the Hilbert space by exponential functions of generators [14]

$$e^{\frac{i}{\hbar}P_0 a}|\psi\rangle = \begin{bmatrix} e^{\frac{i}{\hbar}pa}\Phi_\mu(p) \\ e^{\frac{i}{\hbar}pa}\Phi_\tau(p) \end{bmatrix}$$

$$e^{-\frac{i}{\hbar}H_0 t}|\psi\rangle = \begin{bmatrix} e^{-\frac{i}{\hbar}\omega_\mu(p)t}\Phi_\mu(p) \\ e^{-\frac{i}{\hbar}\omega_\tau(p)t}\Phi_\tau(p) \end{bmatrix}$$

$$e^{\frac{i}{\hbar}K_0 c\theta}|\psi\rangle = \begin{bmatrix} \sqrt{\frac{\omega_\mu(\Lambda_\mu p)}{\omega_\mu(p)}}\Phi_\mu(\Lambda_\mu p) \\ \sqrt{\frac{\omega_\tau(\Lambda_\tau p)}{\omega_\tau(p)}}\Phi_\tau(\Lambda_\tau p) \end{bmatrix}$$

where

$$\omega_{\mu,\tau}(p) \equiv \sqrt{m_{\mu,\tau}^2 c^4 + p^2 c^2}$$

$$\Lambda_{\mu,\tau} p \equiv p\cosh\theta - \frac{\omega_{\mu,\tau}}{c}\sinh\theta$$

and parameter θ is related to the boost velocity v by formula $v = c\tanh\theta$.

The basis of the corresponding representation of the Poincaré Lie algebra is provided by Hermitian operators of total momentum P_0, total energy H_0 and boost K_0. Explicit matrix forms of these generators can be obtained by differentiation

$$P_0 = -i\hbar \lim_{a\to 0}\frac{d}{da}e^{\frac{i}{\hbar}P_0 a} = \begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix} \quad (5)$$

$$H_0 = \begin{bmatrix} \omega_\mu(p) & 0 \\ 0 & \omega_\tau(p) \end{bmatrix} \quad (6)$$

$$K_0 = -i\hbar \begin{bmatrix} \frac{\omega_\mu(p)}{c^2}\frac{d}{dp} + \frac{p}{2\omega_\mu(p)} & 0 \\ 0 & \frac{\omega_\tau(p)}{c^2}\frac{d}{dp} + \frac{p}{2\omega_\tau(p)} \end{bmatrix} \quad (7)$$

The Newton-Wigner (center of energy) position operator is given by formula [15]

$$R_0 = -\frac{c^2}{2}(K_0 H_0^{-1} + H_0^{-1}K_0) = i\hbar \begin{bmatrix} \frac{d}{dp} & 0 \\ 0 & \frac{d}{dp} \end{bmatrix} \quad (8)$$

Position operators for individual particles can be obtained by applying projection operators (3) - (4) to (8)

$$r_\mu = \Pi_\mu R_0 \Pi_\mu = i\hbar \begin{bmatrix} \frac{d}{dp} & 0 \\ 0 & 0 \end{bmatrix} \quad (9)$$

$$r_\tau = \Pi_\tau R_0 \Pi_\tau = i\hbar \begin{bmatrix} 0 & 0 \\ 0 & \frac{d}{dp} \end{bmatrix} \quad (10)$$

B. Particle trajectories

The above formalism allows us to obtain classical trajectories of non-interacting neutrinos. By itself, this calculation is rather trivial. We reproduce it here because it provides a useful template for the more interesting interacting case in section V. Suppose that at time $t = 0$ we prepared a state vector with one μ-neutrino having a normalized momentum-space wave function $\psi(p)$

$$|\psi(0)\rangle \equiv \begin{bmatrix} \psi(p) \\ 0 \end{bmatrix} \quad (11)$$

$$\int dp|\psi(p)|^2 = 1$$

Let us now postulate that this wave function is localized in a narrow region Δp of the momentum space and that the center of the wave packet is at a large positive momentum $\langle p\rangle > 3$ GeV/c. Then we can safely conclude that our particle is ultrarelativistic

$$\langle p\rangle \gg m_\mu c \approx m_\tau c \tag{12}$$
$$\omega_\mu(p) \approx \omega_\tau(p) \approx cp \tag{13}$$

The expectation value of the μ-neutrino position at $t=0$ is

$$\begin{aligned}\langle r_\mu(0)\rangle &\equiv \langle\psi(0)|r_\mu|\psi(0)\rangle \\ &= i\hbar\int dp[\psi^*(p),0]\begin{bmatrix}\frac{d}{dp} & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\psi(p)\\ 0\end{bmatrix} \\ &= i\hbar\int dp\psi^*(p)\frac{d\psi(p)}{dp}\end{aligned} \tag{14}$$

Since r_μ is a Hermitian operator, this expectation value must be real. To make this fact obvious, we can rewrite (14) in an explicitly real form by using the fact that the wave function $\psi(p)$ vanishes at infinity $\psi(-\infty)=\psi(+\infty)=0$

$$\begin{aligned}\langle r_\mu(0)\rangle &= i\hbar|\psi(p)|^2\Big|_{-\infty}^{+\infty} - i\hbar\int dp\psi(p)\frac{d\psi^*(p)}{dp} \\ &= \frac{i\hbar}{2}\left(\int dp\psi^*(p)\frac{d\psi(p)}{dp} - \int dp\psi(p)\frac{d\psi^*(p)}{dp}\right) \\ &= -\hbar\int_{Im} dp\psi^*(p)\frac{d\psi(p)}{dp}\end{aligned}$$

where $\int_{Im}$ means the imaginary part of the integral. At a non-zero time instant t

$$\begin{aligned}&\langle r_\mu(t)\rangle = \langle\psi(0)|e^{\frac{i}{\hbar}H_0 t}r_\mu e^{-\frac{i}{\hbar}H_0 t}|\psi(0)\rangle \\ &= -\hbar\int_{Im} dp[\psi^*(p)e^{\frac{i}{\hbar}\omega_\mu(p)t},0]\begin{bmatrix}\frac{d}{dp} & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\psi(p)e^{-\frac{i}{\hbar}\omega_\mu(p)t}\\ 0\end{bmatrix} \\ &\approx -\hbar\int_{Im} dp\psi^*(p)\frac{d\psi(p)}{dp} + ct\int dp|\psi(p)|^2 \\ &= \langle r_\mu(0)\rangle + ct\end{aligned} \tag{15}$$

If the initial state is in the τ-neutrino sector then, analogously

$$\langle r_\tau(t)\rangle = \langle r_\tau(0)\rangle + ct \tag{16}$$

Within our linear approximation (13) we have neglected the wave function "spreading" effect, which is known to be superluminal but negligibly small [16–22]. Formulas (15) - (16) show that high-energy non-interacting neutrinos propagate with velocities just below the speed of light. However, this result cannot be applied directly to the OPERA and MINOS experiments, because real neutrinos experience an ever-present interaction, which is responsible for the oscillation effect [10]. Our goal in this paper is to find out how this interaction affects neutrino trajectories. Our calculation method is, basically, similar to that outlined above. First, we need to construct a representation of the Poincaré group, which is responsible for oscillations in the Hilbert space $\mathcal{H}$. We are especially interested in the interacting Hamiltonian H, whose construction will be done in section IV. Once the Hamiltonian is known, we can use it in place of H_0 in (15) and (16) to get trajectories of oscillating neutrinos. This calculation will be performed in sections V and VI.

IV. INTERACTION

A. Interacting Hamiltonian

In the Dirac's instant form of dynamics [23, 24], relativistically invariant description of interaction is achieved by adding extra terms to both the energy operator $H=H_0+V$ and the boost operator $K=K_0+Z$, while keeping the total momentum P_0 unchanged. The choice of interactions V and Z must ensure that Poincaré commutators remain the same as in the non-interacting case

$$[H,P_0]=0 \tag{17}$$
$$[K,P_0] = -\frac{i\hbar}{c^2}H \tag{18}$$
$$[K,H] = -i\hbar P_0 \tag{19}$$

In this work we will assume that the Hermitian interaction operator is

$$V = \begin{bmatrix}\xi(p) & f(p)\\ f(p) & \zeta(p)\end{bmatrix}$$

where diagonal elements $\xi(p)$, $\zeta(p)$ and the off-diagonal $f(p)$ are smooth real functions [25]. Then in the flavor basis we can write the full Hamiltonian as a 2×2 momentum-dependent matrix

$$H = H_0 + V = \begin{bmatrix}\Omega_\mu(p) & f(p)\\ f(p) & \Omega_\tau(p)\end{bmatrix} \tag{20}$$

where $\Omega_\mu(p)\equiv\omega_\mu(p)+\xi(p)$ and $\Omega_\tau(p)\equiv\omega_\tau(p)+\zeta(p)$. The corresponding operator of interacting mass is defined as $M=+\sqrt{H^2-P_0^2c^2}/c^2$.

B. Mass (energy) eigenstates

Our primary goal in this section is to calculate the time evolution of neutrino states. This can be done most easily if we find eigenvalues $E_{2,3}(p)$ and eigenstates of H. So, we need to solve equation

$$0 = \begin{bmatrix} \Omega_\mu(p) - E_{2,3}(p) & f(p) \\ f(p) & \Omega_\tau(p) - E_{2,3}(p) \end{bmatrix} \begin{bmatrix} \Phi_\mu^{2,3}(p) \\ \Phi_\tau^{2,3}(p) \end{bmatrix} \tag{21}$$

together with normalization conditions ($i = 2, 3$)

$$|\Phi_\mu^i(p)|^2 + |\Phi_\tau^i(p)|^2 = 1 \tag{22}$$

For the eigenvalues E_2, E_3 we obtain two equations

$$\begin{aligned} f^2(p) &= [\Omega_\mu(p) - E_2(p)]\,[\Omega_\tau(p) - E_2(p)] \\ &= [\Omega_\mu(p) - E_3(p)]\,[\Omega_\tau(p) - E_3(p)] \end{aligned}$$

A necessary requirement for this theory to be relativistically invariant is that energy eigenvalues have the standard momentum dependence

$$E_{2,3}(p) = \sqrt{m_{2,3}^2 c^4 + p^2 c^2} \tag{23}$$

where $m_{2,3}$ are neutrino mass eigenvalues. The true Hamiltonian (20) is not known, so we are free to make our guesses. We will assume that the mass eigenvalues are known: $m_3 > m_2 > 0$. Then, having at our disposal three adjustable real functions $\Omega_\mu(p)$, $\Omega_\tau(p)$ and $f(p)$ we can always choose them in such a way that condition (23) is satisfied and

$$\Omega_\mu(p) + \Omega_\tau(p) = E_2(p) + E_3(p)$$

For example, we can express $\Omega_\tau(p)$ and $f(p)$ in terms of an arbitrarily chosen $\Omega_\mu(p)$

$$\Omega_\tau(p) = E_2(p) + E_3(p) - \Omega_\mu(p) \tag{24}$$

$$f^2(p) = (\Omega_\mu(p) - E_2(p))(E_3(p) - \Omega_\mu(p)) \tag{25}$$

As can be verified by direct substitution in (21) - (22), common eigenvectors of H, M and P_0 are

$$|2, p\rangle = \begin{bmatrix} A(p) \\ -B(p) \end{bmatrix}$$

$$|3, p\rangle = \begin{bmatrix} B(p) \\ A(p) \end{bmatrix}$$

where we introduced notation

$$A(p) \equiv +\sqrt{\frac{\Omega_\tau(p) - E_2(p)}{E_3(p) - E_2(p)}} \tag{26}$$

$$B(p) \equiv +\sqrt{\frac{\Omega_\mu(p) - E_2(p)}{E_3(p) - E_2(p)}} \tag{27}$$

$$A^2(p) + B^2(p) = 1 \tag{28}$$

Parameters A and B can be written in a more standard form [26]

$$\begin{aligned} A(p) &\equiv \cos\theta_{23}(p) \\ B(p) &\equiv \sin\theta_{23}(p) \end{aligned}$$

but we will stick with A and B to keep our formulas short.

Next we need to find a connection between the flavor and mass-energy bases. If $(\Psi_2(p), \Psi_3(p))$ is a state vector written in the basis of mass eigenstates [27], then its expansion in the flavor basis is obtained by a unitary transformation

$$\begin{bmatrix} \Phi_\mu(p) \\ \Phi_\tau(p) \end{bmatrix} = \begin{pmatrix} A(p) & B(p) \\ -B(p) & A(p) \end{pmatrix} \begin{pmatrix} \Psi_2(p) \\ \Psi_3(p) \end{pmatrix} \tag{29}$$

The transformation from the flavor basis to the mass basis is provided by the inverse matrix

$$\begin{pmatrix} \Psi_2(p) \\ \Psi_3(p) \end{pmatrix} = \begin{bmatrix} A(p) & -B(p) \\ B(p) & A(p) \end{bmatrix} \begin{bmatrix} \Phi_\mu(p) \\ \Phi_\tau(p) \end{bmatrix} \tag{30}$$

C. Interacting representation of the Poincaré group

The mass basis is useful because the interacting representation of the Poincaré group takes especially simple form there

$$e^{-\frac{i}{\hbar}Ht} \begin{pmatrix} \Psi_2(p) \\ \Psi_3(p) \end{pmatrix} = \begin{pmatrix} e^{-\frac{i}{\hbar}E_2(p)t}\Psi_2(p) \\ e^{-\frac{i}{\hbar}E_3(p)t}\Psi_3(p) \end{pmatrix} \tag{31}$$

$$e^{\frac{i}{\hbar}Kc\theta} \begin{pmatrix} \Psi_2(p) \\ \Psi_3(p) \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{E_2(\Lambda_1 p)}{E_2(p)}}\Psi_2(\Lambda_1 p) \\ \sqrt{\frac{E_3(\Lambda_2 p)}{E_3(p)}}\Psi_3(\Lambda_2 p) \end{pmatrix}$$

where $\Lambda_i p \equiv p\cosh\theta - (E_i/c)\sinh\theta$ is the usual boost transformation of momentum.

Poincaré generators in the mass basis can be obtained by differentiation similar to (5) - (7)

$$H = i\hbar \lim_{t\to 0} \frac{d}{dt} e^{-\frac{i}{\hbar}Ht} = \begin{pmatrix} E_2(p) & 0 \\ 0 & E_3(p) \end{pmatrix} \tag{32}$$

$$K = -i\hbar \begin{pmatrix} \frac{E_2(p)}{c^2}\frac{d}{dp} + \frac{p}{2E_2(p)} & 0 \\ 0 & \frac{E_3(p)}{c^2}\frac{d}{dp} + \frac{p}{2E_3(p)} \end{pmatrix} \tag{33}$$

$$P_0 = \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} \tag{34}$$

In this representation one can easily verify that commutators (17) - (19) are satisfied. So, our theory is relativistically invariant.

V. INTERACTING TIME EVOLUTION

Obviously, the state vector with one μ-neutrino (11) is not a stationary eigenstate of the Hamiltonian (20). Our goal in this section is to calculate the time evolution of pure flavor states.

A. Time-dependent wave function

In analogy with (12) - (13) and using neutrino parameters from Table I we can approximate

$$\langle p\rangle \gg m_{2,3}c$$
$$E_2(p) = \sqrt{m_2^2c^4+p^2c^2} \approx cp$$
$$E_3(p) = \sqrt{m_3^2c^4+p^2c^2} \approx cp+\gamma(p)$$
$$\frac{dE_2(p)}{dp} \approx \frac{dE_3(p)}{dp} \approx c \tag{35}$$
$$\gamma(p) \approx \frac{(m_3^2-m_2^2)c^3}{2p} \tag{36}$$
$$\frac{d\gamma(p)}{dp} \approx 0 \tag{37}$$

To find the time evolution of the initial state (11) we use (30) to expand it in the mass basis

$$|\psi(0)\rangle = \psi(p)\begin{pmatrix} A(p) \\ B(p)\end{pmatrix}$$

and apply (31)

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}Ht}|\psi(0)\rangle = \psi(p)\begin{pmatrix} A(p)e^{-\frac{i}{\hbar}E_2(p)t} \\ B(p)e^{-\frac{i}{\hbar}E_3(p)t}\end{pmatrix} \tag{38}$$

Wave function components in the flavor basis can be found using transformation (29)

$$\begin{aligned}
&|\psi(t)\rangle \\
&= \psi(p)\begin{pmatrix} A(p) & B(p) \\ -B(p) & A(p)\end{pmatrix}\begin{pmatrix} A(p)e^{-\frac{i}{\hbar}E_2(p)t} \\ B(p)e^{-\frac{i}{\hbar}E_3(p)t}\end{pmatrix} \\
&= \psi(p)\begin{bmatrix} A^2(p)e^{-\frac{i}{\hbar}E_2(p)t}+B^2(p)e^{-\frac{i}{\hbar}E_3(p)t} \\ A(p)B(p)\left(e^{-\frac{i}{\hbar}E_3(p)t}-e^{-\frac{i}{\hbar}E_2(p)t}\right)\end{bmatrix}
\end{aligned} \tag{39}$$

B. Oscillations

The probabilities for finding μ-neutrino and τ-neutrino in the state (39) can be found as expectation values of operators (3) - (4) projecting on the corresponding flavor subspaces. Before evaluating these integrals let us make a few comments about how we are going to deal with momentum integrals in this work. The integrands always contain a momentum-space wave function $\psi(p)$ which was assumed to be localized within a small interval Δp centered at momentum $\langle p\rangle \approx E/c$, where E is particle's energy. Inside this interval we can treat functions $A(p)$, $B(p)$, $E_{2,3}(p)$ and $\gamma(p)$ as constants (denoted simply by A, B, $E_{2,3}$ and γ). These constants can be moved outside the integral sign. In some integrals we will also meet derivatives $dA(p)/dp$, $dB(p)/dp$, $d\Omega_\mu(p)/dp$, etc. We will ignore their variations within Δp as well and replace them by constants denoted dA/dp, dB/dp, $d\Omega_\mu/dp\ldots$. With these considerations in mind we find that flavor probabilities are sinusoidal functions of time [10]

$$\begin{aligned}
\rho_\mu(t) &\equiv \langle\psi(t)|\Pi_\mu|\psi(t)\rangle \approx \left(A^2e^{\frac{i}{\hbar}E_2t}+B^2e^{\frac{i}{\hbar}E_3t}\right)\left(A^2e^{-\frac{i}{\hbar}E_2t}+B^2e^{-\frac{i}{\hbar}E_3t}\right)\int dp|\psi(p)|^2 \\
&= A^4+B^4+2A^2B^2\cos\left(\frac{\gamma t}{\hbar}\right) = 1-\sin^2 2\theta_{23}\sin^2\frac{\gamma t}{2\hbar} \\
\rho_\tau(t) &\equiv \langle\psi(t)|\Pi_\tau|\psi(t)\rangle \approx A^2B^2\left(e^{\frac{i}{\hbar}E_3t}-e^{\frac{i}{\hbar}E_2t}\right)\left(e^{-\frac{i}{\hbar}E_3t}-e^{-\frac{i}{\hbar}E_2t}\right)\int dp|\psi(p)|^2 \\
&= 2A^2B^2-2A^2B^2\cos\left(\frac{\gamma t}{\hbar}\right) = \sin^2 2\theta_{23}\sin^2\frac{\gamma t}{2\hbar} \\
1 &= \rho_\mu(t)+\rho_\tau(t)
\end{aligned} \tag{40}$$

In our ultrarelativistic limit the oscillation period is

$$T = \frac{2\pi\hbar}{\gamma} \approx \frac{4\pi\hbar E}{\Delta m^2c^4} \tag{41}$$

C. Conservation laws

The oscillatory behavior of neutrinos described above may raise doubts about the validity of conservation laws. However, there is no reason for concerns. Conservation laws for the total momentum P_0 and energy H are easily verified using mass basis representation formulas (32), (34) and (38)

$$\langle P_0(t)\rangle \equiv \langle \psi(t)|P_0|\psi(t)\rangle = \langle p\rangle$$
$$\langle H(t)\rangle \equiv \langle \psi(t)|H|\psi(t)\rangle = c\langle p\rangle$$

More work is required to prove another conservation law that says that the center of energy of any isolated physical system moves with a constant velocity along a straight line. This law follows from the definition of the center-of-energy position [28]

$$R = -\frac{c^2}{2}(KH^{-1} + H^{-1}K)$$

and the relationship (written in the Heisenberg representation)

$$K(t) \equiv e^{\frac{i}{\hbar}Ht} K e^{-\frac{i}{\hbar}Ht} = K - P_0 t$$

which is a direct consequence of basic commutators (18) - (19). Combining these two formulas we obtain the following linear time dependence for the center-of-energy expectation value in any state

$$\langle R(t)\rangle = \langle R(0)\rangle + \frac{c^2\langle P_0(0)\rangle}{\langle H(0)\rangle}t = \langle R(0)\rangle + ct \quad (42)$$

To verify this result explicitly for our state (38) we use the matrix form of the boost operator (33) and definition

$$\langle K(0)\rangle = \int dp \psi^*(p) K \psi(p)$$

Then with the help of eq. (28) we obtain

$$\begin{aligned}
\langle K(t)\rangle &\equiv \langle \psi(t)|K|\psi(t)\rangle \\
&= -i\hbar \int dp \psi^*(p) \left(A(p)e^{\frac{i}{\hbar}E_2(p)t}, B(p)e^{\frac{i}{\hbar}E_3(p)t}\right) \begin{pmatrix} \frac{E_2(p)}{c^2}\frac{d}{dp} + \frac{p}{2E_2(p)} & 0 \\ 0 & \frac{E_3(p)}{c^2}\frac{d}{dp} + \frac{p}{2E_3(p)} \end{pmatrix} \psi(p) \begin{pmatrix} A(p)e^{-\frac{i}{\hbar}E_2(p)t} \\ B(p)e^{-\frac{i}{\hbar}E_3(p)t} \end{pmatrix} \\
&\approx \left(Ae^{\frac{i}{\hbar}E_2 t}, Be^{\frac{i}{\hbar}E_3 t}\right) \begin{pmatrix} Ae^{-\frac{i}{\hbar}E_2 t} \\ Be^{-\frac{i}{\hbar}E_3 t} \end{pmatrix} \int dp \psi^*(p) K \psi(p) \\
&- \frac{i\hbar\langle p\rangle}{c} \left(Ae^{\frac{i}{\hbar}E_2 t}, Be^{\frac{i}{\hbar}E_3 t}\right) \begin{pmatrix} \frac{dA}{dp}e^{-\frac{i}{\hbar}E_2 t} - \frac{iA}{\hbar}cte^{-\frac{i}{\hbar}E_2 t} \\ \frac{dB}{dp}e^{-\frac{i}{\hbar}E_3 t} - \frac{iB}{\hbar}cte^{-\frac{i}{\hbar}E_3 t} \end{pmatrix} \int dp|\psi(p)|^2 \\
&= \langle K(0)\rangle - \frac{i\hbar\langle p\rangle}{c}\left(A\frac{dA}{dp} - \frac{iA^2}{\hbar}ct + B\frac{dB}{dp} - \frac{iB^2}{\hbar}ct\right) = \langle K(0)\rangle - \langle p\rangle t
\end{aligned}$$

This means that the center-of-energy $\langle R(t)\rangle = -c^2\langle K(t)\rangle/\langle H\rangle$ moves with the light speed c, as expected from (42).

D. Neutrino trajectories

We find averaged trajectories of the two neutrino species as expectation values of their position operators (9) - (10) scaled by corresponding probabilities $\rho_{\mu,\tau}(t)$

$$\langle r_\mu(t)\rangle = \frac{\langle \psi(t)|r_\mu|\psi(t)\rangle}{\rho_\mu(t)} \quad (43)$$

$$\langle r_\tau(t)\rangle = \frac{\langle \psi(t)|r_\tau|\psi(t)\rangle}{\rho_\tau(t)} \quad (44)$$

For the ν_μ trajectory we obtain

$$
\begin{aligned}
&\langle\psi(t)|r_\mu|\psi(t)\rangle \\
&= -\hbar\int_{Im} dp\psi^*(p)\left(A^2(p)e^{\frac{i}{\hbar}E_2(p)t}+B^2(p)e^{\frac{i}{\hbar}E_3(p)t}\right)\frac{d}{dp}\psi(p)\left(A^2(p)e^{-\frac{i}{\hbar}E_2(p)t}+B^2(p)e^{-\frac{i}{\hbar}E_3(p)t}\right) \\
&\approx -\hbar\left(A^2e^{\frac{i}{\hbar}E_2t}+B^2e^{\frac{i}{\hbar}E_3t}\right)\left(A^2e^{-\frac{i}{\hbar}E_2t}+B^2e^{-\frac{i}{\hbar}E_3t}\right)\int_{Im} dp\psi^*(p)\frac{d}{dp}\psi(p) \\
&-\hbar\ Im\left[\left(A^2e^{\frac{i}{\hbar}E_2t}+B^2e^{\frac{i}{\hbar}E_3t}\right)\left(\frac{dA^2}{dp}e^{-\frac{i}{\hbar}E_2t}-\frac{iA^2ct}{\hbar}e^{-\frac{i}{\hbar}E_2t}+\frac{dB^2}{dp}e^{-\frac{i}{\hbar}E_3t}-\frac{iB^2ct}{\hbar}e^{-\frac{i}{\hbar}E_3t}\right)\right]\int dp|\psi(p)|^2 \\
&= \langle r_\mu(0)\rangle\rho_\mu(t)-\hbar\left(B^2\frac{dA^2}{dp}\sin\frac{\gamma t}{\hbar}-\frac{A^4ct}{\hbar}-\frac{A^2B^2ct}{\hbar}\cos\frac{\gamma t}{\hbar}-A^2\frac{dB^2}{dp}\sin\frac{\gamma t}{\hbar}-\frac{A^2B^2ct}{\hbar}\cos\frac{\gamma t}{\hbar}-\frac{B^4ct}{\hbar}\right) \\
&= \langle r_\mu(0)\rangle\rho_\mu(t)+\rho_\mu(t)ct-\hbar\left(B^2\frac{dA^2}{dp}-A^2\frac{dB^2}{dp}\right)\sin\frac{\gamma t}{\hbar} \\
&= \langle r_\mu(0)\rangle\rho_\mu(t)+\rho_\mu(t)ct+\hbar\frac{dB^2}{dp}\sin\frac{\gamma t}{\hbar}
\end{aligned}
$$

With the help of (27), (35) and (37) we can simplify

$$
\frac{dB^2}{dp}=\frac{d}{dp}\left(\frac{\Omega_\mu-E_2}{\gamma}\right)\approx\frac{1}{\gamma}\left(\frac{d\Omega_\mu}{dp}-c\right)
$$

In what follows we place the origin of our coordinate system at the point where μ-neutrino was created at $t=0$. Then, finally, we obtain our main result for the μ-neutrino trajectory

$$
\langle r_\mu(t)\rangle\approx ct+\frac{\hbar}{\gamma\rho_\mu(t)}\left(\frac{d\Omega_\mu}{dp}-c\right)\sin\frac{\gamma t}{\hbar} \tag{45}
$$

The second term on the right hand side is the interaction correction, which is responsible for the superluminal effect observed in the MINOS and OPERA experiments, according to our model. This term can take both positive and negative values depending on the yet unspecified value $d\Omega_\mu/dp$ and on time t. Thus μ-neutrino position oscillates around the center-of-energy (42). Defining the apparent propagation velocity as $v_\mu(t)\equiv\langle r_\mu(t)\rangle/t$ we obtain the superluminality parameter comparable with experiments

$$
\delta v(t)\equiv\frac{v_\mu(t)-c}{c}=\frac{\hbar}{\gamma\rho_\mu(t)ct}\left(\frac{d\Omega_\mu}{dp}-c\right)\sin\frac{\gamma t}{\hbar} \tag{46}
$$

E. The $\nu_\mu-\nu_\tau$ asymmetry

To get trajectory of the ν_τ component of the state (38) we evaluate (44)

$$
\begin{aligned}
\langle r_\tau(t)\rangle &= -\frac{\hbar}{\rho_\tau(t)}\int_{Im} dp\psi^*(p)A(p)B(p)\left(e^{\frac{i}{\hbar}E_3(p)t}-e^{\frac{i}{\hbar}E_2(p)t}\right)\frac{d}{dp}\psi(p)A(p)B(p)\left(e^{-\frac{i}{\hbar}E_3(p)t}-e^{-\frac{i}{\hbar}E_2(p)t}\right) \\
&= \langle r_\tau(0)\rangle-\frac{\hbar A^2B^2}{\rho_\tau(t)}\ Im\left[\left(e^{\frac{i}{\hbar}E_3t}-e^{\frac{i}{\hbar}E_2t}\right)\left(-\frac{ict}{\hbar}e^{-\frac{i}{\hbar}E_3t}+\frac{ict}{\hbar}e^{-\frac{i}{\hbar}E_2t}\right)\right] \\
&= \langle r_\tau(0)\rangle+\frac{A^2B^2ct}{\rho_\tau(t)}\left(e^{\frac{i}{\hbar}E_3t}-e^{\frac{i}{\hbar}E_2t}\right)\left(e^{-\frac{i}{\hbar}E_3t}-e^{-\frac{i}{\hbar}E_2t}\right) \\
&= \langle r_\tau(0)\rangle+ct
\end{aligned} \tag{47}
$$

This means that, unlike its ν_μ counterpart, the τ-neutrino trajectory always coincides with the center of energy (42).

Interestingly, if the τ-neutrino were created first, i.e., the initial state was

$$
|\psi(0)\rangle\equiv\begin{bmatrix}0\\ \psi(p)\end{bmatrix}
$$

instead of (11), then the ν_τ trajectory would exhibit the oscillatory pattern, while the accompanying μ-neutrino would travel with the constant velocity c.

It seems strange that behaviors of ν_μ and ν_τ depend so much on which species was created originally (at time $t = 0$). For example, suppose that in the case of full mixing ($A^2 = B^2 = 1/2$) we have prepared a pure μ-neutrino state (11) at $t = 0$. According to (39), at time equal to the half-period of oscillation $t = T/2 = \pi\hbar/\gamma$ the system evolves into a τ-neutrino state

$$\begin{aligned} |\psi(T/2)\rangle &\approx \psi(p)e^{-\frac{i}{2\hbar}E_2T} \left[\begin{array}{c} A^2 + B^2 e^{-i\pi} \\ AB\left(e^{-i\pi} - 1\right) \end{array} \right] \\ &= \psi(p)e^{-\frac{i}{2\hbar}E_2T} \left[\begin{array}{c} 0 \\ -1 \end{array} \right] \end{aligned} \quad (48)$$

At the first sight this state is supposed to behave in the same manner as if the τ-neutrino was prepared initially (with the exception of the overall time shift by $T/2$ and spatial shift by $cT/2$), i.e., the μ-component should have a straight trajectory while the ν_τ trajectory should oscillate in disagreement with our results (45) and (47). This "paradox" is caused by approximation used in (48). In a rigorous treatment, the vector components on the right hand side of (48) are not exactly 0 and -1. They have small (but not negligible) p-dependent contributions. So, $|\psi(T/2)\rangle$ is not a pure ν_τ state and it is not required to behave exactly as the pure ν_τ state.

VI. FITTING MODEL PARAMETERS

A. OPERA experiment

Now let us take a closer look at the OPERA experiment whose essential parameters are listed in the third column of table I. The energies of ν_μ were in the broad interval 13.8 - 40.7 GeV. So, using formula (41) we can estimate that oscillation period was about $T \approx 47 - 139$ ms. These values are much higher than the time of flight $T_f \approx 2.4$ ms. Therefore, in (46) we can set $\rho_\mu(t) \approx 1$, $\sin(\gamma t/\hbar) \approx \gamma t/\hbar$ and

$$\delta v(0) = \frac{1}{c}\frac{d\Omega_\mu}{dp} - 1 \quad (49)$$

Recall that function $\Omega_\mu(p)$ is basically a free parameter of our model, which can be adjusted to represent the neutrino speed excess $\delta v(0) \approx 2.37 \times 10^{-5}$ measured by the OPERA team for all studied energies. So, we are going to postulate that $d\Omega_\mu/dp$ is nearly constant

$$\frac{d\Omega_\mu}{dp} \approx 1.0000237c \quad (50)$$

within the entire energy range of interest (3 - 40 GeV, see Table II).

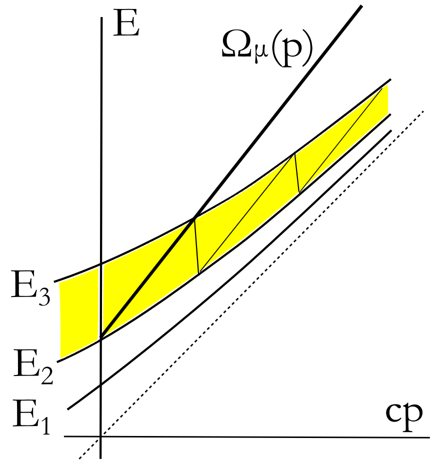

FIG. 1. Neutrino energy diagram. Theoretical consistency requires function $\Omega_\mu(p)$ to remain within the shaded area for a broad range of momenta.

This reasonable assumption presents a serious challenge for our model. The trouble is that the right hand side of (25) must be positive. This can happen only if $\Omega_\mu(p)$ is in the interval $[E_2, E_3]$, which is a very strong restriction meaning that the line representing function $\Omega_\mu(p)$ in Fig. 1 should lie entirely within the shaded area. But this is difficult to achieve, because the tiny width of this area $\gamma < 10^{-12}$ eV cannot accommodate the rather large slope (50) implied by the experiment. The only possibility to squeeze function $\Omega_\mu(p)$ inside the narrow band $[E_2, E_3]$ is to assume that $\Omega_\mu(p)$ has a saw-tooth shape shown by the thin line in the Figure. This is a rather unnatural behavior, and we will try to find other explanations.

B. Three neutrinos

In reality there are three known neutrino flavors (ν_e, ν_μ and ν_τ), which oscillate among each other. So, it seems reasonable to try to resolve the above paradox by allowing all three neutrinos in our theory. The model is similar to the two-neutrino case above. The Hilbert space is

$$\mathcal{H} = \mathcal{H}_e \oplus \mathcal{H}_\mu \oplus \mathcal{H}_\tau$$

The full Hamiltonian is given by the 3×3 matrix with real matrix elements (compare with (20))

$$H = H_0 + V = \left[\begin{array}{ccc} \Omega_e(p) & g(p) & h(p) \\ g(p) & \Omega_\mu(p) & f(p) \\ h(p) & f(p) & \Omega_\tau(p) \end{array} \right] \quad (51)$$

One can show that similar to the two-neutrino case, the ν_μ propagation speed is controlled essentially by the parameter $d\Omega_\mu/dp$. So, our goal is to understand under

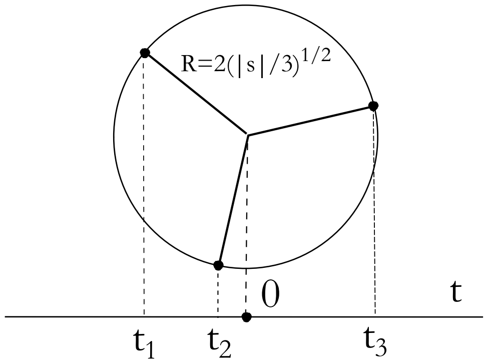

FIG. 2. Graphic solution of the cubic equation (53).

what conditions its value can exceed the speed of light. Eigenvalues of (51) $E_1(p) < E_2(p) < E_3(p)$ are obtained from the cubic equation

$$E^3 + bE^2 + uE + d = 0 \tag{52}$$

where

$$\begin{aligned} b &= -\Omega_e - \Omega_\mu - \Omega_\tau \\ u &= \Omega_e\Omega_\mu + \Omega_e\Omega_\tau + \Omega_\mu\Omega_\tau - Q^2 \\ d &= -\Omega_e\Omega_\mu\Omega_\tau + f^2\Omega_e + h^2\Omega_\mu + g^2\Omega_\tau - 2fgh \\ Q^2 &= f^2 + g^2 + h^2 \end{aligned}$$

Just as in subsection IV B we are going to assume that solutions $E_{1,2,3}(p) = \sqrt{m_{1,2,3}^2c^4 + p^2c^2}$ are known (for the mass eigenvalues $m_{1,2,3}$ see Table II) and see what we can say about $d\Omega_\mu/dp$. First we simplify equation (52) by shifting the variable $E \to t - b/3$

$$\begin{aligned} 0 &= t^3 + st + q \qquad (53) \\ q &= \frac{2b^3}{27} - \frac{bu}{3} + d \\ s &= u - \frac{b^2}{3} \\ &= -\frac{1}{3}(\Omega_e^2 + \Omega_\mu^2 + \Omega_\tau^2) + \frac{1}{3}(\Omega_e\Omega_\mu + \Omega_e\Omega_\tau + \Omega_\mu\Omega_\tau) - Q^2 \qquad (54) \end{aligned}$$

A beautiful theorem [29] says that three real solutions of (53) can be found as $(k = 1, 2, 3)$

$$t_k = 2\sqrt{-\frac{s}{3}}\cos\left[\frac{1}{3}\cos^{-1}\left(\frac{3q}{2s}\sqrt{-\frac{3}{s}}\right) - \frac{2\pi k}{3}\right]$$

Geometrically, they can be understood as projections on the horizontal axis of three equidistant points on the circle of radius $2\sqrt{-s/3}$ (see Fig. 2). A few important properties follow from this observation. First, parameter s must be negative. Second, the maximum distance between the roots t_k is not greater than $4\sqrt{|s|/3}$. At this point we will abandon our rigorous approach and switch to more qualitative considerations, keeping in mind that we are interested in order-of-magnitude estimates only. So, we will ignore all factors, comparable with 1, such as $4/\sqrt{3}$, and claim that $(E_3 - E_1)^2 = (t_3 - t_1)^2 = |s|$. It then follows from (54) that Ω_e, Ω_μ, Ω_τ are related to each other via equation

$$(E_3 - E_1)^2 - Q^2 = \Omega_e^2 + \Omega_\mu^2 + \Omega_\tau^2 - \Omega_e\Omega_\mu - \Omega_e\Omega_\tau - \Omega_\mu\Omega_\tau \tag{55}$$

describing an ellipsoid centered at $\Omega_e = \Omega_\mu = \Omega_\tau = 0$ whose approximate "radius" is $\Sigma = \sqrt{(E_3 - E_1)^2 - Q^2}$. In order to have real solutions for Ω_e, Ω_μ, Ω_τ the left hand side of (55) must be positive. Moreover, absolute values of individual coordinates $|\Omega_e|$, $|\Omega_\mu|$, $|\Omega_\tau|$ of points on the ellipsoid surface cannot exceed Σ. From these conditions we get the following restrictions on matrix elements of the Hamiltonian (51)

$$\begin{aligned} f^2(p), g^2(p), h^2(p) &< (E_3(p) - E_1(p))^2 \\ |\Omega_{e,\mu,\tau}(p)| &< E_3(p) - E_1(p) \approx \frac{(m_3^3 - m_1^2)c^3}{2\langle p\rangle} \end{aligned}$$

This means that the derivative $d\Omega_\mu/dp$ cannot be significantly different from c unless a weird behavior is assumed. So, the three-neutrino case is not qualitatively different from the two-neutrino case discussed in subsection VI A. In both instances the value of our crucial parameter $d\Omega_\mu/dp$ is severely restricted by low neutrino masses.

C. Fourth supermassive neutrino

In this subsection we will explore the idea that there exists a very massive neutrino ν_η, which has not been seen in experiments yet. The mass eigenvalue m_4 related to this particle is assumed to be large, not smaller than few hundreds of MeV/c^2. For simplicity, here we will ignore the existence of ν_e and choose an approximate Hamiltonian in the Hilbert space $\mathcal{H}_\tau \oplus \mathcal{H}_\mu \oplus \mathcal{H}_\eta$ as the following 3×3 matrix

$$H = \begin{bmatrix} E_3(p) & 0 & 0 \\ 0 & \Omega_\mu(p) & g(p) \\ 0 & g(p) & \Omega_\eta(p) \end{bmatrix} \tag{56}$$

where rows and columns are in the order τ, μ, η. One major requirement for this Hamiltonian is to produce energy eigenvalues with correct relativistic momentum dependencies $E_{2,3,4}(p) = \sqrt{m_{2,3,4}^2c^4 + p^2c^2}$. Note that the first row and column are decoupled from the rest of the matrix. So, one energy eigenvalue $E_3(p)$ can be found immediately. Similar to subsection IV B we pretend that solutions $E_{2,4}(p)$ are also known, so that

$$E_4(p) > E_2(p) \approx E_3(p) \quad (57)$$

To make sure that μ-neutrino is superluminal in the entire energy range, we assume that the diagonal element $\Omega_\mu(p)$ has a linear momentum dependence

$$\Omega_\mu(p) = \Omega_\mu(0) + (1.0000237 \times 10^{-5})pc \quad (58)$$

Then from equations analogous to (24) and (25) we have a unique solution for two other matrix elements $\Omega_\eta(p)$ and $g(p)$

$$\Omega_\eta(p) = E_2(p) + E_4(p) - \Omega_\mu(p)$$
$$g^2(p) = [\Omega_\mu(p) - E_2(p)][E_4(p) - \Omega_\mu(p)]$$

which implies that

$$E_2(p) < \Omega_\mu(p) < E_4(p) \quad (59)$$

The resulting energy diagram is shown in Fig. 3. Note that, unlike in Fig. 1, here we assumed $E_2(p) > E_3(p)$, which is also compatible with experimental data from Table II. If we choose function $\Omega_\mu(p)$ only slightly greater than $E_2(p)$, then $\Omega_\eta(p) \approx E_4(p)$ and the mixing angle $\cos\theta_{24}$ will be close to zero

$$\theta_{24} = \cos^{-1}\left(\sqrt{\frac{\Omega_\eta - E_2}{E_4 - E_2}}\right) \approx 0$$

This means that for an initially created pure μ-neutrino the probability of finding η-neutrino $\rho_\eta(t) = \sin^2 2\theta_{24}\sin^2[(E_4 - E_2)t/(2\hbar)]$ will remain low at all times. Thus the presence of the supermassive η-neutrino may have a negligible effect on the observable $\nu_e - \nu_\mu - \nu_\tau$ sector. The frequency of $\nu_\mu - \nu_\eta$ oscillations is extremely high due to the large energy difference $E_4 - E_2$.

Let us estimate the lower bound on the fourth neutrino mass eigenvalue m_4. Assuming that formula (58) remains valid in a broad energy interval up to 40 GeV and that $\Omega_\mu(0) \approx 0$, we obtain $\Omega_\mu(40\ \text{GeV}/c) \approx 40.001$ GeV. According to (59), this value should be lower than

$$E_4(40\ GeV/c) \approx 40\ GeV + \frac{m_4^2 c^4}{80\ GeV}$$

which results in $m_4 > 300$ MeV/c^2.

Of course, the Hamiltonian (56) is only an approximation as it ignores the effect of $\nu_\mu - \nu_\tau$ oscillations. However, based on our experience with the general 3-neutrino case in subsection VI B, it seems possible to perturb this Hamiltonian slightly, so that its non-diagonal matrix elements become non-zero, the correct oscillation pattern is reproduced and, at the same time, the overall relativistic invariance and the superluminality of ν_μ are preserved. We will not attempt explicit construction of such a realistic Hamiltonian in this work.

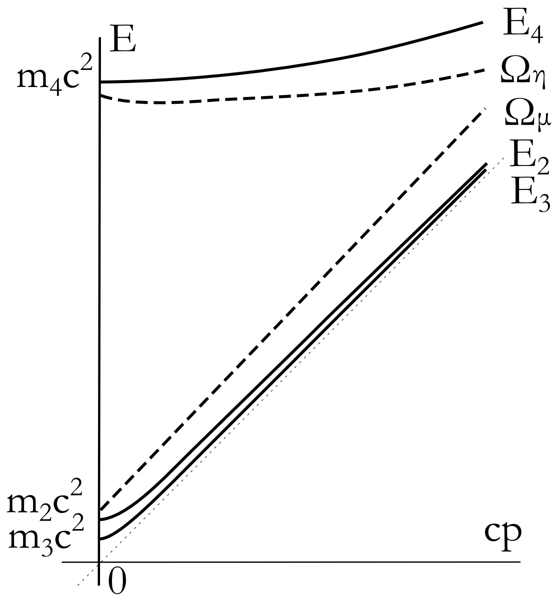

FIG. 3. Energy diagram with a supermassive neutrino ν_η.

VII. SUPERLUMINAL EFFECTS

A. Model predictions

In the preceding section we have postulated existence of a new supermassive neutrino and a linear behavior (58) of the function $\Omega_\mu(p)$ in the energy interval 3 - 40 GeV. Now we will provide more details on how the superluminal effect depends on the neutrino energy E and propagation distance L. Inserting (50) in formula (45) we obtain

$$\langle r_\mu(t)\rangle \approx ct + \frac{2.37 \times 10^{-5}\hbar c}{\gamma\rho_\mu(t)} \sin\frac{\gamma t}{\hbar} \quad (60)$$

The quantity of interest is the ratio (1)

$$\delta v = \frac{1}{L}(\langle r_\mu(L/c)\rangle - L) = \frac{2.37 \times 10^{-5}\hbar c}{L\gamma\rho_\mu(L/c)} \sin\frac{\gamma L}{\hbar c}$$

To evaluate this expression we use formulas (36), (40) and neutrino parameters from Table II. For further simplification we introduce parameter $\beta = L(\text{km})/E(\text{GeV})$, whose values for relevant experiments are listed in Table

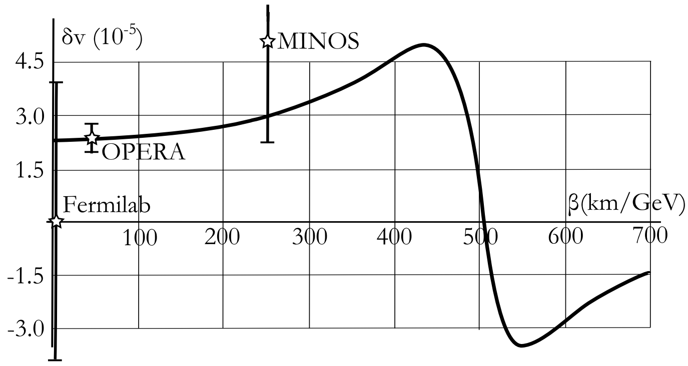

FIG. 4. Deviation of the apparent μ-neutrino velocity from the speed of light as a function of parameter $\beta = L/E$.

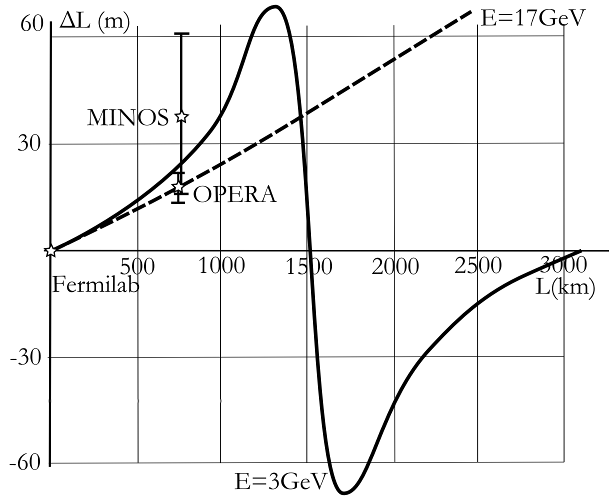

FIG. 5. Separation between the μ-neutrino and $r(t) = ct$ trajectory as a function of traveled distance for two particle energies 3 GeV and 17 GeV. Positive values of ΔL correspond to superluminal propagation.

I. Then δv becomes a universal function of β, which is applicable for all values of L and E

$$\delta v(\beta) = \frac{3.85 \times 10^{-3} \sin(6.2 \times 10^{-3}\beta)}{\beta(1 - 0.97 \sin^2(3.1 \times 10^{-3}\beta))} \tag{61}$$

This function is plotted in Fig. 4. The maximum superluminal effect $\delta v \approx 5 \times 10^{-5}$ occurs at $\beta \approx 430$ km/GeV. Unfortunately, this is exactly the region of β where the probability of finding a μ-neutrino in the beam is at its lowest point $\approx 3\%$. For higher values $\beta > 430$ km/GeV the superluminal effect rapidly decreases, and the propagation becomes subluminal at $\beta > 510$ km/GeV. For even higher β the function $\delta v(\beta)$ oscillates between positive and negative values and gradually decays $\delta v \propto \beta^{-1}$ as β tends to infinity. The values of $\delta v(\beta)$ calculated for actual experimental conditions are reported in the last row of Table I.

It is also interesting to calculate trajectory (60) for a single ν_μ particle with fixed energy E. On average this trajectory coincides with the path $r(t) = ct$ of the center of energy (c.o.e). However, the second term on the right hand side of (60) is responsible for small oscillations around this linear path. Right after the emission neutrino speed exceeds the light speed by the factor $1 + \delta v(0) = 1.0000237$. Then, gradually, neutrino slows down, so that at the end of the first half-period ($T/2$) the c.o.e. catches up. During the second half-period neutrino moves behind the c.o.e., and at $t = T$ their positions coincide again. This cycle repeats indefinitely, so, if averaged over a long time interval, neutrino speed is the same as the speed of light.

It is convenient to measure neutrino position oscillations in terms of neutrino-c.o.e. separation $\Delta L = L\delta v$. This quantity is plotted in Fig. 5 as a function of the traveled distance L for two neutrino energies $E = 3$ GeV and $E = 17$ GeV taken from the MINOS and OPERA experiments, respectively.

B. Comparison with experiments

The values of δv and ΔL along with their error bars obtained in three relevant experiments [1, 2, 5] are compared with theoretical curves in Figs. 4 and 5. The perfect match of the OPERA results is meaningless, because they were used to fit the major parameter of our model $d\Omega_\mu/dp$. The agreement with two other measurements is reasonable, though our model predictions are slightly lower than MINOS observations. This discrepancy can be easily explained in more than one way: For example, in our calculations we have assumed that $\sin^2 2\theta_{23} = 0.97$. If this parameter were chosen to be closer to 1.0, then the superluminal effect would be more pronounced. Another possible explanation is that $d\Omega_\mu(p)/dp$ does not remain constant across the entire energy spectrum (as we assumed in (60) and in Fig. 3) but somewhat increases at its lower end, e.g., around $E \approx 3$ GeV. Our model also predicts that observations of τ-neutrinos in the OPERA setup [11] would not detect any superluminality. This provides yet another opportunity to falsify the model in future experimental studies.

Our results agree well also with SN1987A observations [7, 8]. The corresponding data points $\beta \approx 10^{15}$ km/GeV and $L = 15 \times 10^{12}$ km are not shown in Figs 4 and 5 as these points are far beyond ranges of the plots. Moreover, our calculations did not address $\overline{\nu}_e$ neutrinos directly. Nevertheless, if we assume that our qualitative conclusions are still applicable to $\overline{\nu}_e$, then from (61) we expect parameter δv to be close to zero in agreement with observations. Parameter ΔL is expected to be on the order of few meters, which is negligible at the level of accuracy characteristic for astronomical measurements.

VIII. DISCUSSION

In this article we have formulated a simple model of oscillating neutrinos. This model satisfies all requirements of relativistic quantum theory: A unitary representation of the Poincaré group is constructed explicitly in the neutrino Hilbert space, and this representation takes into account interaction responsible for neutrino oscillations. Relativistic invariance requires that matrix elements $\Omega_\mu(p), \Omega_\tau(p), f(p)\ldots$ of the neutrino Hamiltonian have non-trivial momentum dependencies. This implies that in the classical limit neutrinos are not required to propagate with the constant speed c. Their speed can oscillate around the speed of light, so that in some conditions one can observe a superluminal propagation. The allowed magnitude of superluminality depends on neutrino masses. In this work we have assumed that there exists a fourth neutrino type with a very high mass eigenvalue $m_4 > 0.3$ GeV/c^2. Then the theory developed here permits relatively large superluminal effects observed in MINOS and OPERA experiments.

In this work we have considered only a simplest version of our model. The model can be developed further and thus made more realistic. Obviously, it is desirable to perform a full 3-dimensional treatment of neutrinos taking into account their spins. One can also try to explore more complicated versions of oscillation Hamiltonians (20), (51) and (56). For example, in [30] we have suggested an alternative explanation of neutrino superluminality by assuming that the non-diagonal matrix element of the Hamiltonian (20) is complex: $f(p) = |f(p)|e^{-\frac{i}{\hbar}\chi p}$ with $\chi \approx 18$ meters. Since this explanation went into contradiction with experiments [5, 6], in this work we deliberately set $\chi = 0$. Nevertheless, it is still possible that parameter χ is non-zero (though $|\chi| \ll 18$ m) and that the mechanism of superluminality discussed in [30] has some contribution to the overall effect in addition to the dominant "$d\Omega_\mu/dp$" mechanism discussed in the present work.

It is not clear if Hamiltonians of the type (20) exhaust all Poincaré invariant possibilities. Perhaps it is also possible to have matrix elements non-local in the p-representation, i.e., containing derivatives d/dp. It would be interesting to learn how this generalization could affect the neutrino superluminality.

A. Comments on causality

It is remarkable that our model, while being explicitly Poincaré-invariant, predicts something – the superluminal propagation of particles – which is expressly forbidden by Einstein's special relativity. According to common views, this violation of the universal speed limit is impossible, because it implies a violation of the principle of causality as well. So, it appears that we have a paradox here. To clarify this situation recall that traditional arguments establishing the propagation speed limit invoke Lorentz transformations of special relativity. They say that if (x, t) are space-time coordinates of a physical event in the reference frame at rest, then in the inertial frame moving with velocity $v \equiv c \tanh\theta$ space-time coordinates of the same event are given by formulas

$$x' = x\cosh\theta - ct\sinh\theta \quad (62)$$

$$t' = t\cosh\theta - (x/c)\sinh\theta \quad (63)$$

Special relativity postulates that these formulas remain valid in all circumstances, independent on the physical nature of the event occurring at (x, t) and on interactions responsible for this event. The tacit or explicit assumption used in many discussions of quantum relativistic effects is that space-time arguments of wave functions must transform by the same formulas, i.e., that the position-space wave function in the moving frame is

$$\psi(\theta; x, t) = \psi(0; x\cosh\theta - ct\sinh\theta, t\cosh\theta - (x/c)\sinh\theta) \quad (64)$$

If this were true, then the observed superluminal propagation of neutrinos would be scandalous, because, according to (62) - (64), one would be able to find a moving reference frame in which neutrino arrival in the detector happened *before* its creation in the meson decay process. So, in this moving frame the effect would occur *before* its cause, which is impossible.

However, there are logical gaps in the above arguments. These gaps allow us to suggest that violation of causality in our model is not obvious at all. In our work we have used fully relativistic approaches: the Newton-Wigner's definition of particle's position [31] and Wigner-Dirac formulation of quantum dynamics [23]. In this theory, formula (64) is not valid even in the case of non-interacting particles. The correct non-interacting wave function transformation law is [32]

$$\psi(\theta; x, t) = \langle x|e^{-\frac{i}{\hbar}H_0 t}e^{\frac{i}{\hbar}K_0 c\theta}|\psi\rangle \quad (65)$$

where $|x\rangle$ is an eigenvector of the particle position operator. Clearly, this formula is not the same as (64). Their fundamental difference is exemplified by the well-known effects of superluminal spreading of wave packets and the loss of particle localization in the moving frame [16–21] predicted by (65).

In the interacting case the picture is even more complicated as one needs to use *interacting* energy and boost operators to find the wave function transformation

$$\psi(\theta; x, t) = \langle x|e^{-\frac{i}{\hbar}Ht}e^{\frac{i}{\hbar}Kc\theta}|\psi\rangle \quad (66)$$

We will not analyze this formula in detail here, just mention two remarkable features of (66) that disagree with traditional interpretations of special relativity. First, the neutrino oscillation period observed from a moving frame *does not* scale with velocity according to the usual Einstein's time dilation formula: $T' \neq T \cosh\theta$ [33]. Second, if according to the observer at rest the initial state (at $t = 0$) is prepared as a 100% μ-neutrino then in the moving frame (even at $t = 0$) the probability of finding μ-neutrino is less than 1 and the probability of finding other flavors is greater than 0 [30]. This means that definitions of neutrino flavors are different for different observers. This also implies that the oscillating system lacks clearly identified and observer-independent local events (such as points where $\rho_\mu = 1$), whose space-time coordinates can be used in a rigorous discussion of causality. These unusual features are very similar to properties of unstable particles discussed in [34–37].

Even if the above difficulty with event definitions is resolved, formula (66) cannot provide a clear answer about causality in the moving frame, because in real experiments we are not dealing with free (albeit oscillating) neutrinos: The event that causes neutrino appearance in the detector is the meson decay at $t = 0$. Thus, in order to investigate the cause-effect relationships in different frames we need to include in our description a realistic model of this event, i.e., we need a model that incorporates the unstable meson and its decay products as well as interactions responsible for the meson decay and neutrino oscillations. To the best of author's knowledge, a rigorous quantum relativistic time-dependent description of such a complicated interacting system has not been developed yet.

From a more general standpoint one can argue that superluminal propagation of signals is not forbidden in interacting systems. Just as in the above discussion, the crucial point is that transition to the moving frame should be performed by using a boost operator $K = K_0 + Z$ that depends on interactions. Therefore, in relativistic Hamiltonian systems of interacting particles boost transformations of space-time locations of events are different from simple and universal Lorentz formulas of special relativity (62) - (63) even in the classical (non-quantum) limit [38]. This fact is essential for the proof that instantaneous action-at-a-distance potentials remain instantaneous in all reference frames, so that the principe of causality is not violated even if interactions between particles are not retarded [39].

These arguments lead us to the conclusion that the oscillating neutrino system does not behave in a way expected from a naïve application of special relativity. However, this does not mean that the causality postulate is violated by superluminal effects. A proper discussion of causality requires more realistic modeling of the neutrino preparation and propagation in different reference frames. Such a modeling would be a promising line of further research, but it is beyond the scope of the present paper.

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