Study of Natural Class of Intervals Using $(-\infty, \infty)$ and $(\infty, -\infty)$

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CONTENTS

PREFACE

In this book the authors study the properties of natural class of intervals built using $(-\infty, \infty)$ and $(\infty, -\infty)$. These natural class of intervals behave like the reals R, as far as the operations of addition, multiplication, subtraction and division are concerned. Using these natural class of intervals we build interval row matrices, interval column matrices and $m \times n$ interval matrices. Several properties about them are defined and studied. Also all arithmetic operations are performed on them in the usual way.

The authors by defining so have made it easier for operations like multiplication, addition, finding determinant and inverse on matrices built using natural class of intervals.

We also define polynomials with coefficients from natural class of intervals or polynomial intervals, both these concepts are one and the same, for one can be obtained from the other and vice versa.

The operations of integration and differentiation are defined on these interval polynomials in a similar way as that of usual polynomials.

Interval trigonometric functions are introduced and operations on them are defined.

Finally fuzzy interval polynomials are introduced using the intervals [0, 1] and [1, 0]. We define operations on them. The concept of matrices with polynomial entries are defined and described.

This book has eight chapters. The first chapter is introductory in nature. Chapter two introduces the notion of interval matrices with entries from natural class of intervals. Polynomial intervals are given in chapter three and in chapter four interval trigonometric functions are introduced. Natural class of fuzzy intervals are introduced in chapter five. Calculus on interval polynomials and interval matrices are carried out in chapter six. Applications are suggested in chapter seven. Final chapter gives around 100 problems some of them are at research level. The book "Algebraic structure using natural class of intervals" won the 2011 New Mexico award for Science and Maths.

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Chapter One

INTRODUCTION

In this chapter the notion of natural class of intervals and arithmetic operations on them are introduced. The natural class of intervals are studied and the algebraic structures enjoyed by the arithmetic operations are described. We see the natural class of intervals contains R (Q and Z). Further the natural class of intervals is a group only under addition. Under subtraction it is a groupoid and under multiplication it is a monoid.

Here we introduce the notion of natural class of intervals and the main arithmetic operations on them. The intervals are taken from $(-\infty, \infty)$ and $(\infty, -\infty)$.

Throughout this book R denotes the real field, Q the rational field, Z the ring of integers. Z_n denotes the ring of modulo integers n. $R^+ \cup \{0\}$ denotes the set of positive reals with zero, $Q^+ \cup \{0\}$ the set of positive rationals with zero and $Z^+ \cup \{0\}$ the set of positive integers with zero. These form semifields.

C denotes the complex field. We have $Z \subset Q \subset R \subset C$.

Let $[x, y]$ be an interval from Q or R or Z if $x < y$ (that is x is strictly less than y) then we define $[x, y]$ to be a closed increasing interval or increasing closed interval. We see x and y are included in this interval.

[7, 19], [0, 11], [–8, –2], [–18, 0], $[\sqrt{2}, \sqrt{19}]$ are some examples of closed increasing intervals.

Suppose if [x, y] is replaced by (x, y), both x and y are not included in this interval, we say (x, y) is an increasing open interval or open increasing interval.

We just give some examples of it; (–7, 10), $(-\sqrt{2}, 12)$ (0, 7), (–8, 0) are few examples of open increasing intervals.

Suppose in (x, y) the open bracket is replaced by (x, y] then we define (x, y] as half open-half closed increasing interval or increasing half open half closed interval. Clearly x does not belong to the interval only y belongs to the interval.

We give a few examples of it.

$$(0, 12], (-9, 0], (-\sqrt{21}, 60], (0, \sqrt{41}/3]$$

are some examples of them.

If we replace (x, y] by [x, y) then we define [x, y) as the half closed - half open increasing interval or increasing half closed - half open interval.

Some examples are given below. [27, 48), [0, 17), [–9, 0). $[\sqrt{43}, 101)$ and so on.

Now we see all these intervals are the usual or classical intervals and we have a special type of arithmetic operations on the collection of increasing intervals closed or open or half-open half closed or half closed - half open; ‘or’ used in the mutually exclusive sense only.

That is we can have the collection of closed increasing intervals or open increasing intervals or half open half closed increasing intervals or half closed-half open increasing intervals and we can perform operation on them.

From the very context one can easily understand on which class we perform (or define) the arithmetic operations.

The classical operations are as follows:

Let [a, b] and [c, d] be elements of the collection of closed increasing intervals.

$$[a, b] + [c, d] = [a + c, b + d]$$
$$[a, b] - [c, d] = [a - d, b - c]$$
$$[a, b] \times [c, d] = [\min \{ac, ad, bc, bd\}; \max \{ac, ad, bc, bd\}]$$
$$[a, b] / [c, d] = [\min \{a/c, a/d, b/c, b/d\}, \max \{a/c, a/d, b/c, b/d\}]$$
with $[c, d] \neq [0, 0]$

Division by an interval containing zero is not defined under the basic interval arithmetic.

The addition and multiplication operations are communicative, associative and sub-distributive, the set x (y+z) is a subset of xy + xz.

However the same operations can be defined for open increasing intervals, half open-half closed increasing intervals and half closed - half open increasing intervals. Increasing intervals are classical intervals which is used by us.

Now we proceed onto define decreasing intervals.

Let [x, y] be an interval x and y belongs to Z or Q or R with $x > y$ (x is strictly greater than y) where $[x, y] = \{a \mid x \geq a \geq y\}$ then we define [x, y] to be a decreasing closed interval or closed decreasing interval. The decreasing interval [x, y] is taken from $(\infty, -\infty)$.

We see the temperature may decrease from x°F to y°F or the speed of a car may decrease from x to y and so on. So such intervals also has been existing in nature only we have not so far given them proper representations and systematically develop arithmetic operations on them.

Before we proceed on to move further we give some examples of them.

[9, 0], [5, –8], [0, –18], [19, 2] [–7, –21] and so on are from $(\infty,-\infty)$.

These are examples of closed decreasing intervals. We see in case of decreasing closed intervals both x and y belongs to the interval.

If in the case of decreasing closed intervals [x, y] $(x > y)$ if we replace the closed bracket by open bracket then we get (x, y) $(x > y)$ to be the open decreasing interval or decreasing open interval.

We give a few examples of it.

(0, –11), (20, 0), (–7, –4), (19, 8), (40, –3) and so on.

Now if we replace closed bracket in the interval [x, y] by (x, y] then we define (x, y] to be the half open-half closed decreasing interval or decreasing half open- half closed interval. We see only y belongs to the interval and x does not belong to the interval and these intervals are from $(\infty, -\infty)$.

$(x, y] = \{a \mid x > a \geq y\}$. We give examples of it.

(8, 3], (0, –11], (–11, –29], (40, 0] and so on.

Likewise if the closed bracket of the decreasing interval [x, y] is replaced by the bracket [x, y) then we define [x, y) to be the half closed half open decreasing interval or decreasing half closed - half open interval.

We will give examples of it.

Consider $[0, -8)$, $[19, 0)$, $[-2, -15)$, $[24, 3)$, $[7, -7)$; these intervals are decreasing half closed - half open intervals.

So $N_c(R)$ ($N_c(Z)$ or $N_c(Q)$)
$= \{[a, b] \mid a < b \text{ or } a = b \text{ or } a > b \;\; a, b \in R\}$

denotes the collection of all decreasing, increasing and degenerate closed intervals. If in the interval $[a, b]$; $a = b$ then we call such intervals as degenerate intervals.

Likewise $N_o(R)$ ($N_o(Z)$ or $N_o(Q)$) $= \{(a, b) \mid a, b \in R, a < b \text{ or } a > b \text{ or } a = b\}$ denotes the collection of increasing or decreasing or degenerate open intervals.

$N_{oc}(R)$ ($N_{oc}(Q)$ or $N_{oc}(Z)$) $= \{(a, b] \mid a, b \in R; a > b \text{ or } a < b \text{ or } a = b\}$ denotes the collection of all increasing or decreasing or degenerate open-closed intervals.

$N_{co}(R) = \{[a, b) \mid a, b \in R; a > b \text{ or } a < b \text{ or } a = b\}$ denotes the collection of all increasing or decreasing or degenerate closed-open or half closed - half open intervals.

Clearly from the context one can easily know to which class an interval belongs to.

We just mention a few observations, $N_o(R)$ contains R, Q and Z. and $N_o(Q)$ contains only Q and Z and not R. Further $N_o(Z)$ contains Z and not Q or Z.

This is true if open interval is replaced by $N_{oc}(R)$ or $N_{co}(R)$ or $N_c(R)$.

Now we proceed on define basic operations on intervals. So if we want to study decreasing intervals instead of ($-\infty$ to ∞), i.e., $(-\infty, \infty)$ we consider $(\infty, -\infty)$ every $[a, b]$ with $a > b$ are subsets of $(\infty, -\infty)$.

For any [a, b] and [c, d] in N_c (R) we define

$$[a, b] + [c, d] = [a + c, b + d].$$

If $a < b$ and $c < d$ then $a + c < b + d$ that is sum of increasing intervals is an increasing interval. Likewise if $a > b$ and $c > d$ then $a + c > b + d$ that is sum of decreasing intervals is a decreasing interval. However sum of an increasing interval and a decreasing interval may be an increasing interval or a decreasing interval.

This is illustrated by the following examples.

[0, –8] is a decreasing interval and [–7, 8] is an increasing interval, their sum $[0, -8] + [-7, 8] = [0 + (-7), -8 + 8] = [-7, 0]$ is an increasing interval.

Now [0, 8] is an increasing interval and [2, –20] is a decreasing interval their sum,

$[0, 8] + [2, -20] = [2, -12]$ is a decreasing interval.

But sum of two degenerate intervals is a degenerate interval. Sum of a degenerate interval and an increasing interval can be an increasing interval.

For if $a = a$ and $b < c$ then $[a, a] + [b, c] = [a+b, a+c]$ and $a + b < a + c$ is an increasing interval. Likewise sum of a degenerate interval and a decreasing interval $a = a$ and $b > c$ is

$[a, a] + [b, c] = [a+b\ a+c]$ and $a + b > c + a$, hence a decreasing interval.

However we see (N_c(R), +) is an abelian group under addition and [0, 0] = 0 acts as the additive identity.

For every [a, b] ($a, b \in$ R); [–a, –b] is the additive inverse of [a, b].

Thus we have the following theorem.

THEOREM 1.1: *The natural class of intervals $N_c(R)$ (or $N_c(Z)$ or $N_c(Q)$) is an abelian group with respect to addition.*

Now we proceed onto define the operation of subtraction.

For [a, b] and [c, d] $\in N_c(R)$ (or $N_c(Q)$ or $N_c(Z)$) we have [a, b] – [c, d] = [a–c, b–d].

Clearly [a, b] – [c, d] $\neq$ [c, d] – [a, b]

We see the substraction of degenerate intervals is a degenerate interval. For [a, a] – [c, c] = [a – c, a – c] is a degenerate interval. We see the subtraction of an increasing interval can be an increasing interval or can be a decreasing interval.

For consider [8, 11] and [6, 25]; [8, 11] – [6, 25] = [2, –14] is a decreasing interval.

Now [6, 25] – [8, 11] = [–2, 14] is an increasing interval.

We see $N_c(R)$ with subtraction as a operation is only a groupoid and this groupoid has no identity. Similarly $N_{oc}(R)$ (or $N_{co}(R)$ or $N_o(R)$) under subtraction is a groupoid which has no identity.

Now we see as in case of R or Q or Z we see they are groupoids without identity under the operation of subtraction.

We define a product on $N_c(R)$ (or $N_c(Q)$ or $N_{oc}(Z)$ and so on).

Suppose [x, y] and [a, b] $\in N_c$ (R) then [x, y] [a, b] = [xa, yb] $\in N_c(R)$.

We see the product of degenerate intervals are again degenerate intervals.

Consider [0, –7], [2, –2] in $N_c(R)$. [0, –7] × [2, –2] = [0, 14] we see [0, –7] and [2, –2] are decreasing intervals however the product of [0, –7] and [2, –2] is an increasing interval.

Consider the product of [3, –2] and [0, 7] two intervals, first one decreasing and the other increasing, their product [3, –2] × [0, 7] = [0, –14] is a decreasing interval.

Consider [0, –2] and [7, –9] two decreasing intervals, their product [0, –2] [7, –9] = [0, 18] is an increasing interval.

Consider [–7, 1] and [–4, 2] two increasing intervals their product [–7, 1] [–4, 2] = [28, 2] is a decreasing interval.

Let [3, 0] be a decreasing interval and [–7, 8] be an increasing interval. The product of [3, 0] and [–7, 8] is given by [3, 0] [–7, 8] = [–21, 0] is an increasing interval.

We see $N_c(R)$ is a semigroup under product, further [1, 1] is the multiplicative identity. Some elements in $N_c(R)$ has inverse and all elements of the form [0, a] and [a, 0] have no inverses, infact [a, 0] [0, a] = [0, 0].

We see $(N_c(R), +, \times)$ is a commutative ring with unit. $\{(N_o(Q), +, \times)$, $(N_c(Q), +, \times)$, $(N_{oc}(Q), +, \times)$, $(N_{co}(Q), +, \times)$, $(N_o(R), +, \times)$, $(N_{co}(R), +, \times)$, $(N_{oc}(R), +, \times)\}$ are all commutative rings with unit. Infact if Q or R is replaced by Z still they are rings with unit.

We can define division of two intervals in $N_c(R)$ as follows:

Let [a, b] and [c, d] $\in N_c(R)$.

$\frac{[a,b]}{[c,d]}$ is defined if and only if $c \neq 0$ and $d \neq 0$.

$\frac{[a,b]}{[c,d]}$ = [a/c, b/d] and $\frac{[a,b]}{[c,d]} \in N_c(R)$.

However while dividing two increasing intervals it may become a decreasing interval and vice versa. We will illustrate these situations by some examples.

Let [3, 7] and [2, 19] be any two increasing intervals in $N_c(R)$.

Now

$$[3, 7] / [2, 19] = [3/2, [7/19] \in N_c(R).$$

Clearly [3/2, 7/19] is only a decreasing interval.

Take [5, 7] and [–2, 12] two increasing intervals in $N_c(R)$.

We see [5,7] / [–2, 12] = [–5/2, 7/12] is an increasing interval.

Consider [–7, 0] and [–2, 4] a decreasing and an increasing interval. [7, 0] / [–2, 4] = [–7/2, 0] is an increasing interval.

Consider [–2, –17] and [–8, –20] two decreasing intervals.

$$[-2, -17] / [-8, -20] = [1/4, 17/20] = [0.25, 0.85]$$

is an increasing interval in $N_c(R)$.

Thus all results hold good even if $N_c(R)$ is replaced by $N_c(Q)$, $N_c(Z)$, $N_{oc}(R)$, $N_{oc}(Z)$, $N_{oc}(Q)$, $N_{co}(Z)$, $N_{co}(R)$, $N_{co}(Q)$, $N_o(Q)$, $N_o(Z)$ or $N_o(R)$.

We see the operation of division is non associative. Further we see the operations are distributive.

The main advantage of using these operations is that we see these operations on the natural class of intervals is akin (same as) to the operations on R or Q or Z.

We now proceed onto build matrices using natural class of intervals.

Chapter Two

MATRICES USING NATURAL CLASS OF INTERVALS

In this chapter we proceed onto define matrices using the natural classes of intervals. We show the existing programmes (codes) can be used with simple modifications on interval matrices using natural class of intervals.

DEFINITION 2.1: *Let*

$$X = (a_1, \ldots, a_n) \text{ where } a_i \in N_c(R);\ 1 \leq i \leq n,$$

X is defined as the row (1 × n) matrix of natural class of intervals.

We give examples of this situation.

Example 2.1: Let

$$Y = (a_1, a_2, \ldots, a_8) \text{ where } a_i \in N_c\,(R),\ 1 \leq i \leq 8$$

be the row matrix of natural class of intervals.

Example 2.2: Let

$$A = (a_1, a_2, a_3, a_4) \text{ where } a_i \in N_{oc}(Q), 1 \leq i \leq 4$$

be the row matrix of natural class of intervals.

Example 2.3: Let

$$A = (a_1, a_2, \ldots, a_7) \text{ where } a_i \in N_o(Z), 1 \leq i \leq 7$$

be the row matrix of natural class of intervals.

Now we can define operations on them.

DEFINITION 2.2: *Let*

$$M = \{(a_1, a_2, \ldots, a_n) \mid a_i \in N_{oc}(Q), 1 \leq i \leq n\}$$

be the collection of all $1 \times n$ row matrices. M is a group under addition called the group of natural class of row matrices.

We give examples of them.

Example 2.4: Let

$$M = \{(a_1, a_2, \ldots, a_{12}) \mid a_i \in N_c(Q), 1 \leq i \leq 12\}$$

be the collection of all 1×12 matrices, M is a group under addition.

Example 2.5: Let

$$T = \{(a_1, a_2, a_3, a_4) \mid a_i \in N_{oc}(Z), 1 \leq i \leq 4\}$$

be the collection of all 1×4 row matrices. T is a group under addition. Just we illustrate how addition is performed on T.

Let x = ((3, –2], (-7, 0], (0, –9], (3, 10]) and y = ((–2, 1], (2, –5], (8, 2], (–7, 2]) be in T.

$$x+y = ((3, -2], (-7, 0], (0, -9], (3, 10]) + ((-2, 1], (2, -5], (8, 2], (-7, 2])$$

$$= ((3-2, -2+1], (-7+2, 0+(-5)], (0+8, -9+2], (3-7, 10+2])$$

$$= ((1, -1], (-7, -5], (8, -7], (13, -5]) \in T.$$

Further ((0, 0], (0, 0], (0, 0], (0, 0]) $\in$ T acts as the additive identity.

Also for every $(a_1, a_2, a_3, a_4) \in T$ we see $(-a_1, -a_2, -a_3, -a_4)$ in T acts as the additive inverse.

Now we see one can perform the operation of addition in the following way.

We can recognize the interval

$$\left(\left[a_1^1, a_2^1\right], \left[a_1^2, a_2^2\right], \ldots, \left[a_1^n, a_2^n\right]\right) \text{ as } \left[\left(a_1^1, a_1^2, \ldots, a_1^n\right), \left(a_2^1, a_2^2, \ldots, a_2^n\right)\right].$$

Thus addition can be performed in two ways.

$$\text{If } x = \left(\left[a_1^1, a_2^1\right], \left[a_1^2, a_2^2\right], \ldots, \left[a_1^n, a_2^n\right]\right)$$

$$\text{and } y = \left(\left[b_1^1, b_2^1\right], \left[b_1^2, b_2^2\right], \ldots, \left[b_1^n, b_2^n\right]\right) \text{ in M}$$

where $a_t^i, b_t^i \in$ R (or Q or Z) $1 \le t \le 2$, $1 \le i \le n$;

then x+y

$$= \left(\left[a_1^1, a_2^1\right] + \left[b_1^1, b_2^1\right], \left[a_1^2, a_2^2\right] + \left[b_1^2, b_2^2\right], \ldots, \left[a_1^n, a_2^n\right] + \left[b_1^n, b_2^n\right]\right)$$

$$= \left[\left(a_1^1, a_1^2, \ldots, a_1^n\right) + \left(b_1^1, b_1^2, \ldots, b_1^n\right), \left(a_2^1, a_2^2, \ldots, a_2^n\right) + \left(b_2^1, b_2^2, \ldots, b_2^n\right)\right]$$

$$= \left[\left(a_1^1 + b_1^1, \ldots, a_1^n + b_1^n, a_2^1 + b_2^1, \ldots, a_2^n + b_2^n\right)\right]$$

$$= \left(\left[a_1^1 + b_1^1, a_2^1 + b_2^1\right], \left[a_1^2 + b_1^2, a_2^2 + b_2^2\right], \ldots, \left[a_1^n + b_1^n, a_2^n + b_2^n\right]\right).$$

By this method row matrix of the natural class of intervals acts as usual n-tuples pair.

We see this method of representation will help us to induct the usual program for addition of interval row matrices, in a very simple way which take the same time as that of usual row matrices.

Now we can define product of these row interval matrices, entries taken from the natural class of intervals. $N_c(R)$ or $N_o(R)$ or $N_{oc}(R)$ or $N_{co}(R)$ or R replaced by Q or Z.

It is pertinent to mention here that the operation of division is not defined on $N_o(Z)$ or $N_c(Z)$ or $N_{oc}(Z)$ or $N_{co}(Z)$ when in [a, b] one of a or b is zero; 'or' not used in the mutually exclusive sense.

Let

$X = \{(a_1, \ldots, a_n) \mid a_i \in N_c(R)$ (or $N_o(Q)$ or $N_{oc}(Z)$ or so on); $1 \leq i \leq n\}$ we can define product on X; component wise for each row.

For if $x = (a_1, \ldots, a_n)$ and $y = (b_1, \ldots, b_n)$ then

$$\begin{aligned} x.y &= ((a_1, a_2, \ldots, a_n) \,.\, (b_1, b_2, \ldots, b_n)) \\ &= (a_1b_1, a_2b_2, \ldots, a_nb_n) \in X. \end{aligned}$$

Thus (X, .) is a commutative monoid with ([1, 1], [1, 1], …, [1, 1]) as it multiplicative identity.

We give examples of this situation.

Example 2.6: Let

$$M = \{(a_1, a_2, a_3) \mid a_i \in N_o(Z); 1 \leq i \leq 3\}$$

be the monoid of natural class of row interval matrices. Consider

$$\begin{aligned} x &= (a_1, a_2, a_3) \\ &= ((3, 0), (8, -7), (2, 9)) \end{aligned}$$

$$\begin{aligned} \text{and } y &= (b_1, b_2, b_3) \\ &= ((7, 0), (5, 2), (-3, 1)) \text{ in } M. \end{aligned}$$

$$\begin{aligned} x.y &= (a_1, a_2, a_3) \,.\, (b_1, b_2, b_3) \\ &= ((3, 0), (8, -7), (2, 9)) \,.\, ((7, 0), (5, 2), (-3, 1)) \\ &= ((3, 0).\,(7, 0), (8, -7)\,.\,(5, 2), (2, 9)\,(-3,1)) \\ &= ((0, 0)\,(40, -14), (-6, 9)) \in M. \end{aligned}$$

Thus (M, .) is a commutative monoid. Clearly these monoids have zero divisors, units provided they are built using $N_c(R)$ or $N_o(Q)$.

We give yet another example.

Example 2.7: Let

$$V = \{(a_1, a_2, a_3, a_4, a_5, \ldots, a_{10}) \mid a_i \in N_{oc}(Q); 1 \leq i \leq 10\}$$

be the monoid of row matrices of natural class of intervals. This monoid has units, zero divisors and has no idemponents, except of the form ((1, 0), (0, 1), …, (0, 1)) = x only every entry in x is of the form (0, 1) or (1, 0).

These have idemponents even if these monoids are built using Z or Q or R.

We define for these type of monoids the notion of orthogonal elements. Two elements x, y $\in$ M are orthogonal provided x.y = (0). We also define x to be the complement of y.

We say $(0, x_1)$ is also the complement of $(y_1, 0)$, $x_1, y_1 \in R$ or Q or Z.

In view of this we have the following results.

THEOREM 2.1: *Let*

$$X = \{(a_1, a_2, \ldots, a_n) \mid a_i \in N_{oc}(Q) \text{ or } N_{co}(R) \text{ or } N_o(R) \text{ or } N_c(R) \text{ or so on; } 1 \leq i \leq n\}$$

be the monoid. X has orthogonal elements.

(1) If $x = (a_1, \ldots, a_n)$ and $y = (b_1, \ldots, b_n)$ are such that, if $a_i = (0, x_i]$ then $b_i = (y_i, 0]$ (or if $a_i = (x_i, 0]$ then $b_i = (0, y_i]$) where $x_i, y_i \in Q$, $1 \leq i \leq n$, then x is orthogonal to y or x is complement of y.

(2) If $x = (a_1, \ldots, a_n)$ and $y = (b_1, \ldots, b_n)$ then if $a_i = (x_i, y_i]$ then $b_i = (0, 0]$ if $x_i \neq 0$ or $y_i \neq 0$; $x_i, y_i \in Q$ and if $a_j = (0, 0]$ then $b_j = (c_j, d_j]$ for $1 \leq i, j \leq n$. Then x is orthogonal to y or x is the complement of y.

The proof of this theorem is direct and hence left as an exercise to the reader.

Now we proceed onto define the notion of column matrices of the natural class of intervals.

Let

$$X = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \middle| \; x_i \in N_{oc}(R); 1 \leq i \leq m \right\}$$

be the column matrix of natural class of open closed intervals of reals. $N_{oc}(R)$ can be replaced by $N_{co}(R)$ or $N_c(R)$ or $N_o(R)$ or R can be replaced by Q or Z. Now it is easily verified, X under addition is an abelian group.

Example 2.8: Let

$$W = \left\{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \middle|\; x_i \in N_o(Q); 1 \le i \le 4\right\}$$

be the abelian group of column matrices of natural class of intervals.

Example 2.9: Let

$$M = \left\{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{25} \end{bmatrix} \middle|\; x_i \in N_{co}(R); 1 \le i \le 25\right\}$$

be the abelian group of column matrices of natural class of intervals under addition.

Clearly these column matrices of natural class of intervals can never have product defined on them.

Now we can write the column matrices of natural class of intervals as columns of matrix intervals.

We will just illustrate this situation by some simple examples.

Example 2.10: Let

$$X = \begin{bmatrix} [0,3] \\ [7,2] \\ [-1,0] \\ [4,5] \end{bmatrix}$$

be an interval column matrix with entries from $N_c(R)$.

Now we write X as $X = \left[\begin{bmatrix} 0 \\ 7 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 0 \\ 5 \end{bmatrix}\right]$

this interval is called as the column matrix interval. Thus every column matrix interval is an interval of column matrices and vice versa.

If $A = \left[\begin{bmatrix} a_1 \\ \vdots \\ a_{10} \end{bmatrix}, \begin{bmatrix} b_1 \\ \vdots \\ b_{10} \end{bmatrix}\right]$ be the column matrix interval then

$$A = \begin{bmatrix} [a_1, b_1] \\ [a_2, b_2] \\ \vdots \\ [a_{10}, b_{10}] \end{bmatrix}$$

is the interval column matrix and vice versa.

Thus we can define for the interval row matrix of natural classes also the notion of row matrix interval.

The open or closed or half open-half closed or half closed - half open will be exhibited as follows:

Open interval of row matrices as

$$\begin{aligned} X &= ((a_1, \ldots, a_n), (b_1, \ldots, b_n)) \\ &= ((a_1, b_1), \ldots, (a_n, b_n)) \\ &= ((a_1, \ldots, a_n), (b_1, \ldots, b_n)). \end{aligned}$$

Closed interval of row matrices.

$$\begin{aligned} Y &= ([a_1, \ldots, a_n], [b_1, \ldots, b_n]) \\ &= ([a_1, b_1], \ldots, [a_n, b_n]) \\ &= [(a_1, \ldots, a_n), (b_1, \ldots, b_n)]. \end{aligned}$$

Half open-half closed interval of row matrices.

$$\begin{aligned} Z &= ([a_1, \ldots, a_n], [b_1, \ldots, b_n]] \\ &= ((a_1, \ldots, a_n), (b_1, \ldots, b_n)] \\ &= ((a_1, b_1], (a_2, b_2], \ldots, (a_n, b_n]). \end{aligned}$$

Half closed - half open interval of row matrices.

$$\begin{aligned} M &= [(a_1, \ldots, a_n), (b_1, \ldots, b_n)) \\ &= ([a_1, \ldots, a_n], (b_1, \ldots, b_n)) \\ &= ([a_1, b_1), [a_2, b_2), \ldots, [a_n, b_n)). \end{aligned}$$

Similar techniques in case of interval column matrices or column matrices of natural class of intervals. We can say half open - half closed interval column matrix if

$$X = \left[\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \right]$$

$$= \left(\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}, \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \right]$$

$$= \begin{pmatrix} (a_1, b_1] \\ \vdots \\ (a_n, b_n] \end{pmatrix}.$$

$$Y = \left[\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_M \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \right)$$

$$= \left[\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix}, \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \right]$$

$$= \begin{bmatrix} [a_1, b_1) \\ [a_2, b_2) \\ \vdots \\ [a_m, b_m) \end{bmatrix}$$

is the half closed - half open column matrix interval or interval column matrix.

Likewise we can define open interval column matrix and closed interval column matrix.

$$\text{Let } A = \left(\begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix}, \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \right)$$

$$= \left[\begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix}, \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}\right]$$

$$= \begin{bmatrix} (a_1, b_1) \\ \vdots \\ (a_m, b_m) \end{bmatrix}$$

is the open interval column matrix or column matrix of natural class of open intervals.

$$\text{Let } B \quad = \left[\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}\right]$$

$$= \begin{bmatrix} [a_1, b_1] \\ [a_2, b_2] \\ \vdots \\ [a_n, b_n] \end{bmatrix}$$

is the closed interval column matrix or column matrix of natural class of closed intervals. As we have worked with the collection of column matrix of natural class of intervals and collection of row matrix of natural class of intervals we can work with the collection of open interval column matrices or closed interval column matrices or half open - half closed interval of column matrices or half closed - half open interval of column matrices.

We will denote the collection;

$$N_c^c(R) = \left\{ \text{set of all} \left[\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \right] \middle| \; a_i, b_i \in R, 1 \leq i \leq n \right\}$$

denotes the collection of all closed interval column matrices.

Similarly $N_o^c(R)$ will denote the collection of all open interval column matrices.

$N_{oc}^c(R)$ will denote the collection of all half open- half closed interval column matrices. $N_{oc}^c(R)$ will denote the collection of all half closed - half open interval column matrices.

We can define the operation addition on these collections. We will only illustrate this situation by some examples.

Example 2.11: Let

$$M = \left\{ \left(\begin{bmatrix} a_1 \\ \vdots \\ a_8 \end{bmatrix}, \begin{bmatrix} b_1 \\ \vdots \\ b_8 \end{bmatrix} \right) = X \middle| \; X \in N_c^c(R) \; ; a_i \in R; 1 \leq i \leq 8 \right\}$$

be the collection of closed interval column matrices. M is an abelian group under addition.

For if $X = \left(\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_8 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_8 \end{bmatrix} \right)$ and $Y = \left(\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_8 \end{bmatrix}, \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_8 \end{bmatrix} \right)$ are in M, then $X + Y = \left(\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_8 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_8 \end{bmatrix}\right) + \left(\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_8 \end{bmatrix}, \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_8 \end{bmatrix}\right)$

$$= \left(\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_8 \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_8 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_8 \end{bmatrix} + \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_8 \end{bmatrix}\right)$$

$$= \left(\begin{bmatrix} a_1 + c_1 \\ a_2 + c_2 \\ \vdots \\ a_8 + c_8 \end{bmatrix}, \begin{bmatrix} b_1 + d_1 \\ b_2 + d_2 \\ \vdots \\ b_8 + d_8 \end{bmatrix}\right) \in M.$$

Clearly $(0) = \left(\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right)$ acts as the additive identity, for every $X \in M$, $(0) + X = X + (0) = X$.

Now for every $X = \left(\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_8 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_8 \end{bmatrix}\right)$ in M.

$$-X = \left(\begin{bmatrix} -a_1 \\ -a_2 \\ \vdots \\ -a_8 \end{bmatrix}, \begin{bmatrix} -b_1 \\ -b_2 \\ \vdots \\ -b_8 \end{bmatrix}\right)$$

in M acts as the inverse of X, we see X + (–X) = (0).

Thus M is a group under addition.

Example 2.12: Let

$$N = \left\{ Y = \left(\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_5 \end{bmatrix}, \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_5 \end{pmatrix}\right) \middle| \; Y \in N^c_{co}(R) \; ; a_i \in R; \; 1 \leq i \leq 5 \right\}$$

be the collection of half closed - half open interval column matrices.

N is a group under addition.

Suppose

$$M = \left\{ \begin{bmatrix} [a_1, b_1) \\ [a_2, b_2) \\ \vdots \\ [a_5, b_5) \end{bmatrix} \middle| \; a_i \; b_i \in R, \; 1 \leq i \leq 5 \text{ or } [a_i, b_i) \in N_{co}(R) \right\}$$

be the collection of all column matrices of natural class of half closed-half open intervals. M is a group under addition.

Now we see we can define a map $\eta : N \to M$ as follows.

$$\eta\left[\left(\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_5 \end{bmatrix}, \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_5 \end{pmatrix}\right)\right] = \begin{bmatrix} [a_1, b_1) \\ [a_2, b_2) \\ \vdots \\ [a_5, b_5) \end{bmatrix}.$$

It is easily verified η : N $\to$ M is a group homomorphism. Infact η is one to one and onto so η is a isomorphism.

$$\text{Further ker } \eta = \left\{\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right)\right\}.$$

Thus we can say every interval column matrix group is isomorphic to the group of column matrices using natural class of intervals.

That is

$$V = \left\{\left[\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}\right] = X \;\middle|\; X \in N_o^c(R),\ a_i, b_i \in R;\ 1 \le i \le n\right\}$$

the interval column matrix group is isomorphic with

$$W = \left\{\begin{bmatrix} (a_1, b_1) \\ (a_2, b_2) \\ \vdots \\ (a_n, b_n) \end{bmatrix} \;\middle|\; (a_i, b_i) \in N_o(R);\ 1 \le i \le n\right\}$$

the column matrix with entries from natural class of open intervals from $N_o(R)$.

We have $V \cong W$ with $\eta : V \to W$ where

$$\eta\,(X) = \eta\left(\left[\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}\right]\right) = \begin{bmatrix} (a_1, b_1) \\ (a_2, b_2) \\ \vdots \\ (a_n, b_n) \end{bmatrix} \text{ for every } X \in V.$$

Since $V \cong W$, we can replace one group in the place of other and vice versa. We will say both the groups are equivalent or identical except for the representation.

Next we can say the similar result in case of row matrix of natural class of intervals or interval row matrices.

We see if $M = \{(a_1, \ldots, a_n) \mid [a_1^i, a_2^i] = a_i \in N_c(R);\ 1 \le i \le n\}$ be the group of row interval matrices with entries from the natural class of closed intervals from $N_c(R)$ and

$N = \left\{\left[(a_1^1, \ldots, a_n^1), (a_1^2, \ldots, a_n^2)\right] = \left([a_1^1, \ldots, a_n^1], [a_1^2, \ldots, a_n^2]\right) \,\middle|\, a_i^t \in\right.$ R, $t = 1, 2;\ 1 \le i \le n\}$ be the row interval matrix group, then M is isomorphic to N.

We define $\eta : M \to N$ by

$$\begin{aligned} \eta\,(X) &= \eta\left(\left([a_1^1, a_2^1], \ldots, [a_1^n, a_2^n]\right)\right) \\ &= \left[(a_1^1, \ldots, a_1^n), (a_2^1, \ldots, a_2^n)\right] \\ &= \left((a_1^1, \ldots, a_1^n), (a_2^1, \ldots, a_2^n)\right). \end{aligned}$$

Thus we can without loss of generality work with any one of the groups M or N.

Example 2.13: Let

$$V = \{(a_1, a_2, a_3, a_4) \mid a_i = (a_i^1, a_i^2]\,;\ 1 \le i \le 4 \text{ with } a_i \in N_{oc}(R)\}$$

be the group of interval row matrices with entries from $N_{oc}(R)$.

Consider

$$M = \{[(a_1^1, a_2^1, a_3^1, a_4^1), [a_1^2, a_2^2, a_3^2, a_4^2]] \text{ where } a_j^i \in R,\ i=1, 2;\ 1 \le j \le 4\}$$

be the group of interval row matrices. $M \cong V$.

Now we define interval group of $n \times m$ matrices ($n \neq m$).

Consider the $n \times m$ matrix $m = (m_{ij})$ where $m_{ij} \in N_c(R)$; $1 \le i \le n$ and $1 \le j \le m$, $n \neq m$.

We define M as the $n \times m$ natural class of interval matrix or $n \times m$ interval matrix with entries from the natural class of intervals.

We give examples of them.

Example 2.14: Let

$$M = \begin{pmatrix} (0,9] & (9,2] \\ (8,-2] & (0,-4] \\ (11,13] & (2,-5] \\ (-3,-13] & (-3,7] \end{pmatrix}$$

be the 4×2 half open - half closed natural class of interval matrix 4×2 matrix with entries from $N_{oc}(Z)$.

Example 2.15: Let

$$M = \begin{pmatrix} [3,0] & [7,3] & [5,-3] \\ [-8,-10] & [0,-7] & [9,10] \\ [7,-2] & [3,6] & [-3,8] \\ [-7,0] & [0,-8] & [11,15] \\ [10,-9] & [12,-9] & [0,0] \end{pmatrix}$$

be the 5 × 3 closed interval matrix or 5 × 3 matrix with entries from $N_c(R)$ or natural class of closed interval 5×3 matrix.

Example 2.16: Let

$$V= \begin{pmatrix} [0,7) & [5,-3) & [7,-9) & [14,3) & [10,0) & [12,12) & [11,-1) \\ [5,-2) & [11,15) & [9,9) & [1,-1) & [7,6) & [-1,-1) & [5,-5) \\ [8,8) & [15,-11) & [-1,1) & [0,-7) & [3,0) & [4,3) & [6,6) \end{pmatrix}$$

be a 3 × 7 half closed half open interval matrix or a 3 × 7 interval matrix with entries from $N_{co}(Z)$. We say two interval matrices M and N are of same order if both M and N are t × s matrices and both M and N have entries which belong to $N_c(R)$ or $N_o(R)$ or $N_{oc}(R)$ or $N_{co}(R)$ (or used in the mutually exclusive sense).

We can add two interval matrices M and N only when

(i) They are of same order.
(ii) The entries for both M and N are taken from the same class of natural class of intervals i.e., from $N_c(R)$ (or $N_{co}(R)$ or $N_o(R)$ or $N_{oc}(R)$). (R can be replaced by Z or Q also).

Now if we have matrices from $N_c(R)$ and $N_o(R)$ even of same order we cannot add them. So when all the conditions are satisfied we can add the m × n interval matrices or the natural class of interval matrices.

We will first illustrate this situation by some examples.

Example 2.17: Let

$$M = \begin{pmatrix} [3,2] & [5,-2] & [4,3] \\ [0,7] & [7,-7] & [-8,0] \\ [8,0] & [1,1] & [9,-2] \end{pmatrix}$$

and

$$N = \begin{pmatrix} [0,-7] & [1,-2] & [5,1] \\ [2,0] & [3,-4] & [4,4] \\ [3,5] & [11,-2] & [0,0] \end{pmatrix}$$

be two natural class of interval matrices with entries from $N_c(R)$.

$$\text{Now } M + N = \begin{pmatrix} [3,2] & [5,-2] & [4,3] \\ [0,7] & [7,-7] & [-8,0] \\ [8,0] & [1,1] & [9,-2] \end{pmatrix} +$$

$$\begin{pmatrix} [0,-7] & [1,-2] & [5,1] \\ [2,0] & [3,-4] & [4,4] \\ [3,5] & [11,-2] & [0,0] \end{pmatrix}$$

$$= \begin{pmatrix} [3,2]+[0,-7] & [5,-2]+[1,-2] & [4,3]+[5,1] \\ [0,7]+[2,0] & [7,-7]+[3,-4] & [-8,0]+[4,4] \\ [8,0]+[3,5] & [1,1]+[11,-2] & [9,-2]+[0,0] \end{pmatrix}$$

$$= \begin{pmatrix} [3,-5] & [6,-4] & [9,4] \\ [2,7] & [10,-11] & [-4,4] \\ [11,5] & [12,-1] & [9,-2] \end{pmatrix}.$$

Thus if $V = \{M = (a_{ij});\ 1 \leq i \leq m;\ 1 \leq j \leq n\}$ be the collection of all $m \times n$ interval matrices with entries from $N_{oc}(R)$. V under addition is a group we can replace $N_{oc}(R)$ by $N_o(R)$ or $N_c(R)$ or $N_{co}(R)$ (and R by Q or Z) still the V will be a group under interval matrix addition.

However we cannot define on V the operation of matrix multiplication.

Now if we consider the collection of all interval $n \times n$ matrices V; with entries from $N_o(R)$ (or $N_{oc}(R)$ or $N_{co}(R)$ or $N_c(R)$) then V is a group under matrix addition, but only a non commutative semigroup under multiplication.

We give examples of this situations.

Example 2.18: Let

$$V = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle|\ a_i \in N_{co}(R)\ ;\ 1 \leq i \leq 4 \right\}$$

be a semigroup under multiplication.

If $x = \begin{bmatrix} [0,7) & [0,-7) \\ [-8,0) & [8,3) \end{bmatrix}$ and $y = \begin{bmatrix} [2,0) & [5,5) \\ [3,2) & [0,0) \end{bmatrix}$

be in V then

$$xy = \begin{bmatrix} [0,7) & [0,-7) \\ [-8,0) & [8,3) \end{bmatrix} \begin{bmatrix} [2,0) & [5,5) \\ [3,2) & [0,0) \end{bmatrix}$$

$$= \begin{bmatrix} [0,7)[2,0)+[0,-7)[3,2) & [0,7)[5,5)+[0,-7)[0,0) \\ [-8,0)[2,0)+[8,3)[3,2) & [-8,0)[5,5)+[8,3)[0,0) \end{bmatrix}$$

$$= \begin{bmatrix} [0,0)+[0,-14) & [0,35)+[0,0) \\ [-16,0)+[24,6) & [-40,0)+[0,0) \end{bmatrix}$$

$$= \begin{bmatrix} [0,-14) & [0,35) \\ [8,6) & [-40,0) \end{bmatrix} \text{ is in V.}$$

To show xy ≠ yx.

$$\text{Consider yx} = \begin{bmatrix} [2,0) & [5,5) \\ [3,2) & [0,0) \end{bmatrix} \begin{bmatrix} [0,7) & [0,-7) \\ [-8,0) & [8,3) \end{bmatrix}$$

$$= \begin{bmatrix} [2,0)[0,7)+[5,5)[-8,0) & [2,0)[0,-7)+[5,5)[8,3) \\ [3,2)[0,7)+[0,0)[-8,0) & [3,2)[0,-7)+[0,0)[8,3) \end{bmatrix}$$

$$= \begin{bmatrix} [0,0)+[-40,0) & [0,0)+[40,15) \\ [0,14)+[0,0) & [0,-14)+[0,0) \end{bmatrix}$$

$$= \begin{bmatrix} [-40,0) & [40,15) \\ [0,14) & [0,-14) \end{bmatrix} \neq \text{xy.}$$

V is a semigroup under matrix multiplication.

Infact $\begin{bmatrix} [1,1) & [0,0) \\ [0,0) & [1,1) \end{bmatrix}$ acts as the multiplicative identity.

Further V is also a group under addition of matrices.

$\begin{bmatrix} [0,0) & [0,0) \\ [0,0) & [0,0) \end{bmatrix}$ in V acts as the additive identity.

Now we can define the notion of m × n matrix intervals or natural class of m × n matrix intervals (m ≠ n).

We give examples of this situation.

Example 2.19: Let

$$X = \left(\begin{bmatrix} 9 & 0 & 2 & -1 \\ -4 & -3 & 6 & 7 \\ 5 & 1 & 0 & -5 \end{bmatrix}, \begin{bmatrix} 7 & -8 & 3 & 6 \\ -9 & 0 & -6 & -3 \\ -7 & -1 & 2 & 4 \end{bmatrix} \right)$$

is the 3 × 4 matrix interval of natural class of closed intervals.

Now we can rewrite

$$X \text{ as } \begin{pmatrix} [9,7] & [0,-8] & [2,3] & [-1,6] \\ [-4,-9] & [-3,0] & [6,-6] & [7,-3] \\ [5,-7] & [1,-1] & [0,2] & [-5,4] \end{pmatrix},$$

is the 3 × 4 interval matrices with entries from $N_c(R)$.

Example 2.20: Let

$$M = \left[\begin{pmatrix} 3 & -1 \\ -2 & 5 \\ 0 & 7 \\ -7 & 0 \\ 2 & -2 \end{pmatrix}, \begin{pmatrix} 5 & -7 \\ -5 & 2 \\ 3 & 4 \\ 2 & 0 \\ 2 & -7 \end{pmatrix} \right]$$

be the 5 × 2 matrix interval which are open intervals.

Now M can be written as the natural class of open intervals 5 × 2 matrices with entries from $N_o(R)$ (or $N_o(Z)$ or $N_o(Q)$) as follows.

$$M_1 = \begin{bmatrix} (3,5) & (-1,-7) \\ (-2,-5) & (5,2) \\ (0,3) & (7,4) \\ (-7,2) & (0,0) \\ (2,2) & (-7,-2) \end{bmatrix}$$ is the 5×2 interval matrices.

We see we can go easily from M to M_1 or M_1 to M. Thus we can define a mapping from

$M = (m_{ij}) = ([m^1_{ij}, m^2_{ij}])$, $1 \leq i \leq m$ and $1 \leq j \leq n$ to

$M_1 = [(m^1_{ij}),(m^2_{ij})] = ([m^1_{ij}],[m^2_{ij}])$ as follows.

$$M \mapsto M_1 \text{ by the rule } ([m^1_{ij}, m^2_{ij}])$$

$$= ([m^1_{ij}],[m^2_{ij}]) \text{ or } M_1 \mapsto M$$

$$\text{by } ([m^1_{ij}],[m^2_{ij}]) = [m^1_{ij}, m^2_{ij}].$$

It is easy to verify the following theorem.

THEOREM 2.2: *Let*

$G = \{([m^1_{ij}],[m^2_{ij}])$ *where* $m^t_{ij} \in R$, $t = 1, 2$; $1 \leq i \leq m$ *and* $1 \leq j \leq n\}$ *be the collection of all* $m \times n$ *matrix intervals. G is a group under addition.*

THEOREM 2.3: *Let*

$N = \{([m^1_{ij}],[m^2_{ij}]) = (m_{ij}) \mid m_{ij} \in N_c(R)$ *(or* $N_{oc}(R)$ *or* $N_{co}(R)$ *or* $N_o(R)$*);* $1 \leq i \leq m$ *and* $1 \leq j \leq n\}$ *be the collection of interval* $m \times n$ *matrices with intervals from* $N_c(R)$ *(or* $N_{co}(R)$ *or* $N_{co}(R)$ *or* $N_o(R)$*). N is a group under addition.*

THEOREM 2.4: *The groups G and N mentioned in theorems 2.2 and 2.3 are isomorphic.*

Proof: Define a map $\eta : G \to N$.

$$\eta\ (([m^1_{ij}],[m^2_{ij}])) = ([m^1_{ij}, m^2_{ij}])\,;\ \text{for every}\ ([m^1_{ij}],[m^2_{ij}])\ \text{in G.}$$

η is a group homomorphism, infact an isomorphism. Thus G is isomorphic to N.

Hence we can as per convenience work with G or work with N as both are isomorphic. Now we see if m = n that is we have square (n × n) matrix intervals. Infact the collection of all such square (n × n) matrix intervals forms a group.

Also if we consider the collection P of all interval square matrices with intervals from $N_o(R)$ (or $N_c(R)$ or $N_{oc}(R)$ or $N_{co}(R)$ or R replaced by Q or Z). We see P is a group under addition. We will first illustrate them and then derive related results.

Example 2.21: Let

$$M = \left[\begin{bmatrix} 8 & 3 & 5 \\ -1 & 2 & 7 \\ 0 & -4 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 4 \\ 5 & 1 & 2 \\ 3 & 4 & 2 \end{bmatrix}\right]$$

be a 3 × 3 matrix closed interval. We can rewrite M as the interval matrix with entries from N_c (R) as follows.

$$M = \begin{pmatrix} [8,2] & [3,0] & [5,4] \\ [-1,5] & [2,1] & [7,2] \\ [0,3] & [-4,4] & [2,2] \end{pmatrix}.$$

$$\text{Suppose } P = \left(\begin{bmatrix} 3 & -2 & 8 \\ 8 & 0 & 4 \\ -7 & 5 & -1 \end{bmatrix}, \begin{bmatrix} -9 & 8 & -6 \\ 0 & 2 & 1 \\ 4 & 8 & -5 \end{bmatrix}\right)$$

be a 3 × 3 matrix interval. Then M + P is defined as follows.

$$M + P = \left(\begin{bmatrix} 8 & 3 & 5 \\ -1 & 2 & 7 \\ 0 & -4 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 4 \\ 5 & 1 & 2 \\ 3 & 4 & 2 \end{bmatrix}\right) +$$

$$\left(\begin{bmatrix} 3 & -2 & 8 \\ 8 & 0 & 4 \\ -7 & 5 & -1 \end{bmatrix}, \begin{bmatrix} -9 & 8 & -6 \\ 0 & 2 & 1 \\ 4 & 8 & -5 \end{bmatrix}\right)$$

$$= \left(\begin{bmatrix} 8 & 3 & 5 \\ -1 & 2 & 7 \\ 0 & -4 & 2 \end{bmatrix} + \begin{bmatrix} 3 & -2 & 8 \\ 8 & 0 & 4 \\ -7 & 5 & -1 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 4 \\ 5 & 1 & 2 \\ 3 & 4 & 2 \end{bmatrix} + \begin{bmatrix} -9 & 8 & -6 \\ 0 & 2 & 1 \\ 4 & 8 & -5 \end{bmatrix}\right)$$

$$= \left(\begin{bmatrix} 11 & 1 & 13 \\ 7 & 2 & 11 \\ -7 & 1 & 1 \end{bmatrix}, \begin{bmatrix} -7 & 8 & -2 \\ 5 & 3 & 3 \\ 7 & 12 & -3 \end{bmatrix}\right)$$

$$= \left(\begin{bmatrix} [11,-7] & [1,8] & [13,-2] \\ [7,5] & [2,3] & [11,3] \\ [-7,7] & [1,12] & [-3,1] \end{bmatrix}\right).$$

$$\text{Suppose } X = \left(\begin{bmatrix} [0,2] & [2,1] & [-3,-9] \\ [3,3] & [4,0] & [-7,0] \\ [0,-3] & [7,7] & [11,2] \end{bmatrix}\right) \text{ and}$$

$$\left(\begin{bmatrix} [-2,0] & [7,0] & [-5,-12] \\ [1,2] & [0,7] & [3,0] \\ [3,3] & [11,2] & [-8,0] \end{bmatrix}\right) = Y$$

be any two natural class of closed interval matrices or interval square matrices with intervals from $N_c(R)$. We can add X and Y.

X+Y=

$$\left(\begin{bmatrix} [0,2] & [2,1] & [-3,-9] \\ [3,3] & [4,0] & [-7,0] \\ [0,-3] & [7,7] & [11,2] \end{bmatrix}\right) + \left(\begin{bmatrix} [-2,0] & [7,0] & [-5,-12] \\ [1,2] & [0,7] & [3,0] \\ [3,3] & [11,2] & [-8,0] \end{bmatrix}\right)$$

$$= \left(\begin{bmatrix} [0,2]+[-2,0] & [2,1]+[7,0] & [-3,-9]+[-5,-12] \\ [3,3]+[1,2] & [4,0]+[0,7] & [-7,0]+[3,0] \\ [0,-3]+[3,3] & [7,7]+[11,2] & [11,2]+[-8,0] \end{bmatrix}\right)$$

$$= \begin{pmatrix} [-2,2] & [9,1] & [-8,-21] \\ [4,5] & [4,7] & [-4,0] \\ [3,0] & [18,9] & [3,2] \end{pmatrix}.$$

$$\text{Now } X = \left(\begin{bmatrix} [0,2] & [2,1] & [-3,-9] \\ [3,3] & [4,0] & [-7,0] \\ [0,-3] & [7,7] & [11,2] \end{bmatrix}\right)$$

$$= \left(\begin{bmatrix} 0 & 2 & -3 \\ 3 & 4 & -7 \\ 0 & 7 & 11 \end{bmatrix}, \begin{bmatrix} 2 & 1 & -9 \\ 3 & 0 & 0 \\ -3 & 7 & 2 \end{bmatrix}\right) \text{ and}$$

$$Y = \left(\begin{bmatrix} [-2,0] & [7,0] & [-5,-12] \\ [1,2] & [0,7] & [3,0] \\ [3,3] & [11,2] & [-8,0] \end{bmatrix}\right)$$

$$= \left(\begin{bmatrix} -2 & 7 & -5 \\ 1 & 0 & 3 \\ 3 & 11 & 8 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -12 \\ 2 & 7 & 0 \\ 3 & 2 & 0 \end{bmatrix}\right).$$

Thus we can say the group of n × n matrix intervals is isomorphic with the intervals of n × n matrices provided the intervals are taken as closed (or open or etc) on both collection.

Now using this isomorphism we can always derive all algorithms (programs) for addition, subtraction and multiplication with simple modifications in case of matrix intervals; the time taken for these is the same as that of usual matrices.

We will just illustrate this.

If M = (m_{ij}) = $([m_{ij}^1, m_{ij}^2])$ be the interval matrix of order m × n and N = (n_{ij}) = $([n_{ij}^1, n_{ij}^2])$ be another interval matrix of order m × n both M and N take their entries from $N_c(R)$.

Now to find M + N

$$= (m_{ij}) + (n_{ij})$$

$$= ([m_{ij}^1, m_{ij}^2]) + ([n_{ij}^1, n_{ij}^2])$$

$$= [(m_{ij}^1), (m_{ij}^2)] + [(n_{ij}^1), (n_{ij}^2)]$$

$$= [(m_{ij}^1) + (n_{ij}^1), (m_{ij}^2) + (n_{ij}^2)]$$

(using usual program for m × n matrices we get $(m_{ij}^1)+(n_{ij}^1)$ and $(m_{ij}^2)+(n_{ij}^2)$)

$$= [(s_{ij}^1),(s_{ij}^2)]$$

$$= ([s_{ij}^1, s_{ij}^2]) = (s_{ij}).$$

Thus except for separating them by a ‘,’ the program or algorithm for addition is identical with that of the usual matrices. Now on similar lines M – N, the subtraction is performed.

We now proceed onto define product of two interval matrices.

First the multiplication of interval matrices are defined only when the matrices are square matrices and both of them are of the same type (that is both should take entries from $N_c(R)$ (or $N_o(R)$ or $N_{oc}(R)$ or $N_{co}(R)$ or used in the mutually exclusive sense, then only product can be defined.

$$\text{Suppose } M = (m_{ij}) = ([m_{ij}^1, m_{ij}^2])$$

($1 \leq i, j \leq n$ and $m_{ij}^t \in R$; t =1,2) be a n × n interval matrix and $N = (n_{ij}) = ([n_{ij}^1, n_{ij}^2])$ be a n × n interval matrix both m_{ij} and $n_{ij} \in N_c(R)$.

To find the matrix product

$$\begin{aligned} M \times N &= (m_{ij})\,(n_{ij}) \\ &= \left(\left[m_{ij}^1, m_{ij}^2\right]\right) \times \left(\left[n_{ij}^1, n_{ij}^2\right]\right) \end{aligned}$$

$$= \left[\left(m_{ij}^{1}\right),\left(m_{ij}^{2}\right)\right] \times \left[\left(n_{ij}^{1}\right),\left(n_{ij}^{2}\right)\right]$$

(this is possible as these two matrices are one and the same, as they are identical except for the representation and one can get one from the other and vice versa)

$$= \left[\left(m_{ij}^{1}\right)\left(n_{ij}^{1}\right),\left(m_{ij}^{2}\right)\left(n_{ij}^{2}\right)\right]$$

$$= \left[\left(s_{ij}^{1}\right),\left(s_{ij}^{2}\right)\right]$$

$$= \left(\left[s_{ij}^{1},s_{ij}^{2}\right]\right)$$

$$= (s_{ij}).$$

(where s_{ij}^{t} is the product of the $n \times n$ matrices m_{ij}^{t} with n_{ij}^{t}; $t = 1, 2$).

Thus except for rewriting them the program for the usual matrices can be used for these interval matrices also.

We will illustrate this situation by some examples.

Example 2.22: Let

$$V = \begin{bmatrix} [0,7) & [7,1) & [5,8) \\ [9,0) & [1,2) & [3,7) \\ [11,3) & [10,-1) & [1,1) \end{bmatrix} \text{ and}$$

$$W = \begin{bmatrix} [9,0) & [2,4) & [0,-2) \\ [3,2) & [4,2) & [3,3) \\ [4,4) & [6,6) & [1,1) \end{bmatrix}$$

be interval matrices with entries form $N_{co}(R)$.

To find

$$\mathrm{V.W} = \begin{bmatrix} [0,7) & [7,1) & [5,8) \\ [9,0) & [1,2) & [3,7) \\ [11,3) & [10,-1) & [1,1) \end{bmatrix} . \begin{bmatrix} [9,0) & [2,4) & [0,-2) \\ [3,2) & [4,2) & [3,3) \\ [4,4) & [6,6) & [1,1) \end{bmatrix}$$

$$= \left(\begin{bmatrix} 0 & 7 & 5 \\ 9 & 1 & 3 \\ 11 & 10 & 1 \end{bmatrix}, \begin{bmatrix} 7 & 1 & 8 \\ 0 & 2 & 7 \\ 3 & -1 & 1 \end{bmatrix} \right) \left(\begin{bmatrix} 9 & 2 & 0 \\ 3 & 4 & 3 \\ 4 & 6 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 4 & -2 \\ 2 & 2 & 3 \\ 4 & 6 & 1 \end{bmatrix} \right)$$

$$= \left(\begin{bmatrix} 0 & 7 & 5 \\ 9 & 1 & 3 \\ 11 & 10 & 1 \end{bmatrix} \begin{bmatrix} 9 & 2 & 0 \\ 3 & 4 & 3 \\ 4 & 6 & 1 \end{bmatrix}, \begin{bmatrix} 7 & 1 & 8 \\ 0 & 2 & 7 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 4 & -2 \\ 2 & 2 & 3 \\ 4 & 6 & 1 \end{bmatrix} \right)$$

$$= \left[\begin{pmatrix} 0.9+7.3+5.4 & 0.2+7.4+5.6 & 0.0+7.3+5.1 \\ 9.9+1.3+3.4 & 9.2+1.4+3.3 & 9.0+1.3+3.1 \\ 11.9+10.3+1.4 & 11.2+10.4+1.6 & 11.0+10.3+1.1 \end{pmatrix} \right.$$

$$= \left. \begin{pmatrix} 7.0+1.2+8.4 & 7.4+1.2+8.6 & 7-2+1.3+8.1 \\ 0.0+2.2+1.4 & 0.4+2.2+7.6 & 0.-2+2.3+7.1 \\ 3.0+-1.2+1.4 & 3.4+-1.2+1.6 & 3.-2+-1.3+1.1 \end{pmatrix} \right]$$

$$= \left[\begin{pmatrix} 41 & 58 & 26 \\ 96 & 31 & 6 \\ 133 & 68 & 31 \end{pmatrix}, \begin{pmatrix} 34 & 78 & -3 \\ 8 & 46 & 13 \\ 6 & 16 & -8 \end{pmatrix} \right]$$

$$= \begin{pmatrix} [41,34) & [58,78) & [26,-3) \\ [96,8) & [31,46) & [6,13) \\ [133,6) & [68,16) & [31,-8) \end{pmatrix}.$$

From the working of the example one can easily understand how for any n $\times$ n interval matrix, we can program for the product. The time taken is the same as that of the usual matrix multiplication.

Now we give a method of finding the determinant of interval matrices provided they are square interval matrices.

We first illustrate this situation by some examples.

Example 2.23: Let

$$X = \begin{bmatrix} (0,3) & (7,8) \\ (4,-1) & (9,3) \end{bmatrix}$$

be an interval matrix with entries from $N_o(R)$. To find

$$\begin{aligned} |X| &= (0, 3)\,(9, 3) - (7, 8) \times (4, -1) \\ &= (0, 9) - (28, -8) \\ &= (-28, 11). \end{aligned}$$

So the value of the determinant is also a open interval.

Consider $X = \begin{bmatrix} (0,3) & (7,8) \\ (4,-1) & (9,3) \end{bmatrix}$ be the interval matrices.

$$\text{Now } |X| = \left| \begin{pmatrix} (0,3) & (7,8) \\ (4,-1) & (9,3) \end{pmatrix} \right|$$

$$= \left(\begin{vmatrix} 0 & 7 \\ 4 & 9 \end{vmatrix}, \begin{vmatrix} 3 & 8 \\ -1 & 3 \end{vmatrix} \right)$$

$$= (-28, 17).$$

The second method of finding the determinant can be easily programmed by using matrix interval instead of interval matrix.

We just indicate how to find the determinant of interval matrix.

Let M = $(m_{ij}) = \left(\left[m_{ij}^1, m_{ij}^2\right]\right)$ where $m_{ij} \in N_c(R)$; $1 \le i, j \le n$ be a interval matrix. To find the determinant of M.

$$|M| = |(m_{ij})| = \left|\left(\left[m_{ij}^1, m_{ij}^2\right]\right)\right|$$

$$= \left[\left|m_{ij}^1\right|, \left|m_{ij}^2\right|\right]$$

where $$\left|m_{ij}^1\right| = \left|\begin{pmatrix} m_{11}^1 & \dots & m_{1n}^1 \\ \vdots & & \vdots \\ m_{n1}^1 & \dots & m_{nn}^1 \end{pmatrix}\right|$$ and

$$\left|m_{ij}^2\right| = \left|\begin{pmatrix} m_{11}^2 & \dots & m_{1n}^2 \\ \vdots & & \vdots \\ m_{n1}^2 & \dots & m_{nn}^2 \end{pmatrix}\right|.$$

Since every interval matrix is a matrix interval we can find the determinant by just separating the intervals to get matrix intervals.

We see as in case of usual matrices we can in case of interval matrices also define or write the transpose.

We just illustrate them with examples.

Suppose X $= (a_1, \dots, a_9)$

$$= ([a_1^1, a_1^2], [a_2^1, a_2^2], \dots, [a_9^1, a_9^2])$$

be a interval 1×9 row matrix with $a_i \in N_c(R)$; $1 \leq i \leq 9$.

$$X^t = (a_1, \ldots, a_9)^t$$

$$= ([a_1^1, a_1^2], [a_2^1, a_2^2], \ldots, [a_9^1, a_9^2])^t$$

$$= \begin{bmatrix} [a_1^1, a_1^2] \\ [a_2^1, a_2^2] \\ \vdots \\ [a_9^1, a_9^2] \end{bmatrix}.$$

Thus X^t is the transpose of the interval matrix which is a interval column matrix.

$$\text{Also if } X = (a_1, a_2, \ldots, a_9)$$

$$= ([a_1^1, a_1^2], [a_2^1, a_2^2], \ldots, [a_9^1, a_9^2])^t$$

$$= [(a_1^1, a_2^1, \ldots, a_9^1), (a_1^2, a_2^2, \ldots, a_9^2)]^t$$

$$= [(a_1^1, a_2^1, \ldots, a_9^1)^t, (a_1^2, a_2^2, \ldots, a_9^2)^t]$$

$$= \left[\begin{pmatrix} a_1^1 \\ a_2^1 \\ \vdots \\ a_9^1 \end{pmatrix}, \begin{pmatrix} a_1^2 \\ a_2^2 \\ \vdots \\ a_9^2 \end{pmatrix} \right]$$

$$= \begin{pmatrix} [a_1^1, a_1^2] \\ [a_2^1, a_2^2] \\ \vdots \\ [a_9^1, a_9^2] \end{pmatrix}.$$

Thus the transpose of a row matrix interval is an interval row matrix and we see if X = $(a_1, \ldots, a_n)$ is a interval row matrix then $(X^t)^t = X$.

Now we see if $$Y = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$= \begin{bmatrix} [b_1^1, b_1^2) \\ [b_2^1, b_2^2) \\ \vdots \\ [b_m^1, b_m^2) \end{bmatrix}$$

$$= \left[\begin{bmatrix} b_1^1 \\ b_2^1 \\ \vdots \\ b_m^1 \end{bmatrix}, \begin{pmatrix} b_1^2 \\ b_2^2 \\ \vdots \\ b_m^2 \end{pmatrix} \right]$$

$$\text{Now } Y^t = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}^t = \begin{bmatrix} [b_1^1, b_1^2] \\ [b_2^1, b_2^2] \\ \vdots \\ [b_m^1, b_m^2] \end{bmatrix}^t$$

$$= ([b_1^1, b_1^2], [b_2^1, b_2^2], \ldots, [b_m^1, b_m^2])$$

$$= \left[\begin{pmatrix} b_1^1 \\ b_2^1 \\ \vdots \\ b_m^1 \end{pmatrix}, \begin{pmatrix} b_1^2 \\ b_2^2 \\ \vdots \\ b_m^2 \end{pmatrix}\right]^t$$

$$= [(b_1^1, b_2^1, \ldots, b_m^1), (b_2^1, b_2^2, \ldots, b_m^2))$$

$$= ([b_1^1, b_2^1, \ldots, b_m^1], (b_2^1, b_2^2, \ldots, b_m^2)) .$$

Thus if Y is a interval column matrix we see Y^t is an interval row matrix and $(Y^t)^t = Y$.

Let

$X = (a_{ij})_{m\times n}$, $(m \neq n)$; $a_{ij} = [a_{ij}^1, a_{ij}^2]$, $1 \leq i \leq m$, $1 \leq j \leq n$;

$a_{ij} \in N_o(R)$. (a_{ij} can be in $N_c(R)$ or $N_{oc}(R)$ or $N_{co}(R)$ or R replaced by Z or Q).

$$\text{Thus } X = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}$$

$$= \begin{pmatrix} (a_{11}^1, a_{11}^2) & (a_{12}^1, a_{12}^2) & \ldots & (a_{1n}^1, a_{1n}^2) \\ (a_{21}^1, a_{21}^2) & (a_{22}^1, a_{22}^2) & \ldots & (a_{2n}^1, a_{2n}^2) \\ \vdots & \vdots & & \vdots \\ (a_{m1}^1, a_{m1}^2) & (a_{m2}^1, a_{m2}^2) & \ldots & (a_{mn}^1, a_{mn}^2) \end{pmatrix}$$

be a $m \times n$ interval matrix.

$$\text{Now } X^t = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}^t$$

$$= \begin{pmatrix} (a_{11}^1, a_{11}^2) & (a_{12}^1, a_{12}^2) & \dots & (a_{1n}^1, a_{1n}^2) \\ (a_{21}^1, a_{21}^2) & (a_{22}^1, a_{22}^2) & \dots & (a_{2n}^1, a_{2n}^2) \\ \vdots & \vdots & & \vdots \\ (a_{m1}^1, a_{m1}^2) & (a_{m2}^1, a_{m2}^2) & \dots & (a_{mn}^1, a_{mn}^2) \end{pmatrix}^t$$

$$= \begin{pmatrix} (a_{11}^1, a_{11}^2) & (a_{21}^1, a_{21}^2) & \dots & (a_{m1}^1, a_{m1}^2) \\ (a_{12}^1, a_{12}^2) & (a_{22}^1, a_{22}^2) & \dots & (a_{m2}^1, a_{m2}^2) \\ \vdots & \vdots & & \vdots \\ (a_{1n}^1, a_{1n}^2) & (a_{2n}^1, a_{2n}^2) & \dots & (a_{mn}^1, a_{mn}^2) \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{pmatrix}$$

be a n × m interval matrix. We see if X is a m × n interval matrix then $(X^t)^t = X$.

Finally we find the transpose of an interval square matrix and it is easily seen that only the collection of interval square matrices M with entries from $N_o(R)$ ($N_c(R)$ or $N_{oc}(R)$ or $N_{co}(R)$) is such that the transpose of an interval square matrix is again in M; that is if $X \in M$ then $X^t \in M$.

$$\text{Take } X = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}$$

be an interval $n \times n$ matrix with $a_{ij} \in N_{oc}(R)$; $1 \leq i, j \leq n$.

$$\text{That is } X = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} (a_{11}^1, a_{11}^2] & (a_{12}^1, a_{12}^2] & \ldots & (a_{1n}^1, a_{1n}^2] \\ (a_{21}^1, a_{21}^2] & (a_{22}^1, a_{22}^2] & \ldots & (a_{2n}^1, a_{2n}^2] \\ \vdots & \vdots & & \vdots \\ (a_{n1}^1, a_{n1}^2] & (a_{n2}^1, a_{n2}^2] & \ldots & (a_{nn}^1, a_{nn}^2] \end{pmatrix}$$

$$= \left(\begin{pmatrix} a_{11}^1 & a_{12}^1 & \ldots & a_{1n}^1 \\ a_{21}^1 & a_{22}^1 & \ldots & a_{2n}^1 \\ \vdots & \vdots & & \vdots \\ a_{n1}^1 & a_{n2}^1 & \ldots & a_{nn}^1 \end{pmatrix}, \begin{bmatrix} a_{11}^2 & a_{12}^2 & \ldots & a_{1n}^2 \\ a_{21}^2 & a_{22}^2 & \ldots & a_{2n}^2 \\ \vdots & \vdots & & \vdots \\ a_{n1}^2 & a_{n2}^2 & \ldots & a_{nn}^2 \end{bmatrix} \right)$$

$$\text{Now } X^t = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}^t$$

$$= \begin{pmatrix} a_{11} & a_{21} & \ldots & a_{n1} \\ a_{12} & a_{22} & \ldots & a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \ldots & a_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} (a_{11}^1, a_{11}^2] & (a_{21}^1, a_{21}^2] & \ldots & (a_{n1}^1, a_{n1}^2] \\ (a_{12}^1, a_{12}^2] & (a_{22}^1, a_{22}^2] & \ldots & (a_{n2}^1, a_{n2}^2] \\ \vdots & \vdots & & \vdots \\ (a_{1n}^1, a_{1n}^2] & (a_{2n}^1, a_{2n}^2] & \ldots & (a_{nn}^1, a_{nn}^2] \end{pmatrix}.$$

Thus X^t is also an interval $n \times n$ matrix and $(X^t)^t = X$.

Now we can find eigen values and eigen vectors using these matrices. Clearly to get eigen values in case of interval matrices we need to find interval polynomials or polynomial in the variable x with interval coefficients from $N_c(R)$ or $N_o(R)$ or $N_{oc}(R)$ or $N_{co}(R)$ (or R replaced by Z or Q 'or' used only in the mutually exclusive sense). In the following chapter we introduce interval polynomials.

Chapter Three

POLYNOMIAL INTERVALS (INTERVAL POLYNOMIALS)

Interval polynomials are nothing but $p(x) = \sum a_i x^i$ where $a_i \in N_c(R)$ (or open of half-open closed or half closed open). Here we concentrate only on closed intervals from the natural class of intervals as all properties hold good in case of all type of intervals except in other cases when in applications the solutions takes the value as end points.

Since in this book we are not talking about mathematical models we do not bother about this problem also.

Now

$p(x) = [0, 3]x^7 + [5, 2]x^3 - [7, -3]x^2 + [0, -6]x^5 + [3, 3]x + [2, 9]$ is an interval polynomial we will define the notion of polynomial intervals and show how interval polynomial can be made into polynomial interval and vice versa, so we work both as polynomial interval or interval polynomials.

DEFINITION 3.1: *Let $q(x), p(x) \in R[x]$ (or $Z[x]$ or $Q[x]$ or $C[x]$ or $Z_n[x]$) be any two polynomials. We define $[p(x), q(x)]$*

as the polynomial interval. No restriction on the degree of p(x) or q(x) or on the coefficients of p(x) or q(x) is needed.

We will first illustrate this by some examples.

Example 3.1: Let

$$P(x) = [8x^3 - 4x^2 + 2x - 7,\ 9x^9 - 3x^6 + 4x^5 - 6x^3 + 2x^2 - 7x + 18]$$

be the polynomial interval.

Example 3.2: Let

$$P(x) = [3x^7 - 4x + 2,\ 4x^3 - 5x^2 + 3x - 1] = [a(x), b(x)]$$

be the polynomial interval.

Example 3.3: Let

$$p(x) = [8,\ 7x^2 - 4x + 3]$$

be the polynomial interval.

Example 3.4: Let

$$q\,(x) = [8x^8 - 7x + 3,\ -9]$$

be the polynomial interval.

We have seen polynomial intervals. Now we want to work about structures on these collection of all polynomial intervals; to this end we define the following.

DEFINITION 3.2: *Let*

$$V_R = \{[p(x), q(x)] = P(x) \,/\, p(x), q(x) \in R[x]\}$$

be the collection of all polynomial intervals with coefficients from R or polynomials from R[x]. V_R is a ring of real polynomial intervals.

We see for any two polynomial intervals P(x), S(x) in V_R, we can define addition as follows. Suppose P(x) = [p(x), q(x)] and S(x) = [a(x), b(x)] then define P(x) + S(x) = [p(x), q(x)] + [a(x), b(x)] = [p(x) + a(x), q(x) + b(x)]; since p(x) + a(x) and q(x) + b(x) are in R[x], we see P(x) + S(x) is in V_R. Thus V_R is closed under polynomial interval addition.

Consider $0 = [0, 0] \in V_R$. We call this the zero polynomial interval and $0 = 0x + 0x + \ldots + 0x^m$, $m \in Z^+ \cup \{0\}$.

$$\begin{aligned} \text{We see } P(x) + [0, 0] &= [\,p(x), q(x)] + [0,0] \\ &= [p(x) + 0, q(x) + 0] \\ &= [p(x), q(x)] = P(x). \end{aligned}$$

Thus [0, 0] acts as the additive identity for polynomial interval addition. Further we see for $S(x), P(x) \in V_R$; $P(x) + S(x) = S(x) + P(x)$. Thus the operation of addition on polynomial intervals is commutative.

Also it can be easily verified for $P(x), S(x), T(x) \in V_R$. We have $(P(x) + S(x)) + T(x) = P(x) + (S(x) + T(x))$. That is the operation of polynomial interval addition on V_R is both commutative and associative with $0 = [0, 0]$ as its additive identity.

Thus we can easily prove the following theorem.

THEOREM 3.1: *V_R is an additive abelian group of infinite order.*

For every $P(x) = [p(x), q(x)]$ we have $P(x) = [-p(x), -q(x)]$ is such that $P(x) + (-P(x)) = [p(x), q(x)] + [-p(x), -q(x)] = [p(x) + (-p(x)), q(x) + (-q(x))] = [0, 0]$. Thus for every P(x) in V_R. $-P(x)$ is the additive inverse of P(x).

Now we proceed onto just give simple illustration before we proceed onto define multiplication on V_R. Consider

$P(x) = [p(x), q(x)] = [8x^7 - 3x^2 + 2x - 7, -5x^8 + 15x^7 - 10x^3 + 11x - 1]$ in V_R. $-P(x) = [-p(x), -q(x)] = [-8x^7 + 3x^2 - 2x + 7, 5x^8 - 15x^7 + 10x^3 - 11x + 1] \in V_R$.

We see $P(x) + (-P(x)) = [0,0]$. Thus $-P(x)$ is the inverse of P(x) in V_R with respect to polynomial interval addition.

Now consider P(x), S(x) in V_R; where P(x) = [p(x), q(x)] and S(x) = [s(x), r(x)]. P(x) × S(x) = P(x) . S(x) = [p(x), q(x)] . [s(x), r(x)]

= [p(x) . s(x), q(x) . r(x)] is in V_R as p(x) . s(x) ∈ R[x] and q(x) . r(x) ∈ R [x]. Thus on V_R we have defined a product × or '.'. Further the product of polynomial intervals is commutative as product of polynomials in R[x] is commutative.

We just illustrate by a very simple example.

Consider P(x) = [p(x), q(x)] = $[-2x^3 + x^2 + 1, 5x^3 - 1]$ and S[x] = [a(x), b(x)] = $[x^7 + 1, 2x^2 - 3x +1]$ to be two polynomial intervals in V_R. Now P(x) × S(x) = [p(x), q(x)] × [a(x), b(x)] = [p(x) a(x), q(x) b(x)].

$= [(-2x^3 + x^2 + 1) (x^7 + 1), 5x^3 - 1 \times (2x^2 - 3x + 1)]$

$= [-2x^{10} + x^9 + x^7 - 2x^3 + x^2 + 1, 10x^5 - 2x^2 - 15x^4 + 3x + 5x^3 - 1]$

$= [-2x^{10} + x^9 + x^7 - 2x^3 + x^2 + 1, 10x^5 - 15x^4 + 5x^3 - 2x^2 + 3x - 1]$ is again a polynomial interval in V_R. Further we see as p(x) a(x) = a(x) p(x) and q(x) b(x) = b(x) q(x),

P(x). S(x) = S(x). P(x).

Thus '.' on V_R is a closed commutative operation. It is left for the reader to prove or verify, '.' on V_R is an associative operation. We see V_R is only a commutative semigroup for if [q(x), 0] = S(x) and P(x) = [0, a(x)] are in V_R then P(x). S(x) = S(x). P(x) = [0, a(x)] [q(x), 0] = [0, 0] = [q(x), 0] [0, a(x)].

Since V_R has zero divisors V_R is only a commutative semigroup under polynomial interval multiplication. However the constant polynomial interval [1, 1] acts as the multiplicative identity. For if we take

$P(x) = [a(x), b(x)]$ and $1 = [1, 1] \in V_R$ then $P(x) . 1 = [a(x), b(x)] [1, 1] = [a(x).1, b(x). 1] = [1, 1] [a(x), b(x)] = [1 . a(x), 1. b(x)] = [a(x), b(x)]$. Thus is true for every $P(X)$ in V_R. We call polynomial intervals of the form $[a, b]$ where both a and b are reals as constant polynomial intervals. Thus we see V_R contains the natural class of real intervals $[a, b]$ that is $N_c (R) \subseteq V_R$.

Also the reader can easily verify that the operation '+' and '.' on V_R is distributive; for if $P(x) = [a(x), b(x)]$, $A(x) = [s(x), r(x)]$ and $B(x) = [c(x), d(x)]$ in V_R $(a(x), b(x), s(x), r(x), c(x), d(x)) \in R[x]$) then we have;

$$P(x) \times (A(x) + B(x)) = P(x) \times A(x) + P(x) \times B(x).$$

That is $[a(x), b(x)] ([s(x), r(x)] + [c(x), d(x)])$

$$= [a(x), b(x)] ([s(x) + c(x), r(x) + d(x)])$$

$$= [a(x) (s(x) + c(x)), b(x) (r(x) + d(x))]$$

(Since the operation are distributive in $R[x]$ we have)

$$= [a(x) s(x) + a(x) c(x), b(x) r(x) + b(x) d(x)] \quad \ldots \text{I}$$

Consider $P(x) A(x) + P(x) \times B(x)$

$$= [a(x), b(x)] [s(x), r(x)] + [a(x), b(x)] \times [c(x), d(x)]$$

$$= [a(x) s(x) + a(x) c(x), b(x) r(x) + b(x) d(x)] \quad \ldots \text{II}$$

I and II are identical hence in V_R the operation $\times$ distributes over addition. Thus V_R is a commutative ring with unit and has zero divisors. Only constant polynomial intervals of the form $[a, b]$ ($a \neq 0$, $b \neq 0$ in R) are invertible or have inverse with respect to multiplication of polynomial intervals.

We will now study the properties enjoyed by the real polynomial interval ring V_R.

THEOREM 3.2: *V_R has ideal.*

Proof: Consider X = {[p(x), 0] | p(x) ∈ R[x]} ⊆ V_R. X is an ideal.

Further one can easily verify all polynomial intervals, P(x) = [p(x), q(x)] can generate ideals in V_R.

Consider I = ⟨ [x^2 + 1, x^3 + 1]⟩ = {[p(x), q(x)] | deg p(x) ≥ 2 and deg q(x) ≥ 3} ⊆ V_R. I is an ideal. We can as in case of usual rings find the quotient ring in case of ring of interval polynomial.

Find $\frac{V_R}{I} = \frac{V_R}{\langle [x^2+1, x^3+1] \rangle}$ = {[ax + b, cx^2 + dx + e] + I | a, b, c, d, e ∈ R}, the quotient ring. Clearly $\frac{V_R}{I}$ is the quotient ring of interval polynomials.

This quotient ring of polynomial intervals has zero divisors.

Thus we have infinite number of ideals in V_R. Now we define V_Q = {[p(x), q(x)] = P(x) | p(x), q(x) ∈ Q[x]} to be the rational ring of polynomial intervals. V_Q is a ring, V_Q has also infinite number of ideals in it.

V_c = {P(x) = [p(x), q(x)] | p(x), q(x) ∈ C[x]} is a complex ring of polynomial intervals.

V_c also has infinite number of ideals. Infact we can say Vc is the algebraically closed ring. This notion will be explained in the later part of this chapter.

V_{Z_n} = {P(x) = [p(x), q(x)] | p(x), q(x) ∈ Z_n[x]}

is the modulo integer ring of polynomial of intervals and

$V_z = \{P(x), q(x)] \mid p(x), q(x) \in Z[x]\}$

is the integer ring of polynomial intervals.

We see $V_Z \subseteq V_Q \subseteq V_R \subseteq V_C$ this containment is a proper containment and infact V_Z is a subring of polynomial intervals, V_Q and V_Z are subrings of polynomial intervals of V_R and finally V_Z, V_R, V_Q are subrings of polynomial intervals of V_C. We see clearly these subrings are not ideals so we can say polynomial interval rings have subrings of polynomial intervals which are not ideals. All the polynomial interval rings V_R, V_Q, V_Z and V_C are of infinite order commutative with unit and has zero divisors no idempotents or nilpotents in it.

However if $V_{Z_n} = \{[p(x), q(x)] = P(x) \mid p(x), q(x) \in Z_n[x]\}$ then V_{Z_n} has zero divisors, units idempotents and nilpotents if n is a composite number. V_{Z_n} also has ideals and subrings. V_{Z_n} is the ring of modulo integer polynomial intervals and is of infinite order.

We will give one or two examples.

Consider $V_{Z_{12}} = \{P(x) = [p(x), q(x)] \mid p(x), q(x) \in Z_{12}[x]\}$ is the modulo integer polynomial interval ring of infinite order. Clearly $S = N_c(Z_{12}) = \{[a, b] \mid a, b \in Z_{12}\} \subseteq V_{Z_{12}}$. $S = N_c(Z_{12})$ is only a subring of modulo integer polynomial intervals and is not an ideal. Consider $P = \{p(x) = [p(x), q(x)] \mid p(x), q(x) \in S[x]$ where $S = \{0, 2, 4, 6, 8, 10\} \subseteq Z_{12}\}$; that is all polynomial intervals with coefficients of the polynomials in the polynomial interval is from $S = \{0, 2, 4, \ldots, 10\} \subseteq Z_{12}$. Clearly P is an ideal. Thus $V_{Z_{12}}$ has subrings as well as ideals.

Take $T(x) = [6x + 6x^2 + 4, 6x^3 + 2x + 4]$ and

$S(x) = [6x^3, 6x^2 + 6]$ in $V_{Z_{12}}$. We see $T(x) . S(x) = 0$. Also $[6, 6] \in V_{Z_{12}}$ is a nilpotent element of $V_{Z_{12}}$.

Consider $Z_{11}[x]$, the set of all polynomials in the variable x with coefficients from Z_{11} with the special condition $x^5 = 1$, $x^6 = x$, $x^7 = x^2$ and so on.

Now

$V_{Z_{11}}$ = {[p(x), q(x)] = P(x) | p(x), q(x) in $Z_{11}[x]$ where degp(x) $\leq$ 5 and deg q(x) $\leq$ 5}. $V_{Z_{11}}$ is a modulo integer polynomial interval ring with unit, commutative and of finite order. $V_{Z_{11}}$ also have zero divisors and units but has no idempotents or nilpotents. All polynomial interval rings are commutative rings with unit and zero divisors.

THEOREM 3.3: *Every polynomial interval ring V_Z or V_Q or V_R or V_C or V_{Z_n} contains $N_c(Z)$ or $N_c(Q)$ or $N_c(R)$ or $N_c(C)$ or $N_c(Z_n)$ respectively as a proper subset which is a subring.*

We call this subring as the inherited subring of interval or subring of natural class of intervals.

We say the polynomial interval P(x) = [g(x), h(x)] $\neq$ [0, 0]; divides the polynomial interval R(x) = [p(x), q(x)] if g(x) | p(x) and h(x) | q(x) that is P(x) | R(x). For instance take $[x^2 - 1, 3x^3 - 1]$ = P(x) and

R(x) = $[x^5 + x^3 + x^2 - 2x - 1, 3x^7 - 6x^6 + 2x^4 + 11x^3 - x - 3]$. Clearly P(x) | R(x) and

$$\frac{R(x)}{P(x)} = \frac{[x^5 + x^3 + x^2 - 2x - 1, 3x^7 - 6x^6 + 2x^4 + 11x^3 - x - 3]}{[x^2 - 1, 3x^3 - 1]}$$

$$= \left[\frac{x^5 + x^3 + x^2 - 2x - 1}{x^2 - 1}, \frac{3x^7 - 6x^6 + 2x^4 + 11x^3 - x - 3}{3x^3 - 1}\right]$$

$= [x^3 + 2x + 1, x^4 - 2x^3 + x + 3]$.

We see if $P(x) = [p(x), q(x)] \neq [0, 0]$ and $S(x) = [a(x), b(x)] \in V_R$, we say $P(x) / S(x)$ if $p(x) \nmid a(x)$ or $q(x) \nmid b(x)$ or not used in the mutually exclusive sense.

Consider $P(x) = [x^7 + 1, x^3 + 2x + 7]$ and $S(x) = [x^8 + 1, x^{24} - 1]$ in V_R. We see $P(x) \nmid S(x)$ as $x^7 + 1 \nmid x^8 +1$ and $x^3 + 2x + 7 \nmid x^{24} - 1$.

We see all polynomial intervals $P(x) = [p(x), q(x)]$, ($p(x) = 0$ or $q(x) = 0$) then $P(x)$ does not divide any $S(x) \in V_R$.

Clearly V_R or V_Q or V_Z or V_C are not Euclidean rings as they are not integral domains and they contain zero divisors. However it is pertinent to mention here that V_R or V_Q or V_C or V_Z contains subrings of polynomial intervals which are Euclidean rings.

For instance

$I = \{[q(x), 0] = P(x) / q(x) \in R(x)$ or $Z(x)$ or $Q[x]\} \subseteq V_R$

(or V_Z or V_Q) is a Euclidean subring of polynomial intervals.

$T = \{[q(x), 0] = P(x) / q(x) \in R[x]$ or $Q[x]\}$ is a Euclidean ring as well as principal ideal domain. We say a polynomial interval $P(x) = [p(x), q(x)]$ is primitive if for both $p(x) = p_0 + p_1x + \ldots + p_n x^n$ and $q(x) = q_0 + q_1 x + \ldots + q_m x^m$ in $Z[x]$, we have the greatest common divisor of $p_0, p_1 \ldots p_n$ is 1 and that of the greatest common divisor of $q_0, q_1, \ldots, q_m$ is 1; then we say the polynomial interval is primitive.

If $P(x) = [p(x), q(x)]$ and $S(x) = [a(x), b(x)]$ are two primitive polynomial intervals then

$P(x)\ S(x) = [p(x)\ a(x),\ q(x)\ b(x)]$ is again a primitive polynomial interval.

The content of the polynomial interval
$P(x) = [f(x), g(x)] = [a_0 + a_1x + \ldots + a_nx^n, b_0 + b_1x + \ldots + b_mx^m]$ where a_i's and b_j's are integers $1 \leq i \leq n$, $1 \leq j \leq m$; then the greatest common divisor of the integers $[\{a_0, a_1, \ldots, a_n\}\ \{b_0, b_1, \ldots, b_m\}]$ is an interval in $N_c(Z)$.

We say a polynomial interval $P(x) = [p(x), q(x)]$ is monic if both $p(x)$ and $q(x)$ are monic that is if all coefficients of $p(x)$ and $q(x)$ are integers and the highest coefficient of each of $p(x)$ and $q(x)$ is 1.

Consider $P(x) = [p\ (x), q\ (x)] = [x^9 - 20x^8 + 11x^3 - 12x^2 + x - 45, x^{25} + 14x^{20} - x^{19} + 17x^{10} - x^3 + x^2 - 1]$; $P(x)$ is a monic polynomial interval.

It is easily verified that if $P(x)$ and $S(x)$ are monic polynomial intervals then so is $P(x)\ S(x)$ their product, further if $p(x)$ is a monic polynomial interval then so are the factors.

We say a polynomial interval $P(x) = [p(x), q(x)]$ is reducible if $P(x) = S(x)\ T(x)$ where $S(x) = [a(x), b(x)]$ and $T(x) = [d(x), c(x)] = [a(x)\ d(x), b(x)\ c(x)]$ where deg $(a(x))$ and deg $(d(x))$ are strictly less than deg $p(x)$ and deg $(a(x))$ and deg $d(x)$ are strictly greater than 1 that is $a(x)$ and $d(x)$ are not constant polynomials.

Similar condition for $q(x) = b(x)\ c(x)$ holds good. If $P(x)$ is not reducible we say $P(x)$ is irreducible.

The irreducibility depends on the ring over which the polynomials are defined.

Consider $P(x) = [p(x), q(x)] = [x^2 + 1, 5x^2 + 7] \in V_R$. Clearly $P(x)$ is irreducible over $N_c(R)$ but reducible over $N_c(C)$.

Now we extend the notion of Eisenstein Criterion.

Theorem (Eisenstein Criterion for polynomial intervals) Let $P(x) = [f(x), g(x)] = [a_0 + a_1x + \ldots + a_nx^n, b_0 + b_1x + \ldots + b_mx^m]$ be a polynomial interval with integer coefficients.

Suppose for some prime numbers p_1, p_2 we have $p_1 \not| a_n$, p_1/a_1, p_1/a_2, …, p_1/a_0 and $p_1^2 \not| a_0$ and $p_2 \not| b_n$, p_2/b_1, p_2/b_2, …, p_2/b_0 and $p_2^2 \not| b_0$. Then P (x) is irreducible over the rationals. The proof is direct as in case of usual polynomials [2, 3]. Here the concept of unique factorization domain or integral domain cannot be extended as the polynomial intervals have zero divisors.

We can as in case of polynomials solve the equations in interval polynomials.

For if $P(x) = [p(x), q(x)]$,

we say $[\alpha, \beta]$ is a root of P(x) if $P([\alpha, \beta]) = [p(\alpha), q(\beta)] = [0, 0]$.

Thus if $P(x) = [p(x), q(x)] = [x^2 - 5x + 6, x^3 - 7x - 6]$ then this polynomial interval has the following interval roots.

$[3, -1]$, $[3, 3]$ $[3, -2]$, $[2, -1]$, $[2, 3]$ and $[2, -2]$ are the interval roots of the polynomial interval.

We see $P([3, -1]) = (0, 0)$ and so on.

We can define $V_{Q(\sqrt{2})}$, $V_{Q(\sqrt{7},\sqrt{3})}$, $V_{Q(\sqrt{11},\sqrt{2},\sqrt{5})}$ and so on where these will be called as extended polynomial intervals or extended polynomial interval rings.

So we can say for any $P(x) \in V_Q$ or $V_Q[x]$, (a, b) lying in $N_c(Q(a, b))$ is an interval root if $P([a, b]) = [0, 0]$.

Clearly $Q \subseteq Q(\sqrt{2}) \subseteq Q(\sqrt{11},\sqrt{2},\sqrt{5})$ and $Q \subseteq Q(\sqrt{7},\sqrt{3})$.

We know if $P(x) \in V_F$ or $V_F(x)$ (F a field) then for any interval [a, b] in V_K (K an extension field of F)

$P(x) = [(x–a)\, p_1(x) + p(a), (x–a)\, q_1(x) + q(b)]$ where $[p_1(x), q_1(x)] \in V_K$ and $P(x) = [p(x), q(x)] \in V_F$ and here degree of $p_1(x) = \deg p(x) – 1$ and $\deg q_1(x) = \deg q(x) – 1$.

The proof is direct using the Remainder theorem.

Further we have if K is an extension field of F then $V_F \subseteq V_K$.

For if $[a, b] \in K$ an interval root of $P(x) = [p(x), q(x)] \in V_F$ then in V_K we have

$\{[(x–a), (x–b)] \mid P(x) = [p(x), q(x)]$ that is $[(x–a)/p(x), (x–b)/q(x)]\}$.

We as in case of usual polynomial speak of an interval of multiplicity [m ,n] is the multiplicity of an interval root [a, b] of $P(x) = [p(x), q(x)]$ in V_F for $[a, b] \in N_c(K) \subseteq V_K$, K an extension field of F,

$$\text{if } [(x–a)^m, (x–b)^n] \,/\, P(x)$$

$$\text{that is } \left[{}^{(x-a)^m}\!\big/_{p(x)}, {}^{(x-b)^n}\!\big/_{q(x)}\right]$$

$$\text{where as } [(x–a)^{m+1}, (x–b)^{n+1}] \not/\; P(x)$$

$$\text{that is } [(x–a)^{m+1} \not/\; p(x),\; (x–b)^{n+1} \not/\; q(x)].$$

A polynomial interval $P(x) = [p(x), q(x)] \in V_F$ of interval degree [n, m] over $N_c(F)$ has atmost mn interval roots if any extension interval ring $N_c(K)$ where K is the extension field of F.

This proof is also direct and hence is left as an exercise to the reader.

Further if P(x) = [p(x), q(x)] is a polynomial interval in V_F of interval degree [m, n] ≥ [1, 1] and P(x) is irreducible over $N_c(F)$ then there is an extension ring $N_c(K)$ of $N_c(F)$ (K an extension field of F) such that P(x) has a root in the extended interval. Interested reader can derive all the results for polynomial intervals with appropriate modifications.

Now as in case of usual polynomials we can in case of polynomial intervals also define the notion of derivative and all formal rules of differentiation are true as well.

Let

$P(x) = [f(x), g(x)] = [a_0 x^n + a_1 x^{n-1} + \ldots + a_n, b_0 x^m + b_1 x^{m-1} + \ldots + b_m]$ be a polynomial interval in V_R the derivative of the polynomial interval

$P'(x) = [n a_0 x^{n-1} + (n-1) a_1 x^{n-2} + \ldots + a_{n-1}, mb_0 x^{m-1} + (m-1) b_1 x^{m-2} + \ldots + b_{m-1}]$ in V_R.

For example if P(x) = [p(x), q(x)]

$= [3x^7 - 2x^5 + 2x - 1, 7x^6 + 4x^5 + 3x^2 - 7] \in V_R$;

$P'(x) = [p'(x), q'(x)] = [21x^6 - 10x_4 + 2, 42x_5 + 20x_4 + 6x] \in V_R$.

Now suppose we have P(x) = [p(x), q(x)] and S(x) = [r(x), s(x)] polynomial intervals in V_R then

$(P(x) + S(x))' = P'(x) + S'(x)$.

We will only illustrate this situation. Let P(x) = [p(x), q(x)] and $S(x) = [r(x), s(x)] \in V_R$

$$(P(x) + S(x)) = [p(x), q(x)] + [r(x), s(x)] = [p(x) + r(x), q(x) + s(x)].$$

$$\text{Consider } (P(x) + S(x))' = [p(x) + r(x), q(x) + s(x)]' = [(p(x) + r(x))', (q(x) + s(x))'] = [p'(x) + r'(x), q'(x) + s'(x)]$$

applying the derivative for usual polynomials.

We see if $\alpha \in R$ and $P(x) = [p(x), q(x)] \in V_R$ then we have

$$\alpha P(x) = \alpha [p(x), q(x)] = [\alpha p(x), \beta q(x)].$$

$$\text{If } [\alpha, \beta] \in N_c(R) \text{ then } [\alpha, \beta] P(x) = [\alpha, \beta] [p(x), q(x)] = [\alpha p(x), \alpha q(x)].$$

$$\text{Now } ([\alpha, \beta] [P(x)])' = [\alpha, \beta] P'(x) = [\alpha, \beta] [p(x), q(x)]' = [\alpha p'(x), \beta q'(x)].$$

Further $P(x) = [p(x), q(x)]$ and $S(x) = [r(x), s(x)]$ are in V_R then $(P(x) S(x))' = (([p(x), q(x)] ([r(x), s(x)))'$

$$= [p(x) r(x), q(x) s(x)]'$$

$$= [p'(x) r(x) + p(x) r'(x), q'(x) s(x) + q(x) s'(x)].$$

Thus we can say the polynomial interval $P(x) = [p(x), q(x)] \in V_R$ has multiple interval roots if and only if $P(x)$ and $P'(x)$ have a non trivial common factor which is a polynomial interval.

Also the interested reader can prove. If F is a field of characteristic $p \neq 0$ then the polynomial interval $[x^{p^n} - x, x^{p^m} - x]$ for $n \geq 1$ and $m \geq 1$ has distinct interval roots.

Now having seen some of the properties enjoyed by the polynomial intervals we proceed onto show how every interval polynomial is a polynomial interval and vice versa.

Consider an interval polynomial

$f(x) = [6, 9]x^8 + [3, –2]x^6 + [–3, 1]x^5 + [0, 4]x^4 + 7x^2 + [2, –5]x + [9, 3]$. We see the coefficients of f(x) are from $N_c(Z)$. We can write f(x) as $[6x^8 + 3x^6 – 3x^5 + 7x^2 + 2x + 9, 9x^8 – 2x^6 + x^5 + 4x^4 + 7x^2 – 5x + 3] = [p(x), q(x)]$; thus f(x) is now the polynomial interval.

On similar lines suppose $P(x) = [p(x), q(x)] = [8x^7 – 5x^5 + 2x^4 – 3x^2 + x + 1, 6x^5 – 7x^4 + 3x^3 + 4x^2 – 8x – 9]$ be a polynomial interval; we can write

$P(x) = [8, 8]x^8 + [–5, 6]x^5 + [2, –7]x^4 + [0, 3]x^3 + [–3, 4]x^2 = [1, –8]x + [1, 9]$ which is the interval polynomial.

Thus our claim, that every polynomial interval can be made into an interval polynomial and vice versa is valid.

Now we will study the algebraic structures enjoyed by these polynomial intervals. We know

$V_R = \{[p(x), q(x)] \mid p(x), q(x) \in R[x]\}$ is an additive abelian group.

$V_R^n = \{[p(x), q(x)] \mid p(x), q(x) \in R^n [x]\}$; all polynomials of degree less than or equal to n. Similarly $V_Q^m, V_Z^t, V_{Z_n}^p$ and V_C^s can be defined appropriately.

DEFINITION 3.3: *Let*

$$V_R = \{p(x) = [p(x), q(x)] \mid p(x), q(x) \in R[x]\}$$

be an abelian group of polynomial intervals with respect to addition. V_R is a vector space of polynomial intervals over the *field R (or Q) or polynomial interval vector space over R (or Q).*

We can have subspace of polynomial intervals over R.

Example 3.5: Let

$$V_R = \{P(x) = [p(x), q(x)] / p(x), q(x) \in R[x]\}$$

be the vector space of polynomial intervals over R.

Consider

P = $\{P(x) = [p(x), q(x)] \mid p(x)$ and $q(x)$ are all polynomials of degree less than or equal to five with coefficients from R$\} \subseteq V_R$. P is an abelian group under addition. Further P is a vector space of polynomial intervals over R. Thus P is a subspace of polynomial intervals of V_R over R.

Example 3.6: Let

$$V_R = \{P(x) = [p(x), q(x)] \mid p(x), q(x) \in R[x]\}$$

be a vector space of polynomial intervals over the field Q.

Consider M = $\{P(x) = [p(x), q(x)] \mid p(x), q(x) \in Q[x]\}$ be a subvector space of polynomial intervals of V_R over Q.

Infact V_R has infinite number of vector subspaces of polynomial intervals.

Example 3.7: Let

$$V_{Z_7} = \{P(x) = \{[p(x), q(x)] \mid p(x), q(x) \in Z_7\}$$

be a vector space of polynomial intervals over the field Z_7.

Consider P = {S(x) = [p(x), q(x)] | p(x), q(x) $\in Z_7$ [x]; p(x) and q(x) are of degree less than or equal to 10} $\subseteq V_{Z_7}$; P is a vector subspace of polynomial intervals of finite order.

Infact V_{Z_7} has several vector subspace of polynomial intervals.

Example 3.8: Let

$$V_{Z_2} = \{P(x) = \{[p(x), q(x)] \mid p(x), q(x) \in Z_2 [x]\}$$

be the vector space of polynomial intervals over the field Z_2. Consider

M = {P(x) = [p(x), q(x)] | p(x), q(x) are polynomials of Z_2[x] of degree less than or equal to 7} $\subseteq V_{Z_2}$; M is a subvector space of polynomials intervals of V_{Z_2} over the field Z_2.

We can define for interval polynomials vector space as in case of usual vector space define the notion of linear transformation or linear operator only when those spaces are defined over the same field F. The definition is a matter of routine and we will illustrate this situation only by examples.

Example 3.9: Let

$$V_{Z_5} = \{P(x) = [p(x), q(x)]; p(x), q(x) \in Z_5[x] \text{ of degree less than or equal to } 5\}$$

be a vector space of polynomial intervals over Z_5.

$$W_{Z_5} = \{P(x) = [p(x), q(x)]; p(x), q(x) \in Z_5[x] \text{ of degree less than or equal to } 10\}$$

be a vector space of polynomial interval over Z_5.

Define T : $V_{Z_5} \to W_{Z_5}$ as follows:

$$T([a, b]) = [a, b] \text{ if a and b are in } Z_5.$$

$$\begin{aligned} T(P(x)) &= T([p(x), q(x)]) \\ &= [p(x), q(x)]. \end{aligned}$$

It is easily verified T is a linear transformation of V_{Z_5} into W_{Z_5}. We say if the range space is the same as that of the domain space we define T to be a linear operator on V_R or V_F, F any field.

We will just illustrate this by an example.

Example 3.10: Let

$$V_R = \{P(x) = [p(x), q(x)]; p(x), q(x) \in R[x]\}$$

be a vector space of polynomial intervals.

$$\begin{aligned} &\text{Define } T : V_R \to V_R \\ &\text{by } T(P(x)) = ([p(x), q(x)]) \\ &\qquad\qquad\; = [x\, p(x), x^2\, q(x)]. \end{aligned}$$

T is a linear operator on V_R. We can now give the basis of a polynomial interval vector space. Let

$V_R = \{P(x) \mid p(x) = [p(x), q(x)]$ where $p(x), q(x) \in R[x]\}$
be a polynomial interval vector space over R.

Take B = $\{[1, 0], [x,0], \ldots, [x^n, 0], \ldots, [0, 1], [0, x], \ldots, [0, x^n], \ldots] \subseteq V_R$, B is a basis of polynomial intervals.

Clearly V_R is an infinite dimensional vector space over R.

Example 3.11: Let

$$V_R^5 = \{P(x) = [p(x), q(x)] \text{ where } p(x), q(x) \in R[x];$$

all polynomials with coefficients from R of degree less than or equal to 5.

Consider

S = {[1, 0], [x, 0], [x^2, 0], [x^3, 0], [x^4, 0], [x^5, 0], [0, 1], [0, x], [0, x^2], [0, x^3], [0, x^4], [0, x^5]} $\in V_R^5$;

S is a basis of V_R^5 over R and dimension of V_R^5 over is finite given by 12.

However if R is replaced by Q clearly, V_R^5 is a vector space of interval polynomials of infinite dimension.

Example 3.12: Let

$$V_{Z_{13}}^7 = \{[p(x), q(x)] = P(x) \mid p(x), q(x) \in Z_{13}[x]\}$$

be the collection of all polynomials of degree less than or equal to 7.

Consider

B = {[1, 0], [x, 0], [x^2, 0], [x^3, 0], [x^4, 0], [x^5, 0], [x^6, 0], [x^7, 0], [0, 1], [0, x], [0, x^2], [0, x^3], [0, x^4], [0, x^5], [0, x^6], [0, x^7]} $\subseteq V_{Z_{13}}^7$ is an interval basis of $V_{Z_{13}}^7$. Clearly dimension of $V_{Z_{13}}^7$ is 16.

Thus we can have infinite or finite dimensional polynomial interval vector spaces.

Example 3.13: Consider

$$V_{Z_7} = \{P(x) = [p(x), q(x)] \mid p(x), q(x) \in Z_7(x)\},$$

a vector space of polynomial intervals over the field Z_7.

B = {[1, 0], [x, 0], [x^2, 0], …, [x^n, 0], …, [x^∞, 0], [0, 1], [0, x], …, [0, x^n], …, [0, x^∞]} $\subseteq V_{Z_7}$ is a basis of V_{Z_7}. Clearly V_{Z_7} is an infinite dimensional polynomial interval vector space over Z_7.

We have seen both infinite and finite dimensional polynomial interval vector spaces over a field F.

We can define polynomial interval linear algebras over the field F.

Let V_R = {P(x) = [p(x), q(x)] | p(x), q(x) $\in$ R[x]} be a polynomial interval vector space over the field R. We see V_R is a linear algebra over R as for any P(x) = [p(x), q(x)] and S(x) = [a(x), b(x)] we can define

P(x) S(x) = [p(x), q(x)], [a(x), b(x)] = [p(x), a(x), q(x), s(x)] to be in V_R.

Thus V_R is a polynomial interval linear algebra over R.

Consider

V_R^8 = {P(x) = [p(x), q(x)] | p(x) and q(x) are all polynomial of degree less than or equal to 8 with coefficients from R}.

V_R^8 is only a vector space of polynomial intervals and is not a linear algebra as P(x) S(x) is not in V_R^8 for every P(x) and S(x) in V_R^8.

Thus we see all polynomial interval vector spaces in general are not polynomial interval linear algebra, however every polynomial interval linear algebra is a polynomial interval vector space.

The later part is clear from examples.

When we have linear polynomial interval algebras or linear algebra of polynomial intervals; the dimension etc can be analysed.

Consider $V_R = \{[p(x), q(x)] \mid p(x), q(x) \in R[x]\}$ be the polynomial interval vector space over the field R. We see V_R is a linear algebra of polynomial interval vector space.

Take $B = \{[1, 0], [0, 1], [x, 0], [0, x]\} \subseteq V_R$. B is a basis of V_R as a linear algebra of polynomial intervals.

Thus V_R has dimension 4 as a linear algebra of polynomial intervals but of infinite dimension as a polynomial interval vector space over R.

Consider $V_Q = \{P(x) = [p(x), q(x)] \mid p(x), q(x) \in Q[x]\}$ be the linear algebra of polynomial intervals over the field Q.

Clearly the set $B = \{[1, 0], [0, 1], [x, 0], [0, x]\} \subseteq V_Q$ generates V_Q as a subset of V_Q and B is a linearly independent subset of V_Q.

Thus B is a basis of V_Q and dimension of V_Q over Q is four. Having seen basis of a linear algebra of polynomial intervals we can proceed onto define linear operators and linear transformation; these are simple and easy and hence left as an exercise for the reader.

We are more interested in studying the polynomial intervals than the algebraic structures on them.

We now proceed onto study the polynomial intervals when the coefficients of the polynomial are from $R^+ \cup \{0\}$, $Q^+ \cup \{0\}$ or $Z^+ \cup \{0\}$.

We shall denote these polynomial intervals by $V_{R^+ \cup \{0\}}$, $V_{Q^+ \cup \{0\}}$ and $V_{Z^+ \cup \{0\}}$.

Thus
$V_{R^+ \cup \{0\}} = \{[p(x), q(x)] = P(x) \mid p(x), q(x), \in (R^+ \cup \{0\}) (x)\}$.

For instance $p(x) = \sqrt{3}\, x^7 + 7x^6 + \sqrt{19}\ x^3 + \sqrt{241}$ and

$q(x) = x^{29} + \sqrt{43}\ x^{20} + 17\ x^4 + 101$ are in $(R^+ \cup \{0\}) (x)$ and $P(x) = [p(x), q(x)] \in V_{R^+ \cup \{0\}}$.

We see $V_{R^+ \cup \{0\}}$ is only a semigroup with respect to addition. Infact '0', the zero polynomial serves as the additive identity. Also $V_{Q^+ \cup \{0\}}$ and $V_{Z^+ \cup \{0\}}$ are also only semigroups of polynomial intervals.

We see
$V_{R^+ \cup \{0\}} = \{P(x) = [p(x), q(x)] \mid p(x), q(x) \in (R^+ \cup \{0\}) (x)\}$
is a semiring of interval polynomials.

Infact $V_{R^+ \cup \{0\}}$ is not a semifield as it has zero divisors. However $R^+ \cup \{0\}$ is a semifield contained in $V_{R^+ \cup \{0\}}$, so $V_{R^+ \cup \{0\}}$ is a Smarandache semiring. On similar lines we can say $V_{Q^+ \cup \{0\}}$ and $V_{Z^+ \cup \{0\}}$ are semirings of polynomials intervals which are Smarandache semiring of polynomial intervals.

We can define semivector space of polynomial intervals.

Consider

$V_{R^+ \cup \{0\}} = \{P(x) = [p(x), q(x)] \mid p(x), q(x) \in (R^+ \cup \{0\}) (x)\};$

$V_{R^+ \cup \{0\}}$ is a semivector space of polynomial intervals over the semifield $R^+ \cup \{0\}$ or $Q^+ \cup \{0\}$ or $Z^+ \cup \{0\}$.

Likewise $V_{Q^+ \cup \{0\}}$ and $V_{Z^+ \cup \{0\}}$ are also semivector spaces of polynomial intervals over $R^+ \cup \{0\}$ or $Z^+ \cup \{0\}$ and $Z^+ \cup \{0\}$ respectively. Infact these are also semilinear algebras.

Now we can define substructures and related properties as in case of semivector spaces. This task is left as an exercise to the reader.

All these structures can be easily converted into interval polynomials. So we can say one can get a one to one mapping from polynomial intervals to interval polynomials.

For instance as in case of V_R we see if $[p(x), q(x)] = P(x) \in V_{R^+ \cup \{0\}}$ so $p(x) = 8x^3 + 7x^2 + \sqrt{3}x + \sqrt{19}$ and $q(x) = 18x^5 + 10x^4 + \sqrt{5}x^3 + 2x+1$ then
$[p(x), q(x)] = 8x^3 + 7x_2 + \sqrt{3}x + \sqrt{19}, 18x^5 + 10x^4 + \sqrt{5}x^3 + 2x + 1]$.

$= [0, 18]x^5 + [0, 10]x^4 + [8, \sqrt{5}]x^3 + [7, 0]x^2 + [\sqrt{3}, 2]x + [\sqrt{19}, 1]$ which is the interval polynomial.

Likewise if $[6, 2]x^8 + [\sqrt{7}, 0]x^5 + [0, \sqrt{5}]x^4 + [3, 2]x^3 + [5, 1]x + [10, \sqrt{11}] = P(x)$ be the interval polynomial we can write it as a polynomial interval as $6x^8 + \sqrt{7}x^5 + 3x^3 + 5x + 10 = p(x)$ and $q(x) = 2x^8 + \sqrt{5}x^4 + 2x^3 + x + \sqrt{11}$ and $P(x) = [p(x), q(x)]$ which is a polynomial interval.

Example 3.14: Let

$$V = \{(p_1(x), q_1(x)], [p_2(x), q_2(x), \ldots, [p_9(x), q_9(x)) \mid [p_i(x), q_i(x)] \in V_{R^+ \cup \{0\}}; 1 \leq i \leq 9\}$$

be a semigroup under addition. V is a semivector space over the semifield $S = Z^+ \cup \{0\}$.

Example 3.15: Let

$$M = \left\{ \begin{bmatrix} P_1(x) \\ P_2(x) \\ \vdots \\ P_{10}(x) \end{bmatrix} \middle| P_i(x) \in V_{Z^+ \cup \{0\}}; 1 \leq i \leq 10 \right\}$$

be a semigroup under addition. M is a semivector space over the semifield $S = Z^+ \cup \{0\}$.

Example 3.16: Let

$$V = \left\{ \begin{bmatrix} P_1(x) & P_2(x) & P_3(x) \\ P_4(x) & P_5(x) & P_6(x) \\ P_7(x) & P_8(x) & P_9(x) \end{bmatrix} \middle| P_i(x) \in V_{Q^+ \cup \{0\}}; 1 \leq i \leq 9 \right\}$$

be a semigroup under addition. V is a semivector space over the semifield $S = Z^+ \cup \{0\}$.

Example 3.17: Let

$$V = \left\{ \begin{bmatrix} P_1(x) & P_2(x) & P_3(x) \\ P_4(x) & P_5(x) & P_6(x) \\ \vdots & \vdots & \vdots \\ P_{28}(x) & P_{29}(x) & P_{30}(x) \end{bmatrix} \middle| P_i(x) \in V_{Z^+ \cup \{0\}}; 1 \leq i \leq 30 \right\}$$

be a semigroup under addition. V is the semivector space over the semifield $S = Z^+ \cup \{0\}$.

We give substructures.

Example 3.18: Let

$$V = \{(P_1(x), P_2(x), \ldots, P_8(x) \mid P_i(x) \in V_{Q^+ \cup \{0\}}; 1 \leq i \leq 8\}$$

be a semivector space of interval polynomial over the semifield $S = Z^+ \cup \{0\}$.

Let

$M = \{(P_1(x), 0, P_3(x), 0, P_4(x), P_2(x), 0, 0)$ where $P_i(x) \in V_{Q^+ \cup \{0\}}; 1 \leq i \leq 4\} \subseteq V$ is a semivector subspace of interval polynomial over the semifield $S = Z^+ \cup \{0\}$.

Example 3.19: Let

$$V = \{(P_1(x), P_2(x), \ldots, P_{10}(x) \mid P_i(x) \in V_{Q^+ \cup \{0\}}; 1 \leq i \leq 10\}$$

be a semivector space of interval polynomial over the semifield $S = Q^+ \cup \{0\}$.

Example 3.20: Let

$$V = \left\{\begin{pmatrix} P_1(x) & P_2(x) \\ P_3(x) & P_4(x) \end{pmatrix} \middle| \; P_i(x) \in V_{Q^+ \cup \{0\}}; 1 \leq i \leq 4\right\}$$

be a semivector space of interval polynomial over the semifield $S = Z^+ \cup \{0\}$.

Example 3.21: Let

$$V = \left\{ \begin{bmatrix} P_1(x) \\ P_2(x) \\ \vdots \\ P_{12}(x) \end{bmatrix} \middle| \; P_i(x) \in V_{Q^+ \cup \{0\}} ; 1 \le i \le 12 \right\}$$

be a semivector space of interval polynomial over the semifield $S = Q^+ \cup \{0\}$.

Consider

$$W_1 = \left\{ \begin{bmatrix} P_1(x) \\ P_2(x) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \middle| \; P_1(x), P_2(x) \in V_{Q^+ \cup \{0\}} \right\} \subseteq V,$$

$$W_2 = \left\{ \begin{bmatrix} 0 \\ 0 \\ P_1(x) \\ P_2(x) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \middle| \; P_1(x), P_2(x) \in V_{Q^+ \cup \{0\}} \right\} \subseteq V,$$

$$W_3 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ P_1(x) \\ P_2(x) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \middle| \; P_1(x), P_2(x) \in V_{Q^+ \cup \{0\}} \right\} \subseteq V,$$

$$W_4 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ P_1(x) \\ P_2(x) \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \middle| \; P_1(x), P_2(x) \in V_{Q^+ \cup \{0\}} \right\} \subseteq V,$$

$$W_5 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ P_1(x) \\ P_2(x) \\ 0 \\ 0 \end{bmatrix} \middle| \; P_1(x), P_2(x) \in V_{Q^+ \cup \{0\}} \right\} \subseteq V \text{ and}$$

$$W_6 = \left\{ \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ 0 \\ P_1(x) \\ P_2(x) \end{bmatrix} \middle| \; P_1(x), P_2(x) \in V_{Q^+ \cup \{0\}} \right\} \subseteq V$$

where $W_1, W_2, \ldots, W_6$ are semivector subspaces of V.

Clearly $V = \bigcup_{i=1}^{6} W_i$; $W_i \cap W_j = (0)$ if $i \neq j$; $1 \leq i, j \leq 6$.

Thus V is the direct sum of semivector subspaces of the semivector space over the semifield $S = Q^+ \cup \{0\}$.

Example 3.22: Let

$$V = \left\{ \begin{bmatrix} P_1(x) & P_2(x) \\ P_3(x) & P_4(x) \\ P_5(x) & P_6(x) \\ P_7(x) & P_8(x) \end{bmatrix} \middle| \; P_i(x) \in V_{Q^+ \cup \{0\}}; 1 \leq i \leq 8 \right\}$$

be a semivector space of interval polynomials over the semifield $S = Z^+ \cup \{0\}$.

$$W_1 = \left\{ \begin{bmatrix} P_1(x) & P_2(x) \\ P_3(x) & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \middle| \; P_i(x) \in V_{Q^+ \cup \{0\}}; 1 \leq i \leq 3 \right\} \subseteq V,$$

$$W_2 = \left\{ \begin{bmatrix} P_1(x) & 0 \\ 0 & P_2(x) \\ 0 & 0 \\ 0 & P_3(x) \end{bmatrix} \middle| \; P_i(x) \in V_{Q^+ \cup \{0\}}; 1 \leq i \leq 3 \right\} \subseteq V,$$

$$W_3 = \left\{ \begin{bmatrix} P_1(x) & 0 \\ 0 & 0 \\ P_2(x) & P_3(x) \\ 0 & 0 \end{bmatrix} \middle| \; P_i(x) \in V_{Q^+ \cup \{0\}}; 1 \leq i \leq 3 \right\} \subseteq V,$$

$$W_4 = \left\{ \begin{bmatrix} P_1(x) & 0 \\ 0 & 0 \\ 0 & 0 \\ P_3(x) & P_2(x) \end{bmatrix} \middle| \; P_i(x) \in V_{Q^+ \cup \{0\}}; 1 \leq i \leq 3 \right\} \subseteq V,$$

and

$$W_5 = \left\{ \begin{bmatrix} P_1(x) & 0 \\ 0 & P_2(x) \\ P_3(x) & 0 \\ 0 & P_4(x) \end{bmatrix} \middle| \; P_i(x) \in V_{Q^+ \cup \{0\}}; 1 \le i \le 4 \right\} \subseteq V$$

be semivector subspaces of the semivector space V over the semifield $S = Z^+ \cup \{0\}$.

We see $V = \bigcup_{i=1}^{5} W_i$ but $W_i \cap W_j \neq (0)$ if $i \neq j$; $1 \le i, j \le 5$.

Thus V is only a pseudo direct sum of semivector subspaces of V over S.

We can as in case of usual semivector spaces also define the notion of semivector space of interval polynomial in $V_{R^+ \cup \{0\}}$ or $V_{Q^+ \cup \{0\}}$ or $V_{Z^+ \cup \{0\}}$.

Example 3.23: Let

$$T = \left\{ \begin{bmatrix} P_1(x) \\ P_2(x) \\ \vdots \\ P_{12}(x) \end{bmatrix} \middle| \; P_i(x) \in V_{Q^+ \cup \{0\}}; 1 \le i \le 12 \right\}$$

be a semivector space of interval polynomial over the semifield $S = Q^+ \cup \{0\}$.

Suppose T is defined over the S-semiring $V_{Q^+ \cup \{0\}}$. For if $V_{Q^+ \cup \{0\}} = \{P(x) = (p(x), q(x)) \mid p(x), q(x) \in (Q^+ \cup \{0\})[x]\}$ is a semiring for $(p(x), 0) = P(x)$ and $Q(x) = (0, q(x))$ then $P(x) \cdot Q(x) = (0, 0)$.

So we call such semivector spaces as Smarandache special semivector spaces.

Chapter Four

INTERVALS OF TRIGONOMETRIC FUNCTIONS OR TRIGONOMETRIC INTERVAL FUNCTIONS

Here we for the first time introduce the notion of trigonometric intervals or intervals of trigonometric functions. These collection of trigonometric intervals form a ring under usual addition of trigonometric functions and multiplication of functions.

Before we define this new concept we give a few examples of them, for this will make the reader understand the definition in a easy way.

Example 4.1: Let J = [f(x), g(x)] be an interval where $f(x) = 5 \sin^3 x - 8 \cos^2 x + 4$ and

$$g(x) = \frac{\tan^7 8x - 8\cot 5x}{4\operatorname{cosec}^3 5x}; \text{ we say } J = [f(x), g(x)]$$

is a trigonometric interval.

Example 4.2: Let K = [p(x), q(x)] where

$$p(x) = 18 \sin (9x^2 + 4) + \frac{7 \operatorname{cosec}^3 x}{10 \cot 3x} - 5$$

and

$$q(x) = \frac{\tan x . \sec 4x}{\cot 3x + \cos x}.$$

K is a trigonometric interval function.

Thus with some default we call g(x), f(x), q(x) and p(x) as trigonometric polynomials, that is the variable x is itself a trigonometric function in f(x), p(x), q(x) or g(x). Hence throughout this book by this default we assume a trigonometric polynomials p(x) is a polynomial in $\sin^t x$, $\cos^s x$, $\tan^r cx$, $\cos^n dx$, $\operatorname{cosec}^m ex$ with t, s, r, n, m ≥ 0 and a, b, c, d, e, $\in$ R[x] (that is $\cos^n dx$ can also be like $\cos^5 (20x^2 - 5x + 1)$ here n = 5 and dx = $(20x^2 - 5x + 1)$ and so on.

We are forced to define in this manner mainly for we say an interval J = [p(x), q(x)] is a trigonometric function if all values inbetween p(x) and q(x) is in J; further we cannot compare p(x) with q(x) as it may not be possible in all cases.

We give some more illustrate examples.

Example 4.3: Let J = [sin3x, cos5x]; J be a trigonometric function interval or interval of trigonometric function.

Example 4.4: Let T = [0, 3sin5x] be a trigonometric function of interval.

Example 4.5: Let K = [6 $\cot^3 8x$, 0] be a trigonometric function of interval.

Example 4.6: Let

$$W = \left[\frac{\cos^3 x + 5\sin^2 x}{7 - 8\cot x}, -7\right]$$

be again a trigonometric function of interval.

Example 4.7: Let

$$M = \left[\frac{\operatorname{cosec}^2 x}{1 - \cot 5x}, \frac{7\sin x}{1 + \cot 3x}\right]$$

be a trigonometric function of interval.

Example 4.8: Let J = $\left[9, \frac{8}{1 - \cos^2 x}\right]$ where even if 1 = $\cos^2 x$ can occur as a trigonometric interval, in which case when 1 = $\cos^2 x$, the interval degenerates into [9, ∞].

Example 4.9: Let

$$R = \left[0, \frac{1}{\cos^2 x - \sin^2 x}\right]$$

be a trigonometric interval.

Now having seen examples, we now give a very informal definition. T denotes the collection of all trigonometric functions which is closed under the operations of addition and multiplication as mentioned earlier. T = {f(x) | f(x) is a polynomial in $\sin^a bx$ and (or) $\cot^c dx$ and (or) $\cos^e fx$ and (or) $\tan^r sx$ and (or) $\sec^p(qx)$ and (or) $\operatorname{cosec}^m nx$ with a, c, e, r, p, m $\in Z^+ \cup \{0\}$ and b, d, f, s, q, n $\in$ R[x].

We define $T_I = \{[g(x), f(x)] \mid g(x), f(x) \in T\}$ to be the interval of trigonometric functions or trigonometric function interval or just trigonometric interval.

Clearly T_I contains [a, b] where a, b $\in$ R (reals) we intentionally make this assumption for we see if

$J = \left[\frac{1}{\cos x}, -7\sin x\right]$ and $K = \left[-9\cos x, \frac{-10}{19\sin x}\right]$ are in T_1 then

$$J.K. = \left[\frac{1}{\cos x}, -7\sin x\right] \times \left[-9\cos x, \frac{-10}{19\sin x}\right]$$

$$= \left[\frac{1}{\cos x} \times -9\cos x, -7\sin x \times \frac{-10}{19\sin x}\right]$$

$$= \left[-9, \frac{70}{19}\right] \in T_I.$$

We define product in T_I as follows; if M = [p(x), q(x)] and N = [m(x), n(x)] are in T_I then MN = [p(x), q(x)] [m(x), n(x)]

$$= [p(x).\ m(x), q(x)\ n(x)].$$

It is easily verified that MN = NM. Also we see if M = [p(x), q(x)] and N = [r(x), s(x)] are in T_I, then

$$M + N = [p(x), q(x)] + [r(x), s(x)] = [p(x) + r(x), q(x) + s(x)].$$

We see M + N = N + M and M + N is in T_I. Clearly [0, 0] = 0 $\in T_I$ acts as the additive identity. Further if J = [p(x), q(x)] $\in$ T_I then [–p(x), –q(x)] = k $\in T_I$ acts as the additive inverse of J. Thus

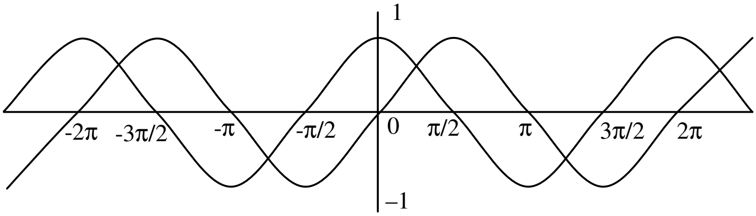
$$K = -J = -[p(x), q(x)] = [-p(x), -q(x)].$$

$1 = [1, 1] \in T_I$ acts as the multiplicative identity for

$$1.J = [1, 1]\,[p(x), q(x)]$$
$$= [p(x), q(x)] = [p(x), q(x)]\,[1, 1] = [\,p(x), q(x)].$$

Thus we can say T_I is a commutative ring with unity of infinite order.

Now we show how the curves on interval trigonometric functions look like. For take $J(x) = [\sin x, \cos x] \in T_I$ the graph associated J is as follows.



Thus the figure shows the curve related with J.

Any curve in T_I can be traced by the interested reader. We see the for value J ([0, 0]) = [0, 1], J ([1, 0]) = [0, 1] and so on.

Let $J = [3, \sin x] \in T_I$. $J = ([3, \pi/2]) = [3, 1]$.

Now we can define differentiation and integration on these trigonometric functions. We will only illustrate these situations by some simple examples.

Example 4.10: Let $J(x) = [3\cos^2 x, \sin 7x] \in T_I$. To find the derivative of J (x). We differentiate component wise

$$\frac{d(J(x))}{dx} = [-6 \cos x \sin x, 7\cos 7x]$$

$$= \frac{d}{dx}([3\cos^2 x, \sin 7x])$$

$$= [\frac{d}{dx}(3\cos^2 x), \frac{d}{dx}(\sin 7x)]$$

$$= [-6\cos x \sin x, 7\cos 7x] \in T_1.$$

Example 4.11: Let

$$P(x) = [\tan 9x + \cos 2x + 8\sin 3x, \frac{\cot x - 1}{\cot x + 1}] \in T_I.$$

To find the derivative of P(x).

$$\frac{d}{dx}(P(x)) = \frac{d}{dx}([\tan 9x + \cos 2x + 8\sin 3x, \frac{\cot x - 1}{\cot x + 1}])$$

$$= [\frac{d}{dx}(\tan 9x + \cos 2x + 8\sin 3x), \frac{d}{dx}\left(\frac{\cot x - 1}{\cot x + 1}\right)]$$

$$= [9\sec^2 9x - 2\sin 2x + 24\cos 3x,$$

$$\frac{(\cot x + 1)(-\operatorname{cosec}^2 x) + (\cot x - 1)\operatorname{cosec}^2 x}{(\cot x + 1)^2}]$$

$$= [9\sec^2 9x - 2\sin 2x + 24\cos 3x, \frac{-2\operatorname{cosec}^2 x}{(\cot x + 1)^2}] \in T_I.$$

Example 4.12: Let

$$P(x) = [\frac{\sec^2 3x}{1 - \cos x}, 3] \in T_I,$$

we find the derivative of P(x);

$$\frac{d}{dx}\,[P(x)] = \frac{d}{dx}\,[\frac{\sec^2 3x}{1-\cos x}, 3\,]$$

$$= \left[\frac{d}{dx}\left(\frac{\sec^2 3x}{1-\cos x}\right), \frac{d(3)}{dx}\right]$$

$$= \left[\frac{(1-\cos x)6\sec^2 3x\tan 3x + \sin x\sec^2 3x}{(1-\cos x)^2}, 0\right].$$

From these examples we see that T_I is such that for every $f(x) \in T_I$ we see $\frac{d}{dx}$ (f(x)) is in T_I.

On similar lines we see define functions that are integrable can be integrated and the resultant is in T_I.

If we work with finite integrals then also the resultant will be in T_I.

Thus T_I is closed under integration.

We will illustrate this situation by some examples.

Example 4.13: Let

$$P(x) = \left[3\sin x, \frac{-\cot 5x}{7}\right] \in T_I.$$

To find

$$\int P(x)\,dx = \int\left[3\sin x, -\cot\frac{5x}{7}\right]dx$$

$$= \left[\int 3\sin x dx, -\int \cot\frac{5x}{7} dx\right] + C \text{ is in } T_I.$$

Example 4:14: Let

$$M(x) = \left[\frac{-8\cos 7x}{3}, \frac{5\tan 7x}{8}\right]$$

be in T_I. To find the integral of M(x).

$$\int M(x)\, dx = \int \left[\frac{-8\cos 7x}{3}, \frac{5\tan 7x}{8}\right] dx$$

$$= \left[\int \frac{-8\cos 7x}{3} dx, \int \frac{5\tan 7x}{8} dx\right] + C \text{ is in } T_I.$$

Example 4:15: Let

$$S(x) = [7 \sin (3x+2), 9]$$

be in T_I.

$$\int S(x)\, dx = \int [7 \sin (3x+2, 9]\, dx$$

$$= \left[\int 7\sin(3x+2)dx, \int 9\, dx\right] + C \text{ is again in } T_I.$$

Using the set T_I we can have the following algebraic structure. We know $(T_I, +)$ is a group under addition which is clearly commutative and is of infinite order. Now using T_I we can also build different types of additive abelian groups. Consider $X = (P_1(x), \ldots, P_n(x))$ where $P_i(x) \in T_I$; $1 \leq i \leq n$. X is defined as the row interval trigonometric function matrix or trigonometric interval function of row matrices. If we consider the collection M of all $1 \times n$ row trigonometric interval matrices then M is a group under addition and M is only a monoid under product.

We will give examples of them.

Example 4.16: Let

$$M = \{(P_1(x), P_2(x), P_3(x), P_4(x), P_5(x)) \mid P_i(x) \in T_I;\ 1 \leq i \leq 5\}$$

be a group of row interval matrix trigonometric functions. M is of infinite order.

Example 4.17: Let

$$P = \{(P_1(x), P_2(x), \ldots, P_{25}(x)) \mid P_i(x) \in T_I;\ 1 \leq i \leq 25\}$$

be a group of row interval trigonometric functions.

We now give example of monoid of row matrix of trigonometric intervals.

Example 4.18: Let

$$N = \{(P_1(x), P_2(x), \ldots, P_8(x)) \mid P_i(x) \in T_I;\ 1 \leq i \leq 8\}$$

be the monoid of row interval matrix of trigonometric functions.

Example 4.19: Let

$$S = \{(M_1(x), M_2(x), \ldots, M_{10}(x)) \mid M_i(x) \in T_I;\ 1 \leq i \leq 10\}$$

be the monoid of row interval matrix of trigonometric functions.

Example 4.20: Let

$$V = \{(P_1(x), P_2(x)) \mid P_i(x) \in T_I;\ 1 \leq i \leq 2\}$$

be the monoid of row interval matrix of trigonometric functions.

These monoids have zero divisors, ideals, subsemigroups and submonoids.

We will give an example or two.

Example 4.21: Let

$$M = \{(P_1(x), P_2(x), P_3(x)) \mid P_i(x) \in T_I;\ 1 \leq i \leq 3\}$$

be a monoid of row interval matrix of trigonometric functions.

Take $S = \{(P(x), P(x), P(x)) \mid P(x) \in T_I\} \subseteq M$; S is a submonoid of row matrix interval of trigonometric functions.

However S is not an ideal.

Consider $W = \{(0, P(x), 0) \mid P(x) \in T_I\} \subseteq M$, W is a submonoid of row matrix interval of trigonometric functions. However W is also an ideal of M.

$V = \{(P(x), 0, P(x)) \mid P(x) \in T_I\} \subseteq M$; V is a submonoid of row matrix interval of trigonometric functions. V is not an ideal.

For if

$X = (q_1(x), q_2(x), q_3(x)) \in M$ with $q_i(x) \in T_I$; $1 \leq i \leq 3\}$, and $v = (p(x), 0, P(x))$ $P(x) \in T_I$ then $v\ X = (q_1(x), q_2(x), q_3(x))$ $(P(x), 0, P(x)) = (q_1(x)\ P(x), 0, q_3(x)\ (P(x))$

$$= (r_1(x), 0, r_2(x));$$

with $r_i(x) \neq r_2(x)$ if $q_i(x) \neq q_2(x)$ in T_I; so $v\ X \notin V$. Hence V is not an ideal of M.

Consider $x = \left(0, \dfrac{\sin^3 x}{1-\cos 3x}, 0\right)$ and

$$y = \left(\frac{7\tan 5x}{1+\tan^2 x}, 0, \frac{8\sec^2 x - 1}{9\tan x + 5}\right) \text{ in M.}$$

It is easily verified xy = 0. Thus M has zero divisors.

$$\text{Take } Y = \left(\frac{\sec^3 5x + 1}{7x - \tan^2 x}, \frac{8}{9}\cot x, \frac{\sec^7 8x}{1+\tan x}\right) \text{ in M.}$$

$$\text{Now } P = \left(\frac{7x - \tan^2 x}{\sec^3 5x + 1}, \frac{9}{8\cot x}, \frac{1+\tan x}{\sec^7 8x}\right) \text{ in M;}$$

is such that YP = PY = (1, 1, 1).

We see all elements x = ($P_1(x)$, $P_2(x)$, $P_3(x)$) in M such that in which atleast one of $P_i(x)$ is zero; $1 \leq i \leq 3$ is such that we cannot find a y in M with xy = yx = (1, 1, 1).

THEOREM 4.1: *Consider*

$$K = \{(P_1(x), P_2(x), \ldots, P_n(x)) \mid P_i(x) \in T_I;\ 1 \leq i \leq n\}$$

be a monoid of trigonometric interval functions. K has ideals, submonoids which are not ideals and zero divisors and units.

The proof is straight forward, hence left as an exercise to the reader.

THEOREM 4.2: *Let*

$$M = \{(P(x), P(x), \ldots, P(x)) \mid P(x) \in T_I\}$$

be a monoid of trigonometric interval functions. M has no zero divisors, no ideals but has units.

This proof is also straight forward and hence left as an exercise to the reader.

Now having seen examples of row matrix monoid of trigonometric intervals we now proceed onto define column matrix group of trigonometric intervals under matrix addition. However these column matrix of trigonometric intervals do not form a monoid under multiplication as product cannot be defined.

We now give examples of them.

Example 4.22: Let

$$V = \left\{ \left[\begin{array}{c} P_1(x) \\ P_2(x) \\ \vdots \\ P_{15}(x) \end{array} \right] \middle| \; P_i(x) \in T_I; \; 1 \leq i \leq 15 \right\}$$

be a group of column matrix trigonometric intervals under addition. Clearly V is not compatible with respect to product.

Example 4.23: Let

$$V = \left\{ \left[\begin{array}{c} P_1(x) \\ P_2(x) \\ \vdots \\ P_7(x) \end{array} \right] \middle| \; P_i(x) \in T_I; \; 1 \leq i \leq 7 \right\}$$

be a group of trigonometric intervals under addition.

Example 4.24: Let

$$V = \left\{ \left[\begin{array}{c} P_1(x) \\ P_2(x) \\ P_3(x) \end{array} \right] \middle| \; P_i(x) \in T_I; \; 1 \leq i \leq 3 \right\}$$

be a group of column matrix of trigonometric intervals under addition.

$$\text{Take } x = \begin{bmatrix} \sin^3 x + \cot 5x \\ \dfrac{-8\sec^2 x}{1+\tan x} \\ \sec^3 7x - 1 \end{bmatrix} \text{ and}$$

$$y = \begin{bmatrix} \sec^3 5x - 5\operatorname{cosec}^2 x - \cot 5x \\ \dfrac{8\sec^2 x}{1+\tan x} \\ 1+\cos 5x \end{bmatrix};$$

$$x+y = \begin{bmatrix} \sin^3 x + \cot 5x \\ \dfrac{-8\sec^2 x}{1+\tan x} \\ \sec^3 7x - 1 \end{bmatrix} + \begin{bmatrix} \sec^3 5x - 5\operatorname{cosec}^2 x - \cot 5x \\ \dfrac{8\sec^2 x}{1+\tan x} \\ 1+\cos 5x \end{bmatrix}$$

$$= \begin{bmatrix} \sin^3 x + \sec^3 5x - 5\operatorname{cosec}^2 x \\ 0 \\ \cos 5x + \sec^3 7x \end{bmatrix} \text{ is in M.}$$

Clearly $x \times y$ is not defined. We now proceed onto say that every element $x \in M$ can generate a subgroup under addition. However these are not the only subgroups of M.

Now if we consider the collection of all $m \times n$ trigonometric interval matrices P(say), P is a group under addition known as the group of $m \times n$ matrix trigonometric intervals.

We will give examples of them.

Example 4.25: Let P = {5 × 7 matrices with entries from T_I} be the group of trigonometric interval matrices under addition.

Example 4.26: Let

$$W = \left\{ \begin{bmatrix} P_1(x) & P_2(x) & P_3(x) \\ P_4(x) & P_5(x) & P_6(x) \end{bmatrix} \middle| \; P_i(x) \in T_I;\ 1 \leq i \leq 6 \right\}$$

be the 2 × 3 group of trigonometric intervals under addition.

Example 4.27: Let

$$W = \left\{ \begin{bmatrix} P_1(x) & P_2(x) \\ P_3(x) & P_4(x) \\ P_5(x) & P_6(x) \end{bmatrix} \middle| \; P_i(x) \in T_I;\ 1 \leq i \leq 6 \right\}$$

be the 3 × 2 matrix of trigonometric intervals group under addition.

Clearly we cannot define product on all these additive groups. If m = n then we see these square matrix intervals of trigonometric functions can be groups under addition and only a semigroup under product. Infact they are monoids under product.

We will give examples of these situations.

Example 4.28: Let

$$W = \left\{ \begin{bmatrix} P_1(x) & P_2(x) \\ P_3(x) & P_4(x) \end{bmatrix} \middle| \; P_i(x) \in T_I;\ 1 \leq i \leq 4 \right\}$$

be the group of 2 × 2 interval of trigonometric functions under addition.

Infact $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ acts as the additive identity.

$$\text{If } X = \begin{bmatrix} p_1(x) & p_2(x) \\ p_3(x) & p_4(x) \end{bmatrix} \text{ and } Y = \begin{bmatrix} q_1(x) & q_2(x) \\ q_3(x) & q_4(x) \end{bmatrix}$$

are in R then

$$X.Y = \begin{bmatrix} p_1(x) & p_2(x) \\ p_3(x) & p_4(x) \end{bmatrix} \begin{bmatrix} q_1(x) & q_2(x) \\ q_3(x) & q_4(x) \end{bmatrix}$$

$$= \begin{bmatrix} p_1(x)q_1(x) + p_2(x)q_3(x) & p_1(x)q_2(x) + p_2(x)q_4(x) \\ p_3(x)q_1(x) + p_4(x)q_3(x) & p_3(x)q_2(x) + p_4(x)q_4(x) \end{bmatrix}$$

is in R. Thus R is a semigroup under product.

We see $I_{2\times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ in R acts as the multiplicative identity. Now R has zero divisors ideals and subsemigroups.

Example 4.29: Let

P = {all 10×10 intervals matrices with intervals from T_I}

be the group under addition and monoid under product.

THEOREM 4.3: *Let*

P = {$n \times n$ interval matrices with intervals from T_I},

P is a monoid and has ideals submonoids, subsemigroups, zero divisors and units.

THEOREM 4.4: *Let M = {n × n interval matrices with intervals from T_I, every element in the matrix is the same} be the monoid. M has no ideals only subsemigroups and has no zero divisors but has units.*

The proof of these theorems are simple and hence left as an exercise to the reader.

Infact we can say if R = {all n × n interval matrices with entries from T_I}, then (R, +, ×) is a ring. Infact a non commutative ring with unit of infinite order. This ring has units, zero divisors, ideals and subrings.

Infact R has subrings which are not ideals.

Example 4.30: Let R = {3 × 3 interval matrices with intervals from T_I} be a ring.

Chapter Five

NATURAL CLASS OF FUZZY INTERVALS

In this chapter we first introduce the natural class fuzzy intervals and define operations on them.

DEFINITION 5.1: *Let*

$V = \{[a, b] \mid a, b \in [0, 1]\}$ (where $a = 0 = b$ or $a = b$ or $a < b$ or $a > b$) be the set of intervals. We define V to be the natural class of fuzzy intervals, which are closed.

Example 5.1: Let

$V = \{[0.7, 0.21], [0, 0.24], [0.3, 0], [0, 1], [1, 0.81]\} = B$ be a subset of natural class of intervals.

We observe as $[0, 1] \subseteq R$ the set of reals, likewise $V \subseteq N_c(R)$.

We denote I_c the collection of natural class of closed fuzzy intervals $I_o = \{(a, b) \mid a, b \in [0, 1]\}$ denotes the natural class of open fuzzy intervals.

$I_{oc} = \{[a, b) \mid a, b \in [0, 1]\}$ denotes the natural class of open closed fuzzy intervals.

$I_{co} = \{(a, b) \mid a, b \in [0, 1]\}$ denotes the natural class of closed open fuzzy intervals.

Now we can define operations on them so that I_o or I_c or I_{co} or I_{oc} becomes a semigroup.

DEFINITION 5.2: *Let $I_c = \{(a, b) \mid a, b \in [0, 1]\}$ be the natural class of closed fuzzy intervals. Define for $x = [a, b]$ and $y = [c, d]$ in I_c the min operation as follows:*

$$\min \{x, y\} = \min \{[a, b], [c, d]\}$$
$$= [\min \{a, c\}, \min \{b, d\}\} \in I_c .$$

Thus $\{I_c, \min\}$ is a semigroup.

Suppose $x = \{[0, 93, 0.271]\}$ and $y = [0.201, 0.758]$ are in I_c, then $\min \{x, y\} = \min \{[0.93, 0.271], [0.201, 0.758]\} = [\min \{0.93, 0.201\}, \min \{0.201, 0.758\}] = [0.201, 0.271]$ and $[0.201, 0.271] \in I_c$.

It is easily verified min on I_c is a semigroup which is commutative. Likewise we define commutative semigroup with min operation on I_{oc}, I_{co} and I_o.

Now instead of min operation on I_c (or I_o or I_{co} or I_{oc}) we can define on I_c the max operation and still I_c under max operation is a commutative semigroup of infinite order.

Example 5.2: Let $I_{oc} = \{(a, b] \mid a, b \in [0, 1]\}$ be the collection of natural class of open closed intervals. Clearly for any $x = (a, b]$ and $y = (c, d]$ we define $\max \{x, y\} = \max \{(a, b], (c, d]\} = (\max \{a, b\}, \max \{b, d\}] \in I_{oc}$. Thus max defined on I_{oc} is a closed binary commutative and associative operation. Thus $\{I_{oc}, \max\}$ is a commutative semigroup.

Likewise we can define max operation on I_{co}, I_o and I_c. Those will be semigroups of infinite order.

Now we can define yet another operation on I_c (or I_o or I_{oc} or I_{co}) which we call as natural product.

DEFINITION 5.3: *Let $I_c = \{(a, b) \mid a, b \in [0, 1]\}$ be the natural class of closed open intervals. Define on I_c the natural product '$\times$' or '.' as follows.*

For $x = [a, b]$ and $y = [c, d]$

we define $$x \times y = x.y = [a, b]\,[c, d] = [a.c,\, b.d] = [a \times c,\, b \times d] \in I_c.$$

Thus $\{I_c, \times\}$ is a semigroup with zero divisors. For $x = [0, 0.732]$ and $y = [0.213, 0]$ in I_c is such that $x.y = [0, 0]$. $[1,1]$ in I_c acts as the multiplicative unit; for if $x = [a, b]$ then $x.[1, 1] = [a, b]\,[1, 1] = [a.1, b.1] = [a, b] \in I_c$.

Thus we can use any of these three operations on I_c (or I_o or I_{oc} or I_{co}) while constructing matrices or polynomials using I_c (or I_o or I_{oc} or I_{co}).

We now proceed onto define fuzzy interval matrices using I_c (or I_o or I_{oc} or I_{co}).

DEFINITION 5.4: *Let*

$X = (a_1, \ldots, a_n)$ where $a_i \in I_c$ (or I_o or I_{oc} or I_{co}); $1 \leq i \leq n$; X is defined as the natural class of fuzzy row intervals matrix or row interval fuzzy matrices with entries from the natural class of fuzzy intervals I_c (or I_o or I_{oc} or I_{co}) (or strictly used only in the mutually exclusive sense) ($1 \leq i \leq n$).

Likewise we define fuzzy column interval matrix

$$y = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \text{ where } b_i \in I_o \text{ (or } I_c \text{ or } I_{oc} \text{ or } I_{co}\text{); } 1 \leq i \leq m$$

or natural class of fuzzy column matrices.

Let $A = (a_{ij})_{m \times n}$ ($m \neq n$) we define A to be a fuzzy $m \times n$ interval matrix if $a_{ij} \in I_o$ (or I_c or I_{oc} or I_{co}) $1 \leq i \leq m$ and

$1 \leq j \leq n$. We define A to be a fuzzy square interval matrix if $m = n$.

We will illustrate these situations by some simple examples before we proceed onto define operations on these collections.

Example 5.3: Let

$$X = ([0.21, 0.001), [0, 0.781), [1, 0.061), [0.22, 0.22), [0.23, 0.7931))$$

be a fuzzy row interval matrix with entries from

$$I_{co} = \{[a, b) \mid a, b \in [0, 1]\}.$$

Example 5.4: Let

$$P = \begin{bmatrix} (0.301, 0.005] \\ (0.12, 0] \\ (1, 0.9921] \\ (0.0.701] \\ (0.17, 0.912] \\ (0.4911, 0.27105] \end{bmatrix}$$

be a fuzzy column interval matrix with entries from

$$I_{oc} = \{(a, b] \mid a, b \in [0, 1]\}.$$

Example 5.5: Let

$$M = \begin{bmatrix} [0,0.3] & [0,0] & [0.32,0.32] & [0,0.75] & [0,0.37] \\ [0.12,0] & [1,1] & [0.33,0] & [0.72,0.33] & [0.71,0.3] \\ [0.3,0.7] & [0,0.121] & [0,0.9301] & [0.31,0.14] & [0.31,0] \end{bmatrix}$$

be a fuzzy 3×5 interval matrix with entries from I_c.

Example 5.6: Let

$$P = \begin{bmatrix} [0,1) & [0,0.3) & [0.358,1) & [0,0.7) \\ [0,0) & [0.3,0.3) & [1,1) & [0.8,1) \\ [1,0.79) & [0.71,0.71) & [0.2,0.2) & [1,0.7) \\ [0.31,0) & [0.26,0) & [0.5,0) & [0.5,6) \end{bmatrix}$$

be a fuzzy 4×4 square interval matrix with entries from

$$I_{co} = \{[a, b) \mid a, b \in [0, 1]\}.$$

Now having seen the four types of fuzzy interval matrices we now proceed onto define operations on them.

Let
$X = \{(a_1, a_2, \ldots, a_n) \mid a_i \in I_{co} = \{[a, b) \mid a, b \in [0, 1]\}; 1 \leq i \leq n\}$ be the collection of all fuzzy interval row matrices with entries from I_{co}. We can define three operations on X and under each of these operations X is a commutative semigroup.

Consider the min operation on X so that if $x = (a_1, \ldots, a_n)$ and $y = (b_1, \ldots, b_n)$ are in X then $\min \{x, y\} = \min \{(a_1, \ldots, a_n), (b_1, b_2, \ldots, b_n)\}$

$$= (\min (a_1, b_1), \min (a_2, b_2), \ldots, \min (a_n, b_n))$$

$= ([\min (a_1^1, b_1^1), \min (a_1^2, b_1^2)), [\min (a_2^1, b_2^1), \min (a_2^2, b_2^2)), \ldots, [\min (a_n^1, b_n^1), \min (a_n^2, b_n^2))$ where

$x = ([a_1^1, a_1^2), [a_2^1, a_2^2), \ldots, [a_n^1, a_n^2))$ and $y = ([b_1^1, b_1^2), [b_2^1, b_2^2), \ldots, [b_n^1, b_n^2))$. Clearly X with min operation is a semigroup known as the fuzzy interval row matrix semigroup with entries from I_{co}.

Now on the same collection X we can define the max operation and under max operation also X is a semigroup which is commutative and is of infinite order.

Take $x = (a_1, a_2, \ldots, a_n)$ and $y = (b_1, b_2, \ldots, b_n)$

where $x = ([a_1^1, a_1^2), [a_2^1, a_2^2), [a_3^1, a_3^2), \ldots, [a_n^1, a_n^2))$ and $y = ([b_1^1, b_1^2), [b_2^1, b_2^2), [b_3^1, b_3^2) \ldots, [b_n^1, b_n^2))$ in X.

Now $\max \{x, y\} = \{\max (a_1, b_1), \ldots, \max (a_n, b_n)\}$

$= \{\max \{[a_1^1, a_1^2), [b_1^1, b_1^2)\}, \ldots, \max \{[a_n^1, a_n^2), [b_n^1, b_n^2)\}\}$
$= [\max \{a_1^1, b_1^1\}), \max \{a_1^2, b_1^2\}), [\max \{a_2^1, b_2^1\}, \max \{a_2^2, b_2^2\}), \ldots, [\max \{a_n^1, b_n^1\}, \max \{a_n^2, b_n^2\})).$

We will just illustrate this situation by examples.

Example 5.7: Let $X = \{(a_1, \ldots, a_5) \mid a_i \in I_{oc}; 1 \leq i \leq 5\}$ be a fuzzy open closed row interval matrix.

Let x = ((0.5, 0.7], (0, 0.3], (1, 0.4], (1, 1], (0.8, 0.2101]) and y = ((0, 0.2], (0.3, 0.101], (0, 0], (1, 0], (0.71, 0.215]) be in X.

max {x, y} = max {((0.5, 0.7], (0, 0.3], (1, 0.4], (1, 1], (0.8, 0.2101]), ((0, 0.2], (0.3, 0.101], (0, 0], (1, 0], (0.71, 0.215])}

= (max {(0.5, 0.7], (0, 0.2]}, max {(0, 0.3], (0.3, 0.101]}, max {(1, 0.4], (0, 0]}, max {(1, 1], (1, 0]}, max {(0.8, 0.2101], (0.71, 0.215]})

= ((max {0.5, 0}, max {0.7, 0.2}], (max {0, 0.3}, max {0.3, 0.101}, (max {1, 0}, max {0.4, 0}], (max {1, 1}, max {1, 0}], (max {0.8, 0.71}, max {0.2101, 0.215}])

= ((0.5, 0.7], (0.3, 0.3], (1, 0.4], (1, 1], (0.8, 0.215]) $\in$ X.

Thus if X be the collection of fuzzy interval row matrices with entries from I_c (or I_o or I_{oc} or I_{co}), then X is a semigroup under max operation of infinite order.

Now we proceed onto define the notion of product on the collection of fuzzy interval row matrices with entries from I_c (or I_o or I_{oc} or I_{co}).

Let
$X = \{(a_1, a_2, \ldots, a_n) \mid a_i \in I_{co} = \{[a, b) \mid a, b \in [0, 1]\}, 1 \leq i \leq n\}$ be the collection of fuzzy interval row matrices.

For any $x = (a_1, a_2, \ldots, a_n) = ([a_1^1, a_2^1), [a_1^2, a_2^2), \ldots, [a_1^n, a_2^n))$ and $y = (b_1, b_2, \ldots, b_n) = ([b_1^1, b_2^1), [b_1^2, b_2^2), \ldots, [b_1^n, b_2^n))$ in X define $x.y = (a_1, a_2, \ldots, a_n) . (b_1, b_2, \ldots, b_n) = ([a_1^1, a_2^1), [a_1^2, a_2^2), \ldots, [a_1^n, a_2^n)) . ([b_1^1, b_2^1), [b_1^2, b_2^2), \ldots, [b_1^n, b_2^n))$

$= ([a_1^1, a_2^1), [b_1^1, b_2^1), [a_1^2, a_2^2), [b_1^2, b_2^2), \ldots, [a_1^n, a_2^n) [b_1^n, b_2^n))$

$= ([a_1^1, b_1^1, a_2^1, b_2^1), [a_1^2, b_1^2, a_2^2, b_2^2), \ldots, [a_1^n, b_1^n, a_2^n, b_2^n))$ is in I_{co}.

Thus (X, product '.') is a semigroup under multiplication.

We now can define max 'or' min operation (or used in the mutually exclusive sense) as follows:

Let $x = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ and $y = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ be any two column interval matrices with entries from $I_c = \{[a, b] \mid a, b \in [0, 1]\}$.

We define $\max \{x, y\} = \max \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \right\}$

(here $a_i = [a_i^1, a_i^2]$ and $b_i = [b_i^1, b_i^2]$, $1 \leq i \leq n$).

$$= \begin{bmatrix} \max\{a_1, b_1\} \\ \max\{a_2, b_2\} \\ \vdots \\ \max\{a_n, b_n\} \end{bmatrix} = \begin{bmatrix} \max\{[a_1^1, a_1^2],[b_1^1, b_1^2]\} \\ \max\{[a_2^1, a_2^2],[b_2^1, b_2^2]\} \\ \vdots \\ \max\{[a_n^1, a_n^2],[b_n^1, b_n^2]\} \end{bmatrix}$$

$$= \begin{bmatrix} \max\{a_1^1, b_1^1\}, \max\{a_1^2, b_1^2\} \\ \max\{a_2^1, b_2^1\}, \max\{a_2^2, b_2^2\} \\ \vdots \\ \max\{a_n^1, b_n^1\}, \max\{a_n^2, b_n^2\} \end{bmatrix}.$$

Now we can define this situation by some example.

Example 5.8: Let

$$X = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_6 \end{bmatrix} \text{ and } Y = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_6 \end{bmatrix} \text{ where } a_i, b_i \in I_o;\ 1 \le i \le 6.$$

$$\text{That is } X = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_6 \end{bmatrix} = \begin{bmatrix} (0, 0.7) \\ (1,0) \\ (0.2,1) \\ (0.31, 0.12) \\ (0.25, 0.14) \\ (0.1,1) \end{bmatrix} \text{ and}$$

$$Y = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_6 \end{bmatrix} = \begin{bmatrix} (0.3,1) \\ (0.56,0) \\ (0,0) \\ (0.31,0.31) \\ (0.76,0) \\ (0.71,1) \end{bmatrix}$$

be two fuzzy interval column matrices

$$\max\{X, Y\} = \max\left\{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_6 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_6 \end{bmatrix}\right\}$$

$$= \max\left\{\begin{bmatrix} (0,0.7) \\ (1,0) \\ (0.2,1) \\ (0.31,0.12) \\ (0.25,0.14) \\ (0.1,1) \end{bmatrix}, \begin{bmatrix} (0.3,1) \\ (0.56,0) \\ (0,0) \\ (0.31,0.31) \\ (0.76,0) \\ (0.71,1) \end{bmatrix}\right\}$$

$$= \begin{bmatrix} (\max\{0,0.3\}, \max\{0.7,1\}) \\ (\max\{1,0.56\}, \max\{0,0\}) \\ (\max\{0.2,0\}, \max\{1,0\}) \\ (\max\{0.31,0.31\}, \max\{0.12,0.31\}) \\ (\max\{0.25,0.76\}, \max\{0.14,0\}) \\ (\max\{0.1,0.71\}, \max\{1,1\}) \end{bmatrix}$$

$$= \begin{bmatrix} (0.3,1) \\ (1,0) \\ (0.2,1) \\ (0.31,0.31) \\ (0.76,0.14) \\ (0.71,1) \end{bmatrix}.$$

Let

$$V = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \middle| a_i \in I_{oc}; 1 \leq i \leq n \text{ where } a_i = (a_i^1, a_i^2] \right\}$$

be the collection of all fuzzy interval column matrices. {V, max} is a semigroup of infinite order which is commutative. We see one can define on the set of fuzzy interval column matrices the operation min. Still the collection will be a semigroup under min operation.

We will first illustrate this situation by some examples.

Example 5.9: Let

$$x = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \text{ and } y = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix} \text{ where } a_i = (a_i^1, a_i^2] \text{ and } b_i = (b_i^1, b_i^2]$$

be in I_{oc}. $1 \leq i \leq 5$.

$$\text{Suppose } x = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_5 \end{bmatrix} = \begin{bmatrix} (0,1] \\ (1,0] \\ (0.8,0] \\ (1,0.2] \\ (0.71,0.9] \end{bmatrix}$$

$$y = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_5 \end{bmatrix} = \begin{bmatrix} (0.31,1] \\ (0,0.58] \\ (1,0] \\ (0,1] \\ (0.37,0.215] \end{bmatrix}$$

be two fuzzy interval column matrices.

$$\text{Now min } (x, y) = \min \left\{ \begin{bmatrix} (0,1] \\ (1,0] \\ (0.8,0] \\ (1,0.2] \\ (0.71,0.9] \end{bmatrix}, \begin{bmatrix} (0.31,1] \\ (0,0.58] \\ (1,0] \\ (0,1] \\ (0.37,0.215] \end{bmatrix} \right\}$$

$$= \begin{bmatrix} \min\{(0,1],(0.31,1]\} \\ \min\{(1,0],(0,0.58]\} \\ \min\{(0.8,0],(1,0]\} \\ \min\{(1,0.2],(0,1]\} \\ \min\{(0.71,0.9],(0.37,0.215]\} \end{bmatrix}$$

$$= \begin{bmatrix} (\min\{0,0.31\},\min\{1,1\}] \\ (\min\{1,0\},\min\{0,0.58\}] \\ (\min\{0.8,1\},\min\{0,0\}] \\ (\min\{1,0\},\min\{0.2,1\}] \\ (\min\{0.71,0.37\},\min\{0.9,0.215\}] \end{bmatrix}$$

$$= \begin{bmatrix} (0,1] \\ (0,0] \\ (0.8,0] \\ (0,0.2] \\ (0.37,0.215] \end{bmatrix}.$$

Now having seen min {x, y}, x and y fuzzy column interval matrices. We proceed onto define semigroup of fuzzy column interval matrices under 'min' operation.

Thus if

$$W = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \middle| a_i \in I_{oc}; a_i = (a_i^1, a_i^2]; 1 \leq i \leq n \right\}$$

be the collection of fuzzy column interval matrices. W under 'min' operation is a semigroup.

Let $x = (a_1, a_2, \ldots, a_n) = ([a_1^1, a_2^1], [a_1^2, a_2^2], \ldots, [a_1^n, a_2^n])$ be a fuzzy row interval matrix. We can as in case of usual matrices define the transpose of x as follows:

$$x^t = ([a_1^1, a_2^1], [a_1^2, a_2^2], \ldots, [a_1^n, a_2^n])^t$$

$$= \begin{bmatrix} [a_1^1, a_1^2] \\ [a_1^2, a_2^2] \\ \vdots \\ [a_n^1, a_n^2] \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}.$$

We see $(x^t)^t = x$.

Now we see if $x = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} (a_1^1, a_1^2) \\ (a_2^1, a_2^2) \\ \vdots \\ (a_n^1, a_n^2) \end{bmatrix}$

where $a_i = (a_i^1, a_i^2) \in I_o$; $1 \leq i \leq n$ then $x^t = \begin{pmatrix} (a_1^1, a_1^2) \\ (a_2^1, a_2^2) \\ \vdots \\ (a_n^1, a_n^2) \end{pmatrix}^t$

$= ((a_1^1, a_1^2), (a_1^2, a_2^2), \ldots, (a_n^1, a_n^2))$

$= (a_1, a_2, \ldots, a_n)$.

We see $(x^t)^t = x = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$.

It is important to mention that as in case of usual column vectors we cannot in case of fuzzy column matrices also define product only 'min' and 'max' operation can be defined.

Suppose $x = \begin{bmatrix} [0,1) & [0.1,1) & [0,0) \\ [0.1,0) & [0,1) & [0.1,0.1) \\ [1,0) & [0.3,0.2) & [0,0.5) \\ [0.2,0.1) & [0.4,0) & [0.4,0.4) \\ [0.5,0.4) & [1,1) & [0.2,0.1) \end{bmatrix}$

$$\text{and } y = \begin{bmatrix} [0,0) & [0.7,0) & [0,0.4) \\ [1,1) & [0.8,0.1) & [0.7,0.2) \\ [0,1) & [0.9,1) & [0.4,0.3) \\ [1,0) & [0,0.3) & [0.7,0.7) \\ [0.5,0.4) & [1,0.4) & [1,0.3) \end{bmatrix}$$

be two 5×3 fuzzy interval matrices.

We can define min (or max) operation on x and y 'or' used in the mutually exclusive sense.

min (x, y)

$$= \min\left\{\begin{bmatrix} [0,1) & [0.1,1) & [0,0) \\ [0.1,0) & [0,1) & [0.1,0.1) \\ [1,0) & [0.3,0.2) & [0,0.5) \\ [0.2,0.1) & [0.4,0) & [0.4,0.4) \\ [0.5,0.4) & [1,1) & [0.2,0.1) \end{bmatrix}, \begin{bmatrix} [0,0) & [0.7,0) & [0,0.4) \\ [1,1) & [0.8,0.1) & [0.7,0.2) \\ [0,1) & [0.9,1) & [0.4,0.3) \\ [1,0) & [0,0.3) & [0.7,0.7) \\ [0.5,0.4) & [1,0.4) & [1,0.3) \end{bmatrix}\right\}$$

$$= \begin{pmatrix} \min([0,1),[0,0)) & \min([0.1,1),[0.7,0)) & \min([0,0),[0,0.4)) \\ \min([0.1,0),[1,1)) & \min([0,1),[0.8,0.1)) & \min([0.1,0.1),[0.7,0.2)) \\ \min([1,0),[0,1)) & \min([0.3,0.2),[0.9.1)) & \min([0,0.5),[0.4,0.3)) \\ \min([0.2,0.1),[1,0)) & \min([0.4,0),[0,0.3)) & \min([0.4,0.4),[0.7,0.7)) \\ \min([0.5,0.4),[0.5,0.4)) & \min([1,1),[1,0.4)) & \min([0.2,0.1),[1,0.3)) \end{pmatrix}$$

$$= \begin{pmatrix} [0,0) & [0.1,0) & [0,0) \\ [0.1,0) & [0,0.1) & [0.1,0.1) \\ [0,0) & [0.3,0.2) & [0,0.3) \\ [0.2,0) & [0,0) & [0.4,0.4) \\ [0.5,0.4) & [1,0.4) & [0.2,0.1) \end{pmatrix}.$$

This we see if

$$V=\{(a_{ij})_{m\times n} \mid a_{ij} = [a^1_{ij}, a^2_{ij}];\ 1 \leq i \leq m \text{ and } 1 \leq j \leq n\ a_{ij} \in I_{co}\}$$

be the collection of all fuzzy interval m × n matrices, then V under the min operation is a semigroup which is commutative.

$$\text{Consider } x = \begin{pmatrix} [0,1) & [0,0) & [0.5,0.2) & [0.2,1) \\ [0.3,0) & [0.7,0.7) & [0.7,0.3) & [0.4,0) \end{pmatrix}$$

$$\text{and } y = \begin{pmatrix} [1,0.2) & [0.3,0) & [1,1) & [0.3,0.71) \\ [0.3,0.1) & [0.2,0.2) & [0.5,0.2) & [0.9,1) \end{pmatrix}$$

be two fuzzy interval 2 × 4 matrices with entries from I_{co}.

Now we can define max operation of x, y = max (x, y) =

$$\max\left\{\begin{pmatrix} [0,1) & [0,0) & [0.5,0.2) & [0.2,1) \\ [0.3,0) & [0.7,0.7) & [0.7,0.3) & [0.4,0) \end{pmatrix},\right.$$

$$\left.\begin{pmatrix} [1,0.2) & [0.3,0) & [1,1) & [0.3,0.71) \\ [0.3,0.1) & [0.2,0.2) & [0.5,0.2) & [0.9,1) \end{pmatrix}\right\}$$

$$= \left(\begin{matrix} \max\{[0,1),[1,0.2)\} & \max\{[0,0),[0.3,0)) \\ \max\{[0.3,0),[0.3,0.1)\} & \max\{[0.7,0.7),[0.2,0.2)\} \end{matrix}\right.$$

$$\left.\begin{matrix} \max\{[0.5,0.2),[1,1)\} & \max\{[0.2,1),[0.3,0.71)\} \\ \max\{[0.7,0.3),[0.5,0.2)\} & \max\{[0.4,0),[0.9,1)\} \end{matrix}\right)$$

$$= \begin{pmatrix} [1,1) & [0.3,0) & [1,1) & [0.3,1) \\ [0.3,0.1) & [0.7,0.7) & [0.7,0.3) & [0.9,1) \end{pmatrix}.$$

We can define max operation on the set of all fuzzy interval $m \times n$ matrices with entries from I_c; V the max operator; where $V = \{(a_{ij})_{m\times n} = a_{ij} = \left(\left[a_{ij}^1, a_{ij}^2\right]\right)_{m\times n}$,

with $a_{ij}^t \in [0,1]$; $1 \leq t \leq 2$ and $1 \leq i \leq m$ and $1 \leq j \leq n\}$.

Thus {V, max} is a semigroup. Now we can transpose any $m \times n$ fuzzy interval matrix A and A^t will be a $n \times m$ fuzzy interval matrix.

$$\text{For if } A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m_1} & a_{m_2} & \cdots & a_{m_n} \end{pmatrix}$$

$$= \begin{pmatrix} [a_{11}^1, a_{11}^2] & [a_{12}^1, a_{12}^2] & \cdots & [a_{1n}^1, a_{1n}^2] \\ [a_{21}^1, a_{21}^2] & [a_{22}^1, a_{22}^2] & \cdots & [a_{2n}^1, a_{2n}^2] \\ \vdots & \vdots & & \vdots \\ [a_{m1}^1, a_{m1}^2] & [a_{m2}^1, a_{m2}^2] & \cdots & [a_{mn}^1, a_{mn}^2] \end{pmatrix}$$

with $a_{ij} = [a_{ij}^1, a_{ij}^2] \in I_c$; $1 \leq i \leq m$ and $1 \leq j \leq n$ be the fuzzy interval $m \times n$ matrix.

$$\text{Now } A^t = \begin{pmatrix} [a_{11}^1, a_{11}^2] & [a_{12}^1, a_{12}^2] & \cdots & [a_{1n}^1, a_{1n}^2] \\ [a_{21}^1, a_{21}^2] & [a_{22}^1, a_{22}^2] & \cdots & [a_{2n}^1, a_{2n}^2] \\ \vdots & \vdots & & \vdots \\ [a_{m1}^1, a_{m1}^2] & [a_{m2}^1, a_{m2}^2] & \cdots & [a_{mn}^1, a_{mn}^2] \end{pmatrix}^t$$

$$= \begin{pmatrix} [a_{11}^1, a_{11}^2] & [a_{21}^1, a_{21}^2] & \cdots & [a_{m1}^1, a_{m1}^2] \\ [a_{12}^1, a_{12}^2] & [a_{22}^1, a_{22}^2] & \cdots & [a_{m2}^1, a_{m2}^2] \\ \vdots & \vdots & & \vdots \\ [a_{1n}^1, a_{1n}^2] & [a_{2n}^1, a_{2n}^2] & \cdots & [a_{mn}^1, a_{mn}^2] \end{pmatrix}$$

is a n × m fuzzy interval matrix and is the transpose of A. Clearly $\left(A^t\right)^t = A$.

Now if A be a fuzzy interval square matrix we can define three operations on A. In the first place transpose of a fuzzy interval square matrix is a square matrix.

$$\text{Let } A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} [a_{11}^1, a_{11}^2] & [a_{12}^1, a_{12}^2] & \ldots & [a_{1n}^1, a_{1n}^2] \\ [a_{21}^1, a_{21}^2] & [a_{22}^1, a_{22}^2] & \ldots & [a_{2n}^1, a_{2n}^2] \\ \vdots & \vdots & & \vdots \\ [a_{n1}^1, a_{n1}^2] & [a_{n2}^1, a_{n2}^2] & \ldots & [a_{nn}^1, a_{nn}^2] \end{pmatrix}$$

$a_{ij} \in I_c$; $1 \leq i, j \leq n$.

Now transpose of A denoted by

$$A^t = \begin{pmatrix} [a_{11}^1, a_{11}^2] & [a_{12}^1, a_{12}^2] & \ldots & [a_{1n}^1, a_{1n}^2] \\ [a_{21}^1, a_{21}^2] & [a_{22}^1, a_{22}^2] & \ldots & [a_{2n}^1, a_{2n}^2] \\ \vdots & \vdots & & \vdots \\ [a_{n1}^1, a_{n1}^2] & [a_{n2}^1, a_{n2}^2] & \ldots & [a_{nn}^1, a_{nn}^2] \end{pmatrix}^t$$

$$= \begin{pmatrix} [a_{11}^1, a_{11}^2] & [a_{21}^1, a_{21}^2] & \ldots & [a_{n1}^1, a_{n1}^2] \\ [a_{12}^1, a_{12}^2] & [a_{22}^1, a_{22}^2] & \ldots & [a_{n2}^1, a_{n2}^2] \\ \vdots & \vdots & & \vdots \\ [a_{1n}^1, a_{1n}^2] & [a_{2n}^1, a_{2n}^2] & \ldots & [a_{nn}^1, a_{nn}^2] \end{pmatrix}.$$

We see A^t is also a $n \times n$ fuzzy interval matrix of $\left(A^t\right)^t = A$.

Now we proceed onto define the max (or min) operation on the collection of all $n \times n$ fuzzy interval matrices.

We will illustrate this situation by some examples.

Example 5.10: Let

$$A = \begin{bmatrix} [0,0.3) & [0,0) & [1,0.7) & [0,0.3) \\ [0,1) & [1,1) & [0.3,0.1) & [1,1) \\ [1,0) & [0.2,0.2) & [0.2,1) & [0,1) \\ [0.4,0.4) & [0.7,0.1) & [0.7,1) & [1,1) \end{bmatrix}$$

$$\text{and } B = \begin{bmatrix} [0.7,0) & [0,0.8) & [0.1,0.1) & [0.9,0.7) \\ [0.4,1) & [0.3,0.1) & [0,0) & [0.2,0.4) \\ [0.2,0.3) & [0.5,0.4) & [1,1) & [0.1,0.2) \\ [0.1,1) & [0.7,0.9) & [0.2,0) & [0.7,0.5) \end{bmatrix}$$

be any two 4×4 fuzzy interval matrices.

We define

$$\min\{A, B\} = \min\left\{\begin{bmatrix} [0,0.3) & [0,0) & [1,0.7) & [0,0.3) \\ [0,1) & [1,1) & [0.3,0.1) & [1,1) \\ [1,0) & [0.2,0.2) & [0.2,1) & [0,1) \\ [0.4,0.4) & [0.7,0.1) & [0.7,1) & [1,1) \end{bmatrix},\right.$$

$$\left.\begin{bmatrix} [0.7,0) & [0,0.8) & [0.1,0.1) & [0.9,0.7) \\ [0.4,1) & [0.3,0.1) & [0,0) & [0.2,0.4) \\ [0.2,0.3) & [0.5,0.4) & [1,1) & [0.1,0.2) \\ [0.1,1) & [0.7,0.9) & [0.2,0) & [0.7,0.5) \end{bmatrix}\right\}$$

$$= \begin{bmatrix} \min\{[0,0.3),[0.7,0)\} & \min\{[0,0),[0,0.8)\} \\ \min\{[0,1),[0.4,1)\} & \min\{[1,1),[0.3,0.1)\} \\ \min\{[1,0),[0.2,0.3)\} & \min\{[0.2,0.2),[0.5,0.4)\} \\ \min\{[0.4,0.4),[0.1,1)\} & \min\{[0.7,0.1),[0.7,0.9)\} \end{bmatrix}$$

$$\begin{bmatrix} \min\{[1,0.7),[0.1,0.1)\} & \min\{[0,0.3),[0.9,0.7)\} \\ \min\{[0.3,0.1),[0,0)\} & \min\{[1,1),[0.2,0.4)\} \\ \min\{[0.2,1),[1,1)\} & \min\{[0,1),[0.1,0.2)\} \\ \min\{[0.7,1),[0.2,0)\} & \min\{[1,1),[0.7,0.5)\} \end{bmatrix}$$

(using the fact min {[a, b), [c, d) = [min {a, c}, min {b, d}))

$$= \begin{bmatrix} [0,0) & [0,0) & [0.1,0.1) & [0,0.3) \\ [0,1) & [0.3,0.1) & [0,0) & [0.2,0.4) \\ [0.2,0) & [0.2,0.2) & [0.2,1) & [0,0.2) \\ [0.1,0.4) & [0.7,0.1) & [0.2,0) & [0.7,0.5) \end{bmatrix}.$$

Thus for a collection of n × n fuzzy interval matrices V, with entries from I_c (or I_o or I_{oc} or I_{co}); V with 'min' operator is a semigroup.

Likewise we can use max operator instead of min operator and V under max operator is also a semigroup.

We give only examples of them in what follows.

$$\text{Let } P = \begin{bmatrix} (0,0.3] & (1,1] & (0.2,0] \\ (0.7,0.2] & (0,0.3] & (1,0] \\ (0.9,0.4] & (0.7,0.7] & (0,1] \end{bmatrix}$$

$$\text{and } S = \begin{bmatrix} (0.2,1] & (0.2,0.4] & (0,0] \\ (0.3,0] & (0.7,1] & (1,0] \\ (0,1] & (0.4,0.3] & (0,0.5] \end{bmatrix}$$

be two fuzzy interval 3 × 3 square matrices.

To find max {P, S} =

$$\max\left\{\begin{bmatrix} (0,0.3] & (1,1] & (0.2,0] \\ (0.7,0.2] & (0,0.3] & (1,0] \\ (0.9,0.4] & (0.7,0.7] & (0,1] \end{bmatrix}, \begin{bmatrix} (0.2,1] & (0.2,0.4] & (0,0] \\ (0.3,0] & (0.7,1] & (1,0] \\ (0,1] & (0.4,0.3] & (0,0.5] \end{bmatrix}\right\}$$

$$= \begin{bmatrix} (0.2,1] & (1,1] & (0.2,0] \\ (0.7,0.2] & (0.7,1] & (1,0] \\ (0.9,1] & (0.7,0.7] & (0,1] \end{bmatrix}.$$

Now we can define yet another operation 'max min' operation. First we will illustrate this by an example.

Example 5.11: Let

$$x = \begin{pmatrix} [0,1] & [1,1] & [0,1] & [1,0.3] \\ [0.3,0] & [0,0] & [1,0] & [0.5,0] \\ [0.2,0.2] & [0,0.3] & [0,0.7] & [0.7,1] \\ [0.1,0] & [0.5,0] & [1,0.3] & [0.8,0] \end{pmatrix} = (x_{ij})$$

$$\text{and } y = \begin{pmatrix} [1,0] & [0.3,1] & [0,0] & [0,0.7] \\ [0.3,0] & [0.2,0] & [1,1] & [0.2,0.2] \\ [0.5,1] & [0,0] & [0.5,0.8] & [0.5,0] \\ [0.7,0.2] & [0.7,1] & [1,0] & [0,0] \end{pmatrix} = (y_{ij})$$

be 4 × 4 fuzzy interval matrices. We show how max min operation is defined on x and y

= max {min {x, y}}, max {min {first row of x, first column of y}}

= min {[1, 1], [0.3,0]}, min {[0,1], [0.5, 1]}, min {[0.1, 0], [0.7, 0.2}} = a_{11} = [a^1_{11}, a^2_{11}]

= max {[0, 0], [0.3, 0], [0, 1], [0.1, 0]}

= {max {0, 0.3, 0, 0.1}, max {0, 0,1, 0}}

= [0.3, 1] = a_{11}.

max min {first row of x, second column of y}

= max {min {[0, 1], [0.3, 1]}, min {[1, 1], [0.2, 0]} min {[0, 1], [0, 0]} min {[0.1, 0], [0.7, 1]}}

= max {[0, 1], [0.2, 0], [0, 0], [0.1, 0]}

= [max {0, 0.2, 0, 0}, max {1, 0, 0, 0}]

= [0.2, 1] = a_{12} = [a^1_{12}, a^2_{12}].

Now max {min {first row of x, third column of y}}

= max {min {[0, 1], [0, 0]}, {min {[1, 1], [1, 1]}, {min {[0, 1], [0.5, 0.8]}, {min {[1, 0.3], [1, 0]}}

= max {[0,0], [1,1], [0,0.8], [1,0]}

= [max {0, 1, 0, 1}, max {0, 1, 0.8, 0}]

= [1, 1] = a_{13} = [a^1_{13}, a^2_{13}].

max {min {first row of x, fourth column of y}}

= max {min {[0, 1], [0, 0.7]}, {min {[1, 1], [0.2, 0.2]}, {min {[0, 1], [0.5, 0]}, {min {[1, 0.3], [0, 0]}}

= max {[0,0.7], [0.2,0.2], [0,0], [0,0]}

$= [\max \{0, 0.2, 0, 0\}, \max \{0.7, 0.2, 0, 0\}]$

$= [0.2, 0.7] = a_{14} = [a^1_{14}, a^2_{14}]$.

max {min {second row of x, first column of y}}

$= \max \{\min \{[0.3, 0], [1, 0]\}, \{\min \{[0, 0], [0.3, 0]\}, \{\min \{[0, 0.3], [0.5, 1]\}, \{\min \{[0.5, 0], [0.7, 0.2]\}\}$

$= \max \{[0.3,0], [0,0], [0,0.3], [0.5,0]\}$

$= [\max \{0.3, 0, 0, 0.5\}, \max \{0, 0, 0.3, 0\}]$

$= [0.5, 0.3] = a_{21} = [a^1_{21}, a^2_{21}]$.

$$\text{Thus max min } (x, y) = \begin{bmatrix} [0.3,1] & [0.2,1] & [1,1] & [0.2, 0.7] \\ [0.5, 0.3] & . & . & . \\ \vdots & \vdots & \vdots & \vdots \\ . & . & . & . \end{bmatrix}.$$

Thus we can have such operations and the collection of square fuzzy interval matrices under max min operation is a semigroup. Now having seen fuzzy interval matrices and operations on them we can write every fuzzy interval matrix M = (m_{ij}) ($m_{ij} \in I_c$ or I_o or I_{oc} or I_{co}) as fuzzy matrix interval.

That is if M = (m_{ij}) = $((m^1_{ij}), (m^2_{ij}))$ then $M = ([m^1_{ij}, m^2_{ij}])$. It is pertinent to mention here that only such representation simplifies the calculations in interval matrices.

First we will illustrate this situation by some examples.

Let

$$A = \begin{bmatrix} [0,0.3) & [0.4,1) & [1,1) & [0.7,0) \\ [1,0.8) & [0.5,0.5) & [0,1) & [1,0) \\ [0.2,0.6) & [0.7,0.6) & [0.4,0.71) & [0.5,0.2) \end{bmatrix}$$

be a fuzzy interval 3×4 matrix with entries from I_{co}.

Now A can be written uniquely as a fuzzy 3×4 matrix interval as

$A = [A_1, A_2)$

$$= \left[\begin{pmatrix} 0 & 0.4 & 1 & 0.7 \\ 1 & 0.5 & 0 & 1 \\ 0.2 & 0.7 & 0.4 & 0.5 \end{pmatrix}, \begin{pmatrix} 0.3 & 1 & 1 & 0 \\ 0.8 & 0.5 & 1 & 0 \\ 0.6 & 0.6 & 0.71 & 0.2 \end{pmatrix} \right).$$

$$\text{Now } A_1 = \begin{pmatrix} 0 & 0.4 & 1 & 0.7 \\ 1 & 0.5 & 0 & 1 \\ 0.2 & 0.7 & 0.4 & 0.5 \end{pmatrix} \text{ and}$$

$$A_2 = \begin{pmatrix} 0.3 & 1 & 1 & 0 \\ 0.8 & 0.5 & 1 & 0 \\ 0.6 & 0.6 & 0.71 & 0.2 \end{pmatrix}$$

are fuzzy matrices. Now this way of representing a fuzzy interval matrix as a matrix interval helps in simplifying all calculations. Thus we can also define a fuzzy matrix interval. $A = [A_1, A_2]$ where A_1 and A_2 are fuzzy matrices of same order and A takes its entries from I_c if $A = (a_{ij}) = [(a^1_{ij}),(a^2_{ij})]$ where $A_1 = (a^1_{ij})$ and $A_2 = (a^2_{ij})$.

Suppose $A = [A_1, A_2)$ where A_1 and A_2 are fuzzy matrices of same order be a fuzzy matrix interval then

$A = (a_{ij}) = [(a^1_{ij}), (a^2_{ij})) = ([a^1_{ij}, a^2_{ij}))$ then entries $a_{ij} \in I_{co}$.

In a similar way we can define $A = (A_1, A_2]$ where A_1 and A_2 are fuzzy matrices of same order then A is a fuzzy matrix interval and A as a fuzzy interval matrix we see $A = ((a^1_{ij}, a^2_{ij}])$ where $A_1 = (a^1_{ij})$ and $A_2 = (a^2_{ij})$ with $(a^1_{ij}, a^2_{ij}] \in I_{oc}$.

Finally we have $A = (A_1, A_2) = ((a^1_{ij}), (a^2_{ij})) = ((a^1_{ij}, a^2_{ij}))$ where $A_1 = (a^1_{ij})$ and $A_2 = (a^2_{ij})$ are fuzzy matrices and $(a^1_{ij}, a^2_{ij}) \in I_o$ and A is both a interval fuzzy matrix when A is represented as $((a^1_{ij}, a^2_{ij}))$ and A is a fuzzy matrix interval if $A = (A_1, A_2)$ where A_1 and A_2 are fuzzy matrices of same order.

With these techniques we can have fuzzy interval matrices and operations on them are similar to fuzzy matrix intervals.

We now proceed onto define fuzzy interval polynomials or polynomials with fuzzy interval coefficients.

Let I_c (or I_{oc} or I_{co} or I_o) be the collection of fuzzy intervals. A fuzzy interval polynomial $p(x) = p_0 + p_1 x + \ldots + p_n x^n$ where $p_i = [a_i, b_i]$ with $a_i, b_i \in [0, 1]$; $0 \leq i \leq n$.

Now we cannot add two fuzzy interval polynomials as the resultant may not be a fuzzy interval polynomial.

For take
$$\begin{aligned} p(x) &= p_0 + p_1 x + p_2 x^2 \\ &= [0.7, 0.9] + [1, 0.8]x + [0.7, 1]x^2 \end{aligned}$$

and
$$\begin{aligned} q(x) &= [1, 0.3] + [0.2, 0.7]x + [0.9, 0.2]x^2 \\ &\quad + [0, 1]x^3 \end{aligned}$$

to be two interval fuzzy polynomials with coefficients from I_c. Now $p(x) + q(x) = ([0.7, 0.9] + [1, 0.8]x + [0.7, 1]x^2) + [1, 0.3] + [0.2, 0.7]x + [0.9, 0.2]x^2 + [0, 1]x^3$

$= ([0.7, 0.9] + [1, 0.3]) + ([1,0.8] + [0.2,0.7])x + ([0.7,1] + [0.9, 0.2])x^2 + [0,1]x^3$

$= [1.7, 1.2] + [1.2, 1.5]x + [1.6, 1.2]x^2 + [0,1]x^3$.

We see p(x) + q(x) is not a interval fuzzy polynomial as the coefficients of p(x) + q(x) are not fuzzy intervals or does not belong to the natural class of closed fuzzy interval I_c.

Thus we are forced to define two types of binary operations on fuzzy interval polynomials.

Suppose

$$S[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in [a_i^1, a_i^2] \in I_c; \right.$$

x a variable or an indeterminate} we say S[x] is the collection of all fuzzy interval polynomial with coefficients from I_c.

We can replace by I_o or I_{oc} or I_{co} and the collection will be a fuzzy interval polynomial with coefficients from I_o or I_{oc} or I_{co} respectively.

We see on S[x] we cannot define usual addition or usual product for fuzzy interval polynomials. We define the 'max' operator as an operation on fuzzy interval polynomials.

Let

$p(x) = [0, 1) + [0.7, 0)x + [0.3, 0.7)x^2 + [1, 0)x^3 + [0.3, 1)x^5$

and

$q(x) = [0, 0.3) + [0.8, 0.5)x^2 + [0.4, 0.7)x^3 + [0, 0.2)x^4 + [0.7, 0)x^5$ be two fuzzy interval polynomials with coefficients from I_{co}.

max (p(x), q(x)) = max {[0, 1) + [0.7, 0)x + [0.3, 0.7)x^2 + [1, 0)x^3 +[0.3, 1)x^5, [0, 0.3) + [0.8, 0.5)x^2 + [0.4, 0.7)x^3 + [0, 0.2)x^4 + [0.7, 0)x^5}

= max {[0, 1), [0, 0.3)} + max {[0.7, 0), [0, 0)}x + max {[0.3, 0.7); [0.8, 0.5)}x^2 + max {[1, 0), [0.4, 0.7)}x^3 + max {[0, 0), [0, 0.2)}x^4 + max {[0.3, 1), [0.7,0)}x^5

= [0, 1) + [0.7, 0)x + [0.8, 0.7)x^2 + [1, 0.7)x^3 + [0, 0.2)x^4 + [0. 7, 1)x^5.

Thus we see the collection of fuzzy interval polynomials under max operation is a semigroup. However under 'min' operation we feel the structure of quality of two polynomials is not properly represented.

For if q(x) = [0, 1] + [0, 0.3]x^3 + [0.2, 1]x^4
and p(x) = [1, 0]x + [0.2, 0.7]x^2 + [0, 0.9]x^5

are fuzzy interval polynomial with entries from I_c, then min {q(x), p(x)} = [0, 0] so we see the operation does not do justice to every term; we feel 'min' operation on fuzzy interval polynomials does not yield a satisfactory result. Thus we can use only max function on fuzzy interval polynomials. Solving roots is not very difficult as the fuzzy interval polynomials are written as a fuzzy polynomial intervals.

For if p(x) = (0, 0.7] + (0.7, 1]x + (0.6, 0.9]x^2 + (0, 0.2]x^3 + (0.7, 0]x^4 + (0.3, 0.2]x^5 be a fuzzy interval polynomial, then

$$\begin{aligned} p(x) &= (p_1(x), p_2(x)) \\ &= (0.7x + 0.6x^2 + 0.7x^4 + 0.3x^5, 0.7 + x \\ &\quad + 0.9x^2 + 0.2x^3 + 0.2x^5] \end{aligned}$$

where $p_1(x)$ and $p_2(x)$ are fuzzy polynomials.

One can at present exploit the existing methods of solving these equations however one has to invent some other ways for the existing methods are not satisfactory. Thus writing fuzzy interval polynomials as fuzzy polynomial intervals we get the roots.

Chapter Six

CALCULUS ON MATRIX INTERVAL POLYNOMIAL AND INTERVAL POLYNOMIALS

In this chapter we introduce the notion of matrix whose entries are interval polynomials and show how in general interval polynomials are differentiated and integrated.

First we know if (a, b] is an interval in $N_{oc}(R)$ then for any integer n. n (a, b] = (na, nb] is in $N_{oc}(R)$; n can be positive or negative. We will first show how differentiation is carried out.

Let p (x) = [6, 0.3) + [0.31, 6.7)x + [8, –9)x^2 + [11, 15)x^3 + [0, –30)x^4 be an interval polynomial then the derivative of p(x) is

$$\frac{d}{dx}(p(x)) = \frac{d}{dx}([6,\ 0.3) + [0.31,\ 6.7)x + [8,\ -9)x^2 + [11,\ 15)x^3 + [0,\ -30)x^4)$$

$$= [0, 0] + [0.\ 31, 6.7) + [16, -18)x + [33, 45)x^2 + [0, -120)x^3.$$

Let $q(x) = [9, 0] + [-8, 1]x + [-5, 10]x^3 + [7, 7]x^4 + [10, -11]x^5 + [7, 10]x^7$ be an interval polynomial.

The derivative of q(x) is given by

$$q'(x) = \frac{dq(x)}{dx} = \frac{d}{dx}([9, 0] + [-8, 1]x + [-5, 10]x^3 + [7, 7]x^4 + [10, -11]x^5 + [7, 10]x^7)$$

$$= [-8, 1] + 3\,[-5, 10]x^2 + 4\,[7, 7]x^3 + 5\,[10, -11]x^4 + 7\,[7, 10]x^6$$

$$= [-8, 1] + [-15, 30]x^2 + [28, 28]x^3 + [50, -55]x^4 + [49, 70]x^6.$$

We can find the second derivative

$$\frac{d}{dx}(q'(x)) = q''(x) = 2\,[-15, 30]x + 3\,[28, 28]x^2 + 4\,[50, -55]x^3 + 6\,[49, 70]x^5$$

$$= [-30, 60]x + [84, 84]x^2 + [200, -220]x^3 + [294, 420]x^5.$$

We can find third, forth or upto seventh derivatives. We show if $q(x) = [q_1(x), q_2(x)]$ represented as the polynomial interval then we can find the derivatives of $q_1(x)$ and $q_2(x)$ separately as follows:

We will show the derivative of an interval polynomial is the same as the derivative of the polynomial interval.

Now

$$q(x) = [9,0] + [-8, 1]x + [-5, 10]x^3 + [7, 7]x^4 + [10, -11]x^5 + [7, 10]x^7$$

$$= [q_1(x), q_2(x)]$$

$$= [9-8x - 5x^3 + 7x^4 + 10x^5 + 7x^7,\ x + 10x^3 + 7x^4 - 11x^5 + 10x^7]$$

$$q'(x) = \frac{d}{dx}\ (q(x)) = \left[q_1'(x), q_2'(x)\right]$$

$$= \left[\frac{d}{dx}(q_1(x)), \frac{d}{dx}(q_2(x))\right]$$

$= [-8 - 15x^2 + 28x^3 + 50x^4 + 49x^6,\ 1 + 30x^2 + 28x^3 - 55x^4 + 70x^6]$

$= [-8, 1] + [-5, 30]x^2 + [28, 28]x^3 + [50, -55]x^4 + [49, 70]x^6$
$= q'(x)$.

Thus we can easily prove that if $p(x) = [p_1(x), p_2(x)]$ is an interval polynomial than the derivative of p(x) is the same as the derivatives of $p_1(x)$ and $p_2(x)$ and vice versa.

Now on similar lines we can define the integration of p(x), where p(x) is the interval polynomial.

$$\int p(x)dx \quad = \int [p_1(x), p_2(x)]\, dx$$

$$= \int p_1(x)\, dx + \int p_2(x)\, dx.$$

We will illustrate this situation by an example.

Let
$p(x) = [9, 2] + [-2, 7]x + [-7, -9]x^2 + [3, 9]x^3 + [8, 10]x^5$ be an interval polynomial.

$p(x)= [p_1(x), p_2(x)]$
$= [9 - 2x - 7x^2 + 3x^3 + 8x^5,\ 2 + 7x - 9x^2 + 9x^3 + 10x^5]$.

$$\int p(x)\, dx \quad = \int [p_1(x), p_2(x)]\, dx.$$

Now

$\int p(x)dx = \int [9, 2]dx + \int [-2, 7]x\, dx + \int [-7, -9]x^2\, dx + \int [3, 9]x^3\, dx + \int [8, 10]x^5\, dx$

$$= [9, 2]\, x + \frac{[-2,7]x^2}{2} + \frac{[-7,-9]x^3}{3} + \frac{[3,9]x^4}{4} + \frac{[8,10]x^6}{6} + C.$$

$= \quad [9, 2]x + [-1, 3.5]x^2 + [-7/3, -3]x^3 + [0.75, 2.25]x^4 +$

$[4/3, 5/3]x^6 + C$

where $C \in N_c(R)$.

Now we can write
$\int p(x)\, dx = [\int p_1(x)\, dx, \int p_2(x)\, dx]$

$= [\int 9-2x - 7x^2 + 3x^3 + 8x^5\, dx, \int 2+ 7x - 9x^2 + 9x^3 + 10x^5 dx]$

$$= \left[9x - \frac{2x^2}{2} - \frac{7x^3}{3} + \frac{3x^4}{4} + \frac{8x^6}{6} + C_1, \right.$$

$$\left. 2x + \frac{7x^2}{2} - \frac{9x^3}{3} + \frac{9x^4}{4} + \frac{10x^6}{6} + C_2 \right]$$

$= [9, 2]\, x + [-1, 3.5]x^2 + [-7/3, -3]x^3 + [3/4, 9/4]x^4$

$+ [8/6, 10/6]x^6 + [C_1, C_2].$

We see $\int p(x)\, dx = [\int p_1(x)\, dx, \int p_2(x)\, dx]$.

The differentiation and integration of interval polynomials is a matter of routine and can be carried out easily with the very simple modification by writing a interval polynomial as the polynomial interval.

Now we proceed onto define the notion of interval matrices whose entries are interval polynomials is one or more variables.

Let $A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$ be a matrix where

$a_{ij} \in N_c(R)[x] = \{p(x) = p_0 = p_1x + \ldots + p_nx^n \mid p_i \in N_c(R)\}$;
$1 \leq i, j \leq n$.

We call A as the interval polynomial matrix or interval matrices with interval polynomial entries.

We will give example of this situation.

$$A = \begin{bmatrix} [0,1)x^3 + [3,0)x^2 & [16,10)x^7 & [1,1)x \\ [6,7)x^3 + [1,1)x^8 & [0,0) & [5,7)x^3 \\ [6,10)x^7 & [1,0)x + [2,3)x^7 & [0,9)x^8 \end{bmatrix}$$

be the interval matrix with polynomial entries from N_{co} (Z) [x].

A interval matrix is differentiated by differentiating every element in the matrix in the classical way.

$$\frac{dA}{dx} = \begin{bmatrix} 3[0,1)x^2 + 2[3,0)x & 7[16,10)x^6 & [1,1) \\ 3[6,7)x^2 + 8[1,1)x^7 & [0,0) & 3[5,7)x^2 \\ 7[6,10)x^6 & [1,0) + 7[2,3)x^6 & 8[0,9)x^7 \end{bmatrix}$$

$$= \begin{bmatrix} [0,3)x^2 + [6,0)x & [112,70)x^6 & [1,1) \\ [18,21)x^2 + [8,8)x^7 & [0,0) & [15,21)x^2 \\ [42,70)x^6 & [1,0) + [14,21)x^6 & [0,72)x^7 \end{bmatrix}.$$

We can differentiate $\frac{dA}{dx}$ the second time

$$\frac{d^2A}{dx} = \begin{bmatrix} 2[0,3)x + [6,0) & 6[112,70)x^5 & [0,0) \\ 2[18,21)x + 7[8,8)x^6 & [0,0) & 2[15,21)x \\ 6[42,70)x^5 & 6[14,21)x^5 & 7[0,72)x^6 \end{bmatrix}$$

and so on. We can find any number of successive derivatives of A.

Now we show we can write A = $[A_1, A_2)$ where A_1 and A_2 are matrix with polynomial entries and differential of A_1 and A_2 remains the same.

Now A = $[A_1, A_2)$

$$= \left[\begin{pmatrix} 3x^2 & 16x^7 & x \\ 6x^3 + x^8 & 0 & 5x^3 \\ 6x^7 & x + 2x^7 & 0 \end{pmatrix}, \begin{bmatrix} x^3 & 10x^7 & x \\ 7x^3 + x^8 & 0 & 7x^3 \\ 10x^7 & 3x^7 & 9x^8 \end{bmatrix}\right).$$

$$\text{Now } \frac{dA}{dx} = \left[\frac{dA_1}{dx}, \frac{dA_2}{dx}\right)$$

$$= \left[\begin{bmatrix} 6x & 112x^6 & 1 \\ 18x^2 + 8x^7 & 0 & 15x^2 \\ 42x^6 & 1 + 14x^6 & 0 \end{bmatrix}, \begin{bmatrix} 3x^2 & 70x^6 & 1 \\ 21x^2 + 8x^7 & 0 & 21x^2 \\ 70x^6 & 21x^6 & 72x^7 \end{bmatrix}\right)$$

$$= \begin{bmatrix} [6,0)x + [0,3)x^2 & [112,70)x^6 & [1,1) \\ [18,21)x^2 + [8,8)x^7 & [0,0) & [15,21)x^2 \\ [42,70)x^6 & [1,0) + [14,21)x^6 & [0,72)x^7 \end{bmatrix} = \frac{dA}{dx}\ .$$

$$\text{Thus } \frac{dA}{dx} = \left[\frac{dA_1}{dx}, \frac{dA_2}{dx}\right).$$

Suppose we have the polynomial ring in two variables say x, y with interval coefficients that is

$N_c(R)[x, y] = \{p(x, y) \mid$ the coefficients of $p(x, y)$ is in $N_c(R)\}$. It is easily verified $N_c(R)[x,y]$ is just a ring and an integral domain. Likewise we can have polynomial ring in three variables say $N_c(R)[x, y, z]$ and any n variables; the only difference being that R [x, y, z] take their coefficients from R (reals) where as $N_c(R)[x, y, z]$ take their interval coefficients from $N_c(R) = \{[a, b] \mid a, b \in R\}$ (It is important to note that $N_c(R)$ can be replaced by $N_{oc}(R)$ or $N_{co}(R)$ or $N_o(R)$ and R can also be replaced by Z or Q).

Now if we have interval polynomials with more than one variable then we can only define the partial derivative on them. This is direct, however we will illustrate this situation by some examples / illustrations.

Let
$p(x, y) = (0, 7) + (2, 5)x^3y + (–7, 0)x^2 y^3 + (0, –14)x^4 y^4 + (9, 8)x^5 y^2 \in N_o(Q)[x, y]$

Now the partial derivative of p(x, y) with respect x and y are as follows:

$$\frac{\partial(p(x,y))}{\partial x} = 0 + 3\,(2, 5)x^2 y + 2\,(–7, 0)xy^3 + 4\,(0, –14)x^3 y^4 + 5\,(9, 8)x^4 y^2$$

$$= (6, 15)\,x^2 y + (–14, 0)xy^3 + (0, –56)x^3 y^4 + (45, 40)x^4 y^2.$$

$$\frac{\partial(p(x,y))}{\partial y} = 0 + (2, 5)x^3 + 3\,(–7, 0)x^2 y^2 + 4(0, –14)x^4 y^3 + 2\,(9, 8)x^5 y$$

$$= (2, 5)x^3 + (–21, 0)x^2 y^2 + (0, –56)x^4 y^3 + (18, 16)x^5 y.$$

Now we find the second derivative

$$\frac{\partial}{\partial y}\left(\frac{\partial(p(x,y))}{\partial x}\right) = (6, 15)x^2 + 3\,(-14, 0)xy^2 + 4\,(0, -56)x^3y^3 + 2\,(45, 40)x^4y$$

$$= (6, 15)x^2 + (-42, 0)xy^2 + (0, -224)x^3y^3 + (90, 80)x^4y$$

$$\frac{\partial}{\partial x}\left(\frac{\partial(p(x,y))}{\partial y}\right) = 3\,(2, 5)x^2 + 2\,(-21, 0)xy^2 + 4\,(0, -56)x^3y^3 + 5\,(18, 16)x^4y$$

$$= (6, 15)x^2 + (-42, 0)xy^2 + (0, -224)x^3y^3 + (90, 80)x^4y.$$

It is easily verified

$$\frac{\partial}{\partial x}\left(\frac{\partial(p(x,y))}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial(p(x,y))}{\partial x}\right).$$

Thus

$$\frac{\partial^2 p(xy)}{\partial x\,\partial y} = \frac{\partial^2 p(xy)}{\partial y\,\partial x}.$$

Thus we see the partial derivatives of interval polynomials behave like partial derivatives in usual polynomials.

Now we proceed onto give matrix whose entries are interval polynomials in two variables.

Consider

$$A = \begin{bmatrix} [0,3]xy+[6,0]x^2 & [7,0] & [8,9]x^3y^9 & [9,8]y^3x+[7,9]y^2 \\ [7,0]x^3y+[2,2]xy^3 & [8,1]x^3y^2 & [5,0]x^2y+[7,2]xy^3 & [8,9] \\ [5,1]x^7+[3,2]xy^5 & [3,12]x^3y+[0,7]y^5 & [7,9]x^3y^4+[1,1]y & [9,9]x^3 \\ [3,2]x^6y^3 & [5,2]x^7y^2+[3,1]y & [3,12]x^8y^3 & [8,-11]y^5 \end{bmatrix}$$

be a interval polynomial matrix with entries from $N_c(R)[x, y]$. We find the first and second partial derivatives of A with respect to x and y.

$$\frac{\partial A}{\partial x} = \begin{pmatrix} [0,3]y+[12,0]x & 0 & [24,27]x^2y^9 & [9,8]y^3 \\ [21,0]x^2y+[2,2]y^3 & [24,3]x^2y^2 & [10,0]xy+[7,2]y^3 & 0 \\ [35,7]x^6+[3,2]y^5 & [9,36]x^2y & [21,27]x^2y^4 & [27,27]x^2 \\ [18,12]x^5y^3 & [35,14]x^6y^2 & [24,96]x^7y^3 & 0 \end{pmatrix}.$$

$$\frac{\partial^2 A}{\partial y\,\partial x} = \begin{pmatrix} [0,3] & 0 & [216,243]x^2y^8 & [27,24]y^2 \\ [21,0]x^2+[6,6]y^2 & [48,6]x^2y & [10,0]x+[21,6]y^2 & 0 \\ [15,10]y^4 & [9,36]x^2 & [84,108]x^2y^3 & 0 \\ [54,36]x^5y^2 & [70,28]x^6y & [72,288]x^7y^2 & 0 \end{pmatrix}.$$

$$\frac{\partial^2 A}{\partial x^2} = \begin{pmatrix} [12,0] & 0 & [48,54]xy^9 & 0 \\ [42,0]xy & [48,6]xy^2 & [10,0]y & 0 \\ [210,42]x^5 & [18,72]xy & [42,54]xy^4 & [54,54]x \\ [90,60]x^4y^3 & [210,84]x^5y^2 & [168,672]x^6 & 0 \end{pmatrix}.$$

Now $$\frac{\partial A}{\partial y} = \begin{pmatrix} [0,3]x & [0,0] \\ [7,0]x^3 + [6,6]xy^2 & [16,2]x^3y \\ [15,10]xy^4 & [3,12]x^3 + [0,35]y^4 \\ [9,6]x^6y^2 & [10,4]x^7y + [3,1] \end{pmatrix.$$

$$\begin{matrix} [72,81]x^3y^8 & [27,24]y^2x + [14,18]y \\ [5,0]x^2 + [21,6]xy^2 & 0 \\ [28,36]x^3y^3 + [1,1] & 0 \\ [9,36]x^8y^2 & [40,-55]y^4 \end{matrix}\Bigg).$$

$$\frac{\partial^2 A}{\partial x\, \partial y} = \Bigg(\begin{matrix} [0,3] & 0 \\ [21,0]x^2 + [6,6]y^2 & [48,6]x^2y \\ [15,10]y^4 & [9,36]x^2 \\ [54,36]x^5y^2 & [70,28]x^6y \end{matrix}$$

$$\begin{matrix} [216,243]x^2y^8 & [27,24]y^2 \\ [10,0]x + [21,6]y^2 & 0 \\ [84,108]x^2y^3 & 0 \\ [72,288]x^7y^2 & 0 \end{matrix}\Bigg).$$

Clearly $\frac{\partial^2 A}{\partial x\, \partial y} = \frac{\partial^2 A}{\partial y\, \partial x}$.

Now we find $\frac{\partial^2 A}{\partial y^2}$

$$= \begin{pmatrix} 0 & 0 & [516,648]x^3y^7 & [54,48]yx + [14,18] \\ [12,12]xy & [16,2]x^3 & [42,12]xy & 0 \\ [60,40]xy^3 & [0,140]y^3 & [84,108]x^3y^2 & 0 \\ [18,12]x^6y & [10,4]x^7 & [18,72]x^8y & [160,-220]y^3 \end{pmatrix}.$$

We will give an example of a matrix with interval polynomials in three variables and give their partial derivatives.

$$\text{Let } M = \begin{bmatrix} [0,3]xyz^3 + [2,1]x^3z^3 & [4,5]x^3y^3z + [0,1]xy \\ [0,10]x^2y^2 + [3,2]xy^2z^3 & [7,2]x^3z^2 + [3,7]x^6 \end{bmatrix}.$$

Clearly elements of M are from $N_c(Q)[x, y, z]$ = {all polynomials in the three variables x, y, z with coefficients from $N_c(Q)$}.

Now we find

$$\frac{\partial M}{\partial x} = \begin{bmatrix} [0,3]yz^3 + [6,3]x^2z^3 & [12,15]x^3y^3z + [0,1]y \\ [0,20]xy^2 + [3,2]y^2z^3 & [21,6]x^2z^2 + [18,42]x^5 \end{bmatrix}.$$

$$\frac{\partial M}{\partial y} = \begin{bmatrix} [0,3]xz^3 & [12,15]x^3y^2z + [0,1]y \\ [0,20]x^2y + [6,4]xyz^3 & [0,0] \end{bmatrix}.$$

$$\frac{\partial M}{\partial z} = \begin{bmatrix} [0,9]xyz^2 + [6,3]x^3z^2 & [4,5]x^3y^3 \\ [9,6]xy^2z^2 & [14,4]x^3z \end{bmatrix}.$$

$$\frac{\partial^2 M}{\partial y\,\partial x} = \begin{bmatrix} 0 & [36,45]x^3y^2z+[0,1] \\ [0,40]xy+[6,4]yz^3 & 0 \end{bmatrix}.$$

$$\frac{\partial^3 M}{\partial z\,\partial y\,\partial x} = \begin{bmatrix} 0 & [36,45]x^3y^2 \\ [18,12]yz^2 & 0 \end{bmatrix}.$$

$$\frac{\partial^2 M}{\partial z\,\partial x} = \begin{bmatrix} [18,9]x^2z^2 & [12,15]x^3y^3 \\ [9,6]y^2z^2 & [42,12]x^2z \end{bmatrix}.$$

$$\frac{\partial^3 M}{\partial y\,\partial z\,\partial x} = \begin{bmatrix} 0 & [36,45]x^3y^2 \\ [18,12]yz^2 & 0 \end{bmatrix}.$$

Thus interested reader can find higher derivatives as it is a matter of routine.

Now we can also integrate a matrix which entries are interval polynomials in the single variable x.

Let

$$A = \begin{bmatrix} [0,3]x^3+[7,1]x^7+[2,1] & [6,5]x^3+[7,0]x & [-3,12] \\ [9,2]x^5+[2,3]x^2 & [8,1]x^7+[3,2]x^3 & [7,5]x^3+[3,2]x^2+[7,5] \\ [21,5]x^9+[3,10]x^2+[1,3]x & [9,1]x^9 & [9,5]x^5+[13,2]x^4+[3,2]x+[0,1] \end{bmatrix}$$

be a matrix with interval polynomial entries, with elements from $N_c(R)[x]$ the integral of A denoted by

$\int$ Adx =

$$\begin{bmatrix} [0,3/4]x^4 + [7/8,1/8]x^8 + [2,1]x & [6/4,5/4]x^4 + [7/2,0]x^2 & [-3,12]x \\ [9/6,1/3]x^6 + [2/3,1]x^3 & [1,1/8]x^8 + [3/4,2/4]x^4 & [7/5,5/4]x^4 + [1,2/3]x^3 + [7,5]x \\ [21/10,5/10]x^{10} + [3/3,10/3]x^3 + [1/2,3/2]x^2 & [9/10,1/10]x^{10} & [9/6,1/3]x^6 + [13/5,2/5]x^5 + [3/2,1]x^2 + [0,1]x \end{bmatrix}$$

$$+ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \text{ with } a_i \in N_c(R);\ 1 \le i \le 9.$$

We can integrate matrix with interval polynomial entries from $N_c(R)$.

Thus the matrix of interval polynomial integration and differentiation are carried as a matter of routine. We can also write the interval matrix M of polynomials from $N_c(R)$ (or $N_{oc}(R)$ or $N_o(R)$ or $N_{co}(R)$ as matrix polynomial intervals and all operations on them can be carried out as a matter of routine.

We will illustrate this situation by a simple examples.

Suppose A = $\begin{bmatrix} [0,5) + [2,-10)x^3 + [6,0)x^5 & [5,2) + [11,3)x + [7,11)x^2 + [0,2]x^3 \\ [9,1) + [20,1)x^2 + [7,3)x^4 & [3,0) + [12.4)x^3 + [1,0)x^7 \end{bmatrix}$

be a interval matrix polynomial with interval polynomials from $N_{co}(R)[x]$ then

$$\frac{dA}{dx} = \begin{bmatrix} [6,-30)x^2 + [30,0)x^4 & [11,3) + [14,22)x + [0,6)x^2 \\ [40,2)x + [28,12)x^3 & [36,12)x^2 + [7,0)x^6 \end{bmatrix}.$$

Now this interval matrix polynomial can be rewritten as matrix interval polynomial of A = $[A_1\ A_2)$

$$= \left[\begin{pmatrix} 2x^3 + 6x^5 & 5 + 11x + 7x^2 \\ 9 + 20x^2 + 7x^4 & 3 + 12x^3 + x^7 \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} 5 - 10x^3 & 2 + 3x + 11x^2 + 2x^3 \\ 1 + x^2 + 3x^4 & 4x^3 \end{pmatrix} \right).$$

$$\frac{dA}{dx} = \left[\frac{dA_1}{dx}, \frac{dA_2}{dx} \right)$$

$$= \left[\begin{pmatrix} 6x^2 + 30x^4 & 11 + 14x \\ 40x + 28x^3 & 36x^2 + 7x^6 \end{pmatrix}, \begin{pmatrix} -30x^3 & 3 + 22x + 6x^2 \\ 2x + 12x^3 & 12x^2 \end{pmatrix} \right)$$

$$= \begin{bmatrix} [6,-30)x^2 + [30,0)x^4 & [11,3) + [14,22)x + [0,6)x^2 \\ [40,2)x + [28,12)x^3 & [36,12)x^2 + [7,0)x^6 \end{bmatrix}.$$

We see $\frac{dA}{dx} = \left[\frac{dA_1}{dx}, \frac{dA_2}{dx} \right)$.

Thus we see we can without any difficulty write the derivative of any interval polynomial matrix as derivative of matrix interval polynomial and both are equal.

Now we find

$$\int Adx = \begin{bmatrix} [0,5)x + [2/4,-10/4)x^4 + [1,0)x^6 & [5,2)x + [11/2,3/2)x^2 + [7/3,11/3)x^3 + [0,2/4)x^4 \\ [9,1)x + [20/3,1/3)x^3 + [7/5,3/5)x^5 & [3,0)x + [3,1)x^4 + [1/8,0)x^8 \end{bmatrix}$$

$$\int A_1\, dx + \int A_2\, dx = \left[\int \begin{pmatrix} 2x^3 + 6x^5 & 5 + 11x + 7x^2 \\ 9 + 20x^2 + 7x^4 & 3 + 12x^3 + x^7 \end{pmatrix} dx, \int \begin{pmatrix} 5 - 10x^3 & 2 + 3x + 11x^2 + 2x^3 \\ 1 + x^2 + 3x^4 & 4x^3 \end{pmatrix} dx \right)$$

$$= \left[\begin{pmatrix} \frac{2x^4}{4} + \frac{6x^6}{6} & 5x + \frac{11x^2}{2} + \frac{7x^3}{3} \\ 9x + \frac{20x^3}{3} + \frac{7x^5}{5} & 3x + \frac{12x^4}{4} + \frac{x^8}{4} \end{pmatrix}, \begin{pmatrix} 5x - \frac{10x^4}{4} & 2x + \frac{3x^2}{2} + \frac{11x^3}{3} + \frac{2x^4}{4} \\ x + \frac{x^3}{3} + \frac{3x^5}{5} & \frac{4x^4}{4} \end{pmatrix} \right)$$

$$= \left[\begin{pmatrix} \left[\frac{2}{4} - \frac{10}{4}\right)x^4 + [0,5)x + [6,0)x^6 \\ [9,1)x + \left[\frac{20}{3}, \frac{1}{3}\right)x^3 + \left[\frac{7}{5}, \frac{3}{5}\right)x^5 \end{pmatrix}\right.$$

$$\left.\begin{pmatrix} [5,2)x + \left[\frac{11}{2}, \frac{3}{2}\right)x^2 + \left[\frac{7}{3}, \frac{11}{3}\right)x^3 + \left[0, \frac{1}{2}\right)x^4 \\ [3,0)x + [3,1)x^4 + \left[\frac{1}{4}, 0\right)x^8 \end{pmatrix}\right]$$

is the integral of the matrix interval polynomials.

Now having seen integration and differentiation of interval matrix polynomials and matrix integral polynomials we proceed onto give their applications.

Chapter Seven

APPLICATIONS OF INTERVAL MATRICES AND POLYNOMIALS BUILT USING NATURAL CLASS OF INTERVALS

This chapter has two sections. First section indicates the derivation of some classical results in case of interval matrices using the natural class of intervals from $N_c(R)$ or $N_o(R)$ or $N_{oc}(R)$ or $N_{co}(R)$ (R replaced by Z or Q). Second section of this chapter suggest some applications.

7.1 Properties of Interval Matrices

In this section the notion of finding determinant of interval matrices and finding inverse of interval matrices are illustrated by examples. We find the determinant of an interval matrix A.

Example 7.1.1: Let

$$A = \begin{bmatrix} [3,2] & [4,0] \\ [5,2] & [0,-7] \end{bmatrix} = [A_1, A_2]$$

be a 2×2 interval matrix.

$$|A| = \left\| \begin{bmatrix} [3,2] & [4,0] \\ [5,2] & [0,-7] \end{bmatrix} \right\| = [3, 2]\ [0, -7], - [4, 0]\ [5, 2]$$

$$= [0, -14] - [20, 0]$$

$$= [-20, -14].$$

$$\det A = |A| = \det [A_1, A_2]$$

$$= [\det A_1, \det A_2]$$

$$= \left[\det \begin{bmatrix} 3 & 4 \\ 5 & 0 \end{bmatrix}, \det \begin{bmatrix} 2 & 0 \\ 2 & -7 \end{bmatrix} \right]$$

$$= [-20, -14]$$

$$= \det A = |A|.$$

Example 7.1.2: Let

$$A = \begin{bmatrix} [3,0) & [0,0) & [1,1) \\ [2,-1) & [2,-1) & [3,1) \\ [0,4) & [1,0) & [0,1) \end{bmatrix}$$

be a 3 × 3 interval matrix with entries from $N_{co}(Q)$. To find the determinant of A.

$$\det A = \left\| \begin{bmatrix} [3,0) & [0,0) & [1,1) \\ [2,-1) & [2,-1) & [3,1) \\ [0,4) & [1,0) & [0,1) \end{bmatrix} \right\|$$

$$= [3, 0) \left\| \begin{bmatrix} [2,-1) & [3,1) \\ [1,0) & [0,1) \end{bmatrix} \right\| - [0, 0)$$

$$\left\|\begin{bmatrix}[2,-1) & [3,1) \\ [0,4) & [0,1)\end{bmatrix}\right\| + [1, 1) \left\|\begin{bmatrix}[2,-1) & [2,-1) \\ [0,4) & [1,0)\end{bmatrix}\right\|$$

= [3, 0) [[2, -1) [0, 1) – [3, 1) [1, 0)] – [0, 0) + [1, 1) [[2, -1) [1, 0) – [2, -1) [0, 4)]

= [3, 0) [[0, -1) – [3, 0)] + [1, 1) ([2, 0) – [0, -4))

= [3, 0) [-3, -1) + p1, 1) [2, 4)

= [-9, 0) + [2, 4) = [-7, 4).

Now consider

$$A = [A_1, A_2] = \left[\begin{pmatrix}3 & 0 & 1 \\ 2 & 2 & 3 \\ 0 & 1 & 0\end{pmatrix}, \begin{pmatrix}0 & 0 & 1 \\ -1 & -1 & 1 \\ 4 & 0 & 1\end{pmatrix}\right).$$

$$\det A = \det [A_1, A_2] = [\det A_1, \det A_2]$$

$$= \left[\det\begin{pmatrix}3 & 0 & 1 \\ 2 & 2 & 3 \\ 0 & 1 & 0\end{pmatrix}, \det\begin{pmatrix}0 & 0 & 1 \\ -1 & -1 & 1 \\ 4 & 0 & 1\end{pmatrix}\right]$$

$$= [3\ (2.0 - 3.1) + 1\ (2.1 - 2.0),\ 1 \times 4)$$

$$= [-9 + 2, 4) = [-7, 4).$$

Thus $\det A = \det [A_1, A_2)$

$$= [\det A_1, \det A_2) = [-7, 4).$$

Example 7.1.3: Let

$$A = \begin{bmatrix} (1,2) & (2,0) & (-1,1) & (3,0) \\ (2,1) & (1,1) & (-2,0) & (3,1) \\ (3,0) & (1,2) & (2,-1) & (1,0) \\ (1,1) & (-1,0) & (0,2) & (2,1) \end{bmatrix}$$

be a 4×4 interval matrix with entries from $N_o(Q)$.

$$\det A = |A| = \left| \begin{bmatrix} (1,2) & (2,0) & (-1,1) & (3,0) \\ (2,1) & (1,1) & (-2,0) & (3,1) \\ (3,0) & (1,2) & (2,-1) & (1,0) \\ (1,1) & (-1,0) & (0,2) & (2,1) \end{bmatrix} \right|$$

$$= (1, 2) \left| \begin{pmatrix} (1,1) & (-2,0) & (3,1) \\ (1,2) & (2,-1) & (1,0) \\ (-1,0) & (0,2) & (2,1) \end{pmatrix} \right|$$

$$- \ (2, 0) \left| \begin{pmatrix} (2,1) & (-2,0) & (3,1) \\ (3,0) & (2,-1) & (1,0) \\ (1,1) & (0,2) & (2,1) \end{pmatrix} \right|$$

$$+ (-1, 1) \left| \begin{pmatrix} (2,1) & (1,1) & (3,1) \\ (3,0) & (1,2) & (1,0) \\ (1,1) & (-1,0) & (2,1) \end{pmatrix} \right|$$

$$- (3,0) \left| \begin{pmatrix} (2,1) & (1,1) & (-2,0) \\ (3,0) & (1,2) & (2,-1) \\ (1,1) & (-1,0) & (0,2) \end{pmatrix} \right|$$

$$= (1,\ 2)\ \Bigg[\ (1,1)\left|\begin{bmatrix} (2,-1) & (1,0) \\ (0,2) & (2,1) \end{bmatrix}\right| - (-2,0)\ \left|\begin{pmatrix} (1,2) & (1,0) \\ (-1,0) & (2,1) \end{pmatrix}\right|$$

$$+ (3,1)\ \left|\begin{pmatrix} (1,2) & (2,-1) \\ (-1,0) & (0,2) \end{pmatrix}\right|\ \Bigg]$$

$$- (2,\ 0)\ \Bigg[\ (2,1)\left|\begin{bmatrix} (2,-1) & (1,0) \\ (0,2) & (2,1) \end{bmatrix}\right|$$

$$- (-2,0)\ \left|\begin{pmatrix} (3,0) & (1,0) \\ (1,1) & (2,1) \end{pmatrix}\right|$$

$$+ (3,1)\ \left|\begin{pmatrix} (3,0) & (2,-1) \\ (1,1) & (0,2) \end{pmatrix}\right|\ \Bigg]$$

$$+ (-1,\ 1)\ \Bigg[\ (2,1)\left|\begin{bmatrix} (1,2) & (1,0) \\ (-1,0) & (2,1) \end{bmatrix}\right| -\ (1,1)\ \left|\begin{pmatrix} (3,0) & (1,0) \\ (1,1) & (2,1) \end{pmatrix}\right|$$

$$+ (3,1)\ \left|\begin{pmatrix} (3,0) & (1,2) \\ (1,1) & (-1,0) \end{pmatrix}\right|\ \Bigg]$$

$$- (3,\ 0)\ \Bigg[\ (2,1)\left|\begin{bmatrix} (1,2) & (2,-1) \\ (-1,0) & (0,2) \end{bmatrix}\right| -$$

$$(1,1)\ \left|\begin{pmatrix} (3,0) & (2,-1) \\ (1,1) & (0,2) \end{pmatrix}\right| +\ (-2,0)\ \left|\begin{pmatrix} (3,0) & (1,2) \\ (1,1) & (-1,0) \end{pmatrix}\right|\ \Bigg]$$

= (1, 2) [[(4, –1) – (0, 0)] – (–2, 0) [(2, 2) – (–1, 0)]
+ (3, 1) [(0, 4) – (–2,0)]] – [(2, 0) [(2, 1) (4, –1) – (0, 0)]

$- (-2,0)\ [(6, 0) - (1, 0)]$
$+ (3, 1)\ [(0,0) - (2, -1)]] +$
$(-1, 1)\ [(2, 1)\ [(2, 2) - (-1, 0)]$
$- (1,1)\ [(6, 0) - (1, 0)]$
$+ (3, 1)\ [(-3, 0) - (1, 2)]]$

$- (3, 0)\ [(2, 1)\ [(0, 4) - (-2, 0)] - (1, 1)\ [(0, 0) - (2, -1)]$
$+ (-2, 0)\ [(-3, 0) - (1, 2)]]$

$= (1, 2)\ [(4, -1) - (-6, 0) + (6, 4)\ (-2, 0)\ [(8, -1) -$
$(-10, 0) - (6, -1)]$
$+ (-1, 1)\ [(6, 2) - (5, 0) + (-12, -2)]$
$- (3, 0)\ [(4, 4) + (2, -1) + (8, 0)]$

$= (1, 2)\ (16, +3) - (-2, 0)\ (12, 0) + (-1, 1)\ (-11, 0) -$
$(3, 0)\ (14, 3)$

$= (16, 6) - (24, 0) + (11, 0) - (42, 0)$

$= (-39, 6),$

Now let $A = [A_1, A_2]$ we find matrix interval determinant

$$|A| = \det A = \det [A_1, A_2] = [|A_1|, |A_2|]$$

$$= \left[\left| \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 1 & -2 & 3 \\ 3 & 1 & 2 & 1 \\ 1 & -1 & 0 & 2 \end{bmatrix} \right|, \left| \begin{bmatrix} 2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 2 & -1 & 0 \\ 1 & 0 & 2 & 1 \end{bmatrix} \right| \right]$$

$$= \left[1 \begin{vmatrix} 1 & -2 & 3 \\ 1 & 2 & 1 \\ -1 & 0 & 2 \end{vmatrix} - 2 \begin{vmatrix} 2 & -2 & 3 \\ 3 & 2 & 1 \\ 1 & 0 & 2 \end{vmatrix} + (-1) \begin{vmatrix} 2 & 1 & 3 \\ 3 & 1 & 1 \\ 1 & -1 & 2 \end{vmatrix} - 3 \begin{vmatrix} 2 & 1 & -2 \\ 3 & 1 & 2 \\ 1 & -1 & 0 \end{vmatrix}, \right.$$

$$2\begin{vmatrix}1 & 0 & 1\\ 2 & -1 & 0\\ 0 & 2 & 1\end{vmatrix} - 0 \Bigg| \qquad \Bigg| + 1\begin{vmatrix}1 & 1 & 1\\ 0 & 2 & 0\\ 1 & 0 & 1\end{vmatrix} - 0 \Bigg| \qquad \Bigg|\Bigg]$$

= [16 – 2 (12) + (–1) (–11) – 3 (14), 2 [1 (–1 – 0) + 1 [2, –0] + 1 [–2]]]

= [–39, 6].

Thus we see the determinant of an interval matrix is the same as that of the matrix interval.

Now we find the inverse of interval matrix and matrix interval.

Example 7.1.4: Let

$$M = \begin{bmatrix}[2,3] & [1,5]\\ [4,2] & [7,1]\end{bmatrix}$$

be a 2×2 interval matrix. To find inverse of M.

$$\text{Clearly } |A| = \begin{vmatrix}[2,3] & [1,5]\\ [4,2] & [7,1]\end{vmatrix}$$

$$= ([2, 3]\,[7, 1] – [1, 5]\,[4, 2])$$

$$= [[14, 3] – [4, 10]]$$

$$= [10, –7] \neq [0, 0].$$

$$A^{-1} = \frac{1}{[10,-7]}\begin{bmatrix}[7,1] & [-4,-2]\\ [-1,-5] & [2,3]\end{bmatrix}^{t}$$

$$= \frac{1}{[10,-7]}\begin{bmatrix} [7,1] & [-1,-5] \\ [-4,-2] & [2,3] \end{bmatrix}$$

$$= \begin{bmatrix} [7/10,-1/7] & [-1/10,5/7] \\ [-4/10,2/7] & [2/10,-3/7] \end{bmatrix}$$

Now $A.A^{-1} = \begin{bmatrix} [2,3] & [1,5] \\ [4,2] & [7,1] \end{bmatrix} \begin{bmatrix} [7/10,-1/7] & [-1/10,5/7] \\ [-4/10,2/7] & [2/10,-3/7] \end{bmatrix}$

$$= \begin{bmatrix} [1,1] & [0,0] \\ [0,0] & [1,1] \end{bmatrix}$$

$= I_{2\times2}$.

Now we show if $A = [A_1, A_2] = \left[\begin{bmatrix} 2 & 1 \\ 4 & 7 \end{bmatrix}, \begin{bmatrix} 3 & 5 \\ 2 & 1 \end{bmatrix}\right]$

Then $A^{-1} = [A_1^{-1}, A_2^{-1}]$.

Thus $|A| = |[A_1, A_2]|$

$= [|A_1|, |A_2|]$

$= [10, -7]$.

Now $A^{-1} = \left[\frac{1}{10}\begin{bmatrix} 7 & -1 \\ -4 & 2 \end{bmatrix}, \frac{1}{-7}\begin{bmatrix} 1 & -5 \\ -2 & 3 \end{bmatrix}\right]$

$$= \left(\begin{bmatrix} 7/10 & -1/10 \\ -4/10 & 2/10 \end{bmatrix}, \begin{bmatrix} -1/7 & 5/7 \\ 2/7 & -3/7 \end{bmatrix}\right)$$

Now A, $A^{-1} = \left[\begin{bmatrix} 2 & 1 \\ 4 & 7 \end{bmatrix}, \begin{bmatrix} 3 & 5 \\ 2 & 1 \end{bmatrix}\right] \times$

$$\left[\begin{bmatrix} 7/10 & -1/10 \\ -4/10 & 2/10 \end{bmatrix}, \begin{bmatrix} -1/7 & 5/7 \\ 2/7 & -3/7 \end{bmatrix}\right]$$

$$= \left[\begin{bmatrix} 2 & 1 \\ 4 & 7 \end{bmatrix}\begin{bmatrix} 7/10 & -1/10 \\ -4/10 & 2/10 \end{bmatrix}, \begin{bmatrix} 3 & 5 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -1/7 & 5/7 \\ 2/7 & -3/7 \end{bmatrix}\right]$$

$$= \left[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right]$$

$$= \begin{bmatrix} [1,1] & [0,0] \\ [0,0] & [1,1] \end{bmatrix} = I_{2\times2}.$$

Hence the claim.

Example 7.1.5: Let

$$M = \begin{bmatrix} (3,1) & (-1,2) & (-2,-2) \\ (2,-1) & (0,3) & (-1,0) \\ (3,0) & (-5,-2) & (0,1) \end{bmatrix} = [M_1, M_2]$$

be a 3×3 interval matrix with entries from $N_o(R)$. To find M^{-1} by elementary row transformation.

$$M = \begin{bmatrix} (1,1) & (0,0) & (0,0) \\ (0,0) & (1,1) & (0,0) \\ (0,0) & (0,0) & (1,1) \end{bmatrix} \begin{bmatrix} (3,1) & (-1,2) & (-2,-2) \\ (2,-1) & (0,3) & (-1,0) \\ (3,0) & (-5,-2) & (1,0) \end{bmatrix}$$

$$= \begin{bmatrix} (3,1) & (-1,2) & (-2,-2) \\ (2,-1) & (0,3) & (-1,0) \\ (3,0) & (-5,-2) & (0,1) \end{bmatrix} = [M_1, M_2]$$

$$= \left(\begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -1 \\ 3 & -5 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 0 \end{bmatrix} \right)$$

$$= \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} M_1, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} M_2 \right)$$

(In M_1 $R_1 \to R_1 - R_2$ and in M_2 $R_2 \to R_1 + R_2$)

$$\left(\begin{bmatrix} 1 & -1 & -1 \\ 2 & 0 & -1 \\ 3 & -5 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} \right)$$

$$= \left(\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} M_1, \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} M_2 \right)$$

(Making $R_2 \to R_2 + (-2) R_1$
and $R_3 \to R_3 + (-3) R_1$ in M_1 and $R_2 \to R_2 + 2R_3$)

We get

$$\left(\begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 1 \\ 0 & -2 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 2 & -2 \\ 0 & 5 & -2 \\ 0 & -2 & 1 \end{bmatrix} \right)$$

$$= \left(\begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ -3 & 3 & 1 \end{bmatrix} M_1, \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} M_2 \right)$$

(Apply $R_2 = R_1$ (1/2) in M_1 and $R_1 \to R_1 + (-2) R_2$ and $R_3 \to R_3 + 2R_2$ in M_2)

$$\left(\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 1/2 \\ 0 & -2 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)$$

$$= \left(\begin{bmatrix} 1 & -1 & 0 \\ -1 & 3/2 & 0 \\ -3 & 3 & 1 \end{bmatrix} M_1, \begin{bmatrix} -1 & -2 & -4 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix} M_2 \right)$$

Now making $R_1 \to R_1 + R_2$ and $R_3 \to R_3 + 2R_2$ in M_1 and $R_1 \to R_1 + 2R_3$ in M_2.

We get

$$\left(\begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)$$

$$= \left(\begin{bmatrix} 0 & 1/2 & 0 \\ -1 & 3/2 & 0 \\ -5/4 & 3/2 & 1/4 \end{bmatrix} M_1, \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix} M_2 \right)$$

(Applying $R_3 \to \frac{1}{4} R_3$ in M_1 no operation on M_2)

$$\left(\begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$$= \left[\begin{bmatrix} 0 & 1/2 & 0 \\ -1 & 3/2 & 0 \\ -5/4 & 3/2 & 1/4 \end{bmatrix} M_1, \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix} M_2\right].$$

(Applying $R_2 \to R_1 + ½\ R_3$, $R_2 \to R_2 – ½\ R_3$ in M_1) we get

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$$= \left(\begin{bmatrix} -5/8 & 5/4 & 1/8 \\ -3/8 & 3/4 & -1/8 \\ -5/4 & 3/2 & 1/4 \end{bmatrix} M_1, \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix} M_2\right).$$

We see $M^{-1} = (M_1^{-1}, M_2^{-1})$

$$= \left(\begin{bmatrix} -5/8 & 5/4 & 1/8 \\ -3/8 & 3/4 & -1/8 \\ -5/4 & 3/2 & 1/4 \end{bmatrix}, \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}\right)$$

$$= \begin{pmatrix} (-5/8,3) & (5/4,2) & (1/8,6) \\ (-3/8,1) & (3/4,1) & (-1/8,2) \\ (-5/4,2) & (3/2,2) & (1/4,5) \end{pmatrix}.$$

We will show $MM^{-1} = \begin{pmatrix} (1,1) & (0,0) & (0,0) \\ (0,0) & (1,1) & (0,0) \\ (0,0) & (0,0) & (1,1) \end{pmatrix}$

Consider MM^{-1} =

$$\begin{pmatrix} (3,1) & (-1,2) & (-2,-2) \\ (2,-1) & (0,3) & (-1,0) \\ (3,0) & (-5,-2) & (0,1) \end{pmatrix} \begin{pmatrix} (-5/8,3) & (5/4,2) & (1/8,6) \\ (-3/8,1) & (3/4,1) & (-1/8,2) \\ (-5/4,2) & (3/2,2) & (1/4,5) \end{pmatrix}$$

$$= \begin{pmatrix} (1,1) & (0,0) & (0,0) \\ (0,0) & (1,1) & (0,0) \\ (0,0) & (0,0) & (1,1) \end{pmatrix}.$$

We see by this method of writing an interval matrix as a matrix interval find the inverses.

Here we give examples of finding eigen values and eigen vectors of interval matrices with entries from $N_c(R)$ (or $N_o(R)$ or $N_{co}(R)$ or $N_{oc}(R)$).

Example 7.1.6: Let

$$M = \begin{pmatrix} [0,8] & [2,1] \\ [0,0] & [9,1] \end{pmatrix}$$

be a 2×2 interval matrix with entries from $N_c(R)$.

To find interval eigen values and interval eigen vectors of M.

$$|M-\lambda I| = \left| \begin{pmatrix} [0,8] & [2,1] \\ [0,0] & [9,1] \end{pmatrix} - \lambda I_{2\times 2} \right|$$

$$= \left| \begin{pmatrix} [0,8]-\lambda[1,1] & [2,1] \\ [0,0] & [9,1]-\lambda[1,1] \end{pmatrix} \right|$$

$$= \left| \begin{pmatrix} [-\lambda, 8-\lambda] & [2,1] \\ [0,0] & [9-\lambda, 1-\lambda] \end{pmatrix} \right|$$

$$= [-\lambda, 8-\lambda]\ [9-\lambda, 1-\lambda] - [2,1]\ [0,0]$$

$$= [(9-\lambda)(-\lambda), (8-\lambda)(1-\lambda)]$$

$$= [0, 0].$$

$\lambda=9$, $\lambda=8$ and $\lambda = 1$

Thus the interval eigen values are [9, 8] and [9, 1].

Now we find interval eigen values in case of 3×3 interval square matrices.

Example 7.1.7: Let

$$M = \begin{pmatrix} [0,2] & [0,0] & [1,0] \\ [1,3] & [1,2] & [0,0] \\ [0,0] & [0,0] & [3,5] \end{pmatrix}$$

be a 3×3 interval matrix with entries from $N_c(R)$.

Let λ be such that $|M - \lambda I| = 0$. I is the 3×3 interval identity interval matrix is given by

$$I = \begin{pmatrix} [1,1] & [0,0] & [0,0] \\ [0,0] & [1,1] & [0,0] \\ [0,0] & [0,0] & [1,1] \end{pmatrix}$$

$$= \left| \begin{pmatrix} [0,2] & [0,0] & [1,0] \\ [1,3] & [1,2] & [0,0] \\ [0,0] & [0,0] & [3,5] \end{pmatrix} - \lambda \begin{pmatrix} [1,1] & [0,0] & [0,0] \\ [0,0] & [1,1] & [0,0] \\ [0,0] & [0,0] & [1,1] \end{pmatrix} \right|$$

$$= \left|\begin{pmatrix} [0,2]-\lambda[1,1] & [0,0] & [1,0] \\ [1,3] & [1,2]-\lambda[1,1] & [0,0] \\ [0,0] & [0,0] & [3,5]-\lambda[1,1] \end{pmatrix}\right|$$

$$= [-\lambda,\ 2-\lambda]\ \left|\begin{matrix} [1-\lambda, 2-\lambda] & [0,0] \\ [0,0] & [3-\lambda, 5-\lambda] \end{matrix}\right| - [0,0]\left|\quad\right|$$

$$+\ [1,0]\left|\begin{matrix} [1,3] & [1-\lambda, 2-\lambda] \\ [0,0] & [0,0] \end{matrix}\right|$$

$= [-\lambda, 2-\lambda]\ [1-\lambda, 2-\lambda]\ [3-\lambda, 5-\lambda] = [0,0]$

$= [-\lambda\ (1-\lambda)\ (3-\lambda), (2-\lambda)2\ (5-\lambda)] = [0,0]$

$[\lambda = \{0, 1, 3\}$,

$\{2, 2, 5\}]$.

Hence the interval eigen values are [0, 2], [0, 2], [0, 5], [1, 2], [1, 2], [1, 5], [3, 2], [3, 2] and [3, 5].

Thus one of the reasons for introducing polynomial intervals is that they can be used in solving the characteristic equations where the coefficients are intervals.

Thus one can as in case of usual matrix theory find for interval matrices the eigen values and eigen vectors without any difficulty.

We now show by examples how this is done.

Example 7.1.8: Let

$$M = \begin{bmatrix} [6,-2] & [7,0] & [1,1] \\ [0,0] & [2,1] & [0,0] \\ [4,-1] & [0,0] & [1,5] \end{bmatrix}$$

be a 3 × 3 interval matrix with entries from $N_c(R)$. Now M can be written as the matrix interval as $M = [M_1, M_2]$.

$$= \left[\begin{bmatrix} 6 & 7 & 1 \\ 0 & 2 & 0 \\ 4 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -2 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 5 \end{bmatrix} \right]$$

where

$$M_1 = \begin{bmatrix} 6 & 7 & 1 \\ 0 & 2 & 0 \\ 4 & 0 & 1 \end{bmatrix} \text{ and}$$

$$M_2 = \begin{bmatrix} -2 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 5 \end{bmatrix}$$

are usual 3 ×3 matrix and $M = [M_1, M_2]$ is the matrix interval or natural class of matrix interval.

We can find the eigen values of M as the separate eigen values of M_1 and M_2 separately.

$$[M-\lambda I] = |[M_1, M_2] = [\lambda_1 I, \lambda_2 I]|$$

$$= [|M_1 - \lambda_1 I|, |M_2 - \lambda_2 I|]$$

$$= \left[\left\| \begin{bmatrix} 6 & 7 & 1 \\ 0 & 2 & 0 \\ 4 & 0 & 1 \end{bmatrix} - \lambda_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\|, \left\| \begin{bmatrix} -2 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 5 \end{bmatrix} - \lambda_2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\| \right]$$

$$= \left[\begin{vmatrix} 6-\lambda_1 & 7 & 1 \\ 0 & 2-\lambda_1 & 0 \\ 4 & 0 & 1-\lambda_1 \end{vmatrix}, \begin{vmatrix} -2-\lambda_2 & 0 & 1 \\ 0 & 1-\lambda_2 & 0 \\ -1 & 0 & 5-\lambda_2 \end{vmatrix} \right]$$

$$= \left[(6-\lambda_1) \begin{vmatrix} 2-\lambda_1 & 0 \\ 0 & 1-\lambda_1 \end{vmatrix} - 7\,|0| + \right.$$

$$\left. 1 \begin{vmatrix} 0 & 2-\lambda_1 \\ 4 & 0 \end{vmatrix}, (-2-\lambda_2) \begin{vmatrix} 1-\lambda_2 & 0 \\ 0 & 5-\lambda_2 \end{vmatrix} \text{ - } 0 + 1 \begin{vmatrix} 0 & 1-\lambda_2 \\ -1 & 0 \end{vmatrix} \right]$$

$$= [(6-\lambda_1)\ (2-\lambda_1)\ (1-\lambda_1) + 4\ (2-\lambda_1),\ -\ (2+\lambda_2)\ (1-\lambda_2)\ (5-\lambda_2) + (1-\lambda_2)].$$

The roots after simplification are

$$= \left[\{2, \frac{7 \pm i\sqrt{15}}{2}\}, \{1, \frac{3 \pm \sqrt{45}}{2}\} \right]$$

$$= \left\{ [2,1], \left[2, \frac{3 \pm \sqrt{45}}{2} \right] \right\}.$$

For the interval matrix is defined over R.

However we see we cannot find all the characteristic roots as a pair of roots are complex.

Thus even in case of interval matrices of natural class of intervals we see we can solve for eigen values of eigen vectofrs using matrix intervals. From the example it is clear the how working is carried out in a simple way.

We can generalize this situation and illustrate it for any $n \times n$ interval matrix.

$$\text{Let } A = [(a_{ij})]_{n\times n}$$

$$= ([a_{ij}^1, a_{ij}^2])_{n\times n}$$

where $1 \leq i, j \leq n$ with $a_{ij}^t \in R$. $t = 1,2$.

Now how to find the eigen values

$$|A - \lambda I_{n\times n}| = |[(a_{ij})]_{n\times n} - \lambda I_{n\times n}|$$

$$= [|(a_{ij}^1) - \lambda_1 I_{n\times n}|, |(a_{ij}^2) - \lambda_2 I_{n\times n}|]$$

$$= \left(\left| \begin{bmatrix} a_{11}^1 - \lambda_1 & a_{12}^1 & \dots & a_{1n}^1 \\ a_{21}^1 & a_{22}^1 - \lambda_1 & \dots & a_{2n}^1 \\ \vdots & \vdots & & \vdots \\ a_{n1}^1 & a_{n2}^2 & \dots & a_{nn}^1 - \lambda_1 \end{bmatrix} \right|, \left| \begin{bmatrix} a_{11}^2 - \lambda_2 & a_{12}^2 & \dots & a_{1n}^2 \\ a_{21}^2 & a_{22}^1 - \lambda_2 & \dots & a_{2n}^2 \\ \vdots & \vdots & & \vdots \\ a_{n1}^2 & a_{n2}^n & \dots & a_{nn}^n - \lambda_2 \end{bmatrix} \right| \right)$$

$$= [(0,0)]$$

= [nth degree polynomial in λ_1, nth degree polynomial in λ_2].

If $\{\alpha_1^1, \dots, \alpha_n^1\}$ and $\{\alpha_1^2, \dots, \alpha_n^2\}$ are roots then we get

$[\alpha_1^1, \alpha_1^2]$, $[\alpha_1^1, \alpha_2^2]$, …, $[\alpha_1^1, \alpha_n^2]$, $[\alpha_2^1, \alpha_1^2]$, …, $[\alpha_2^1, \alpha_n^2]$, …, $[\alpha_n^1, \alpha_n^2]$ are the interval roots of $|A - \lambda I_{n\times n}| = (0, 0)$.

Now calculating the interval eigen vector is also a matter of routine for we case of matrix interval and find the solution.

Hence by using the natural class of intervals we can easily make the interval matrix and the interval polynomial into matrix interval and polynomial intervals respectively.

We just show if $p(x) = [a_0, b_0] + [a_n, b_n]x + \ldots + [a_n, b_n]x^n$ be an interval polynomial with $[a_i, b_i] \in N_c(R)$; $0 \le i \le n$; then the polynomial interval corresponding of p(x) is

$[a_0 + a_1x + \ldots + a_nx^n,\ b_0 + b_1x+\ldots + b_nx^n]$; $a_i, b_i \in R$; $0 \le i \le n$.

Now if

$$M = \begin{bmatrix} [a_1^1, b_1^1] & \ldots & [a_n^1, b_n^1] \\ [a_1^2, b_2^2] & \ldots & [a_n^2, b_n^2] \\ \vdots & & \vdots \\ [a_1^n, b_1^n] & \ldots & [a_n^n, b_n^n] \end{bmatrix}$$

be the interval matrix $[a_i^t, b_i^t] \in N_c(R)$; $1 \le t \le n$, $1 \le i \le n$; this interval matrix can be written as a matrix interval as follows.

$$M = \left(\begin{bmatrix} a_1^1 & a_2^1 & \ldots & a_n^1 \\ a_1^2 & a_2^2 & \ldots & a_n^2 \\ \vdots & \vdots & & \vdots \\ a_1^n & a_2^n & \ldots & a_n^n \end{bmatrix}, \begin{bmatrix} b_1^1 & b_2^1 & \ldots & b_n^1 \\ b_1^2 & b_2^2 & \ldots & b_n^2 \\ \vdots & \vdots & & \vdots \\ b_1^n & b_2^n & \ldots & b_n^n \end{bmatrix} \right)$$

$= (M_1, M_2)$ is a matrix interval where $a_i^t, b_i^t \in R$, $1 \le t \le n$ and $1 \le i, j \le n$.

Now finding eigen values for these matrix interval is easy or it is carried out as in case of usual matrices however many choices of solutions (i.e., eigen values). For 2×2 matrices we can have 4 interval choices as solution. In case of 3×3 matrices we have 9 interval solutions.

7.2 Possible Applications of these New Natural Class of Intervals

In this section we give the probable applications of these natural class of intervals in due course of time when this concept becomes familiar with researchers. When these new class of intervals are used in finite element analysis certainly the time used to code will not be NP hard.

Secondly unlike the usual interval matrix operations when these natural class of intervals are used the time for finding determinant, matrix multiplication and finding the inverse we see the time is as that of coding the usual matrices.

These structures can also be used in modeling.

Thus we see lot of applications can be found and the existing intervals operation can be replaced by the natural operation for these intervals.

Chapter Eight

SUGGESTED PROBLEMS

In this chapter we introduce problems some of which are very difficult only a few problems are easily solvable. Many problems suggested can be viewed as research problems.

Further these concepts of new class of intervals works akin to the reals but arriving results akin to reals for these new class of intervals is not easy.

1. Find for the interval matrix

$$A = \begin{bmatrix} [0,1) & [1,0.7) & [0.1,0) & [0,1) \\ [1,0) & [0.2,0.4) & [1,0.1) & [0.2,1) \\ [0.3,0.1) & [0.5,0.1) & [0,0) & [1,1) \\ [1,0.2) & [0.7,0.9) & [1,1) & [0,0) \end{bmatrix}, \text{ elements of A are}$$

from $N_{co}(R)$.

(i) The characteristic interval polynomials.
(ii) Characteristic interval values
(iii) Characteristic interval vectors.

2. Prove the collection of all 2×2 interval matrices with intervals from $N_{oc}(R)$ is a ring with zero divisors and is non commutative.

3. Prove the collection of all 2×7 interval matrices V with intervals from $N_c(R)$ is a semigroup under addition.
   (i) Does V have ideals?
   (ii) Can V has subsemigroups which are not ideals?

4. Let
   M = {all 3×3 interval matrices with entries from $N_{co}(R)$};
   (i) Is M a commutative ring?
   (ii) Can M have ideals?
   (iii) Can M have subrings which are not ideals?
   (iv) Is M a S-ring?
   (v) Can M have S-zero divisors?
   (vi) Can M have S-units?

5. Find the interval eigen values and interval eigen vectors of

$$M = \begin{pmatrix} (0,3] & (0,0] & (1,0] & (2,3] \\ (0,-31] & (1,2] & (0,0] & (5,-1] \\ (0,0] & (1,1] & (2,1] & 0,4] \\ (1,0] & (2,2] & (0,1] & (1,1] \end{pmatrix}.$$

6. Find the differential of the interval polynomial.
   $p(x) = (6,2)x^9 - (7,-3)x^6 + (3,5)x^4 + (-3,-10)x^3 + (2,-7)x + (3,3)$.
   a) Is it possible to solve and find the interval roots of p(x).

7. Find the interval roots of the polynomial
   $p(x) = [0,-7) + [2,0)x^7 + [3,3)x^5 + [4,-1)x^3 + [1,0)x$.

8. Obtain some interesting properties about interval polynomial ring $N_{oc}(R)[x]$.

9. Find some applications of interval polynomial ring $N_{co}(Z)[x]$.

10. Compare Z[x] and $N_o(Z)[x]$.

11. Is $N_{oc}(Z)[x]$ a ring which satisfies ascending chain condition?

12. Can $N_{oc}(R)[x]$ have ideals?

13. Can $N_c(Q)[x,y]$ have minimal ideals?

14. Find principal ideals if any in $N_{oc}(Z)$.

15. Prove
$V = \{(a_1, \ldots, a_n) \mid a_i \in N_{co}(R); 1 \leq i \leq n; a_i = [a_i^1, a_i^2]; a_i^1, a_i^2 \in R\}$ is a interval vector space over R.
    a) Is V finite dimensional?
    b) Find a basis of V over R.
    c) Find interval subspaces of V over R.
    d) Is it possible to write V as a direct sum?

16. Give some nice applications of
$M = \{(a_{ij}) = ([a_{ij}^1, a_{ij}^2)) \mid a_{ij}^1, a_{ij}^2 \in R; 1 \leq i, j \leq n\}$.
    a) Prove M is a non commutative ring with unit and zero divisors.
    b) Is M a vector space over R of finite dimension?
    c) Find a basis of M over R.

17. Give some applications of interval fuzzy matrices.

18. Prove interval matrices and matrix intervals of same type are isomorphic.

19. Prove the program of finding the determinant of an interval matrix is the same as the program of finding the determinant of matrix intervals.

20. Does $A = \begin{bmatrix} [0,6] & [8,6] & [7,1] \\ [4,-5] & [9,2] & [3,3] \\ [8,8] & [11,11] & [0,0] \end{bmatrix}$ have inverse?

a) Find interval eigen values of A.

21. Let

$$X = \begin{bmatrix} [0,3)x^3 + [7,7)x^2 + [2,0) & [9,8)x^6 & [0,31)x^{13} + [7,7)x^2 + [3,-1) \\ [1,2)x^8 + [-4,5)x^6 + [3,7)x & [0,9)x + [1,1)x^4 & [8,1)x^{10} + [5,4)x^2 + [3,2) \\ [1,1)x^3 + [3,5)x + [1,0) & [11,8)x^7 + [2,1) & [10,9)x^2 + [1,5)x^3 + [1,1)x \\ [9,8)x^8 + [1,0)x^4 + [1,-9)x^2 & [14,5)x^5 & [9,9)x^9 + [1,2)x + [3,4) \end{bmatrix}$$

be a interval polynomial matrix.

(i) Find the first 3 derivatives of X.

(ii) Write $X = [X_{ij}^1, X_{ij}^2)$ as the polynomial matrix interval.

(iii) Find the integral of X.

22. Find application of finite interval analysis using these polynomial interval matrices?

23. Can these interval matrices be used in any other application of stiffness matrices?

24. Give any other interesting properties enjoyed by interval matrices?

25. Give an example of irreducible interval polynomial with coefficients from $N_{oc}(Q)$.

26. Can these interval matrices be applied to rounding error analysis?

27. Give nice applications of fuzzy interval matrices.

28. Construct a model in which the fuzzy interval matrices are used.

29. Give a model in which fuzzy interval matrices cannot be used only interval matrices can be used.

30. Find the roots of $p(x) = [8,0) + [3,5)x^2 - [7,2)x^3$. How many roots exist of p(x)?

31. Let $p(x) = x^2 - (9, 16] \in N_{oc}(R)[x]$ find roots of p(x).

32. Let $p(x) = x^2 + (3, -4) \in N_o(Z)[x]$; find roots of p(x) in $N_o(Z)$?

33. Let $p(x) = x^3 + (6, 9] \in N_{oc}(R)[x]$ be an interval polynomial; does roots of $p(x) \in N_{oc}(R)[x]$.

34. Can every interval polynomial in $N_{oc}(R)[x]$ be made monic? Justify your answer.

35. Give $p(x) \in N_{oc}(R)[x]$ such that p(x) is irreducible.

36. Give $p(x) \in N_c(Z)[x]$ which is reducible.

37. Let $p(x) = N_o(Z)[x]$ which is non monic yet linearly reducible.

38. Does an interval polynomial of degree n in $N_c(R)[x]$ have more than n roots? Justify.

39. Can we say every interval polynomial p(x) of degree n in $N_{oc}(R)[x]$ has atmost n^2 roots if p(x) is completely reducible?

40. Is it true, "Every nth degree polynomial p(x) in $N_c(R)[x]$ (or $N_{oc}(R)[x]$ and so on) have n^2 and only n^2 roots?

41. Is $N_o(C)$ ($N_{oc}(C)$ or $N_{co}(C)$ or $N_c(C)$) an algebraically closed interval ring? Justify.

42. Obtain any other interesting properties of $N_c(R)[x]$.

43. Let M be a $n \times n$ interval matrix with entries from $N_c(R)$. Does M have only n eigen values?

44. Let M is a 2×2 interval matrix with entries from $N_o(R)$. Prove M has 4 eigen values.

45. Can a $n \times n$ interval matrix have less than n^2 interval eigen values? Justify.

46. Let $M = \begin{bmatrix} [0,2] & [0,0] & [0,0] \\ [7,3] & [1,2] & [0,0] \\ [0,4] & [0,0] & [0,3] \end{bmatrix}$ be a 3×3 interval matrix with entries from $N_c(R)$. Can M have 3^2 eigen values or less? (Justify your claim).

47. Find the characteristic values of

$A = \begin{pmatrix} [0,6] & [7,0] \\ [0,4] & [2,0] \end{pmatrix}$ the entries of A are from $N_c(R)$.

Can A have 4 characteristic values?

48. Solve $p(x) = (0,8] + (8,1]x + (1,2]x^2 + (1,2]x^3$, the coefficients are from $N_{oc}(Q)$.

49. Let $M = (a_{ij})$ where $[a_{ij}^1, a_{ij}^2) \in N_{co}(R)$; $1 \leq i, j \leq 9$. Find all characteristic values of M.

50. Let $X = \begin{pmatrix} [9,2] & [1,0] & [1,0] \\ [0,1] & [0,2] & [0,3] \\ [0,1] & [3,0] & [1,0] \end{pmatrix}$ be a 3×3 interval matrix with entries from $N_c(R)$.
    (i) Find the eigen values of X.
    (ii) Does X have 9 eigen value or less? Justify.

51. Find the inverse of $A = \begin{bmatrix} [0,3) & [1,2) & [10,5) \\ [11,1) & [3,4) & [1,1) \\ [2,2) & [5,5) & [2,10) \end{bmatrix}$ in $N_{co}(R)$.

52. Give an example of a 5×5 interval matrix which has no inverse.

53. Let $N = \begin{pmatrix} (0,1) & (1,0) & (0,2) & (2,0) \\ (4,0) & (0,4) & (0,1) & (1,0) \\ (0,2) & (2,0) & (0,3) & (3,0) \\ (1,0) & (0,1) & (0,1) & (1,0) \end{pmatrix}$ be an interval matrix with entries from $N_o(Z)$.
    Does N^{-1} exist? Justify your answer.

54. Can N in problem (53) have 16 eigen values or less than 16 eigen values? Prove your answer.

55. Give a $n \times n$ interval matrix which has no inverse (entries from $N_{oc}(R)$).

56. Give an example of an interval polynomial which has repeated roots.

57. Prove if an interval polynomial $p(x) = p_0 + p_1x + \ldots + p_nx^n$ where $p_i \in N_{oc}(R)[x]$; $0 \leq i \leq n$ has repeated roots then $p'(x)$ and $p(x)$ has a common root.

58. Find ideals in $N_{oc}(R)[x]$.

59. Find subrings of $N_c(R)[x]$ which are not ideals.

60. Can $N_{co}(Z)[x]$ have ideals?

61. Can $N_{co}(Q)$ have ideals?

62. Prove $N_{oc}(R)$ has ideals.

63. Find ideals in $N_o(Q)$.

64. Prove if $N_o(Q)[x]$ is a polynomial ring.

65. Prove set of all 2×2 interval matrices M with entries from $N_{oc}(R)$ is a non commutative ring.

    (i) Find right ideals which are not left ideals in M.
    (ii) Find two sided ideals in M.
    (iii) Can M have zero divisors?

66. Can the ring of 3×3 interval matrices P with entries from $N_{oc}(Z)$ have invertible matrices.

    (i) Find left ideals in P.
    (ii) Find subring in P which are not ideals.
    (iii) Can P have zero divisors?
    (iv) Can P have idempotents?

67. Define Jacobson radical for $N_{oc}(R)$.
    Find Jacobson radical of $N_{oc}(R)$.

68. Find p(x) and q(x) interval polynomials in $N_c(R)[x]$, which are reducible.

69. Let $p(x) = x^2 - [6,9] \in N_c(R)[x]$ find the ideal I generated by p(x). Find $\frac{N_c(R)[x]}{I} = M$. Is M a field? (Justify).

70. Let $p(x) = x^3 + [6, -3]x^2 + [-6, 40]x + [9,7] \in N_c(Q)[x]$. Let J be the ideal generated by p(x). Find $\frac{N_c(Q)[x]}{J}$.

71. Let $M = \langle p(x) = x^3 - [3,2], q(x) = x^4 + [9, -3]x^2 + [8,9]\rangle$ be the ideal generated by M. Find $\frac{N_c(R)[x]}{M}$.

72. Give an ideal I in $N_{oc}(Q)[x]$ so that $\frac{N_{oc}(Q)[x]}{I}$ is not a field.

73. Can $N_c(Z)[x]$ have ideals I such that $\frac{N_c(Z)[x]}{I}$ is a field?

74. Can $N_c(Z)[x]$ have ideals I such that $\frac{N_c(Z)[x]}{I}$ is a finite field?

75. Let P = {all 3 × 2 interval matrices with entries from $N_{oc}(R)$} be a semigroup.
    (i) Find subsemigroups of P.
    (ii) Can P have subsemigroups which are ideals?
    (iii) Is P a group under '+'? Justify.
    (iv) Can P be written as a direct sum of subgroups?

76. Let B = {all 5 × 5 interval matrices with entries from $N_c(R)$} be a semigroup under multiplication.
    (i) Prove B is non commutative.
    (ii) Does B contain an interval matrix which has 5^2 distinct eigen values?
    (iii) Does B contain an interval matrix for which the characteristic interval polynomial is reducible?
    (iv) Can B have an interval matrix which has only 5 eigen values?
    (v) Can B contain an interval matrix which is diagonalizable?

77. Let S = $\begin{bmatrix} [-3,0] & [2,2] & [0,4] & [3,5] \\ [2,-4] & [1,4] & [4,0] & [5,3] \\ [7,-8] & [5,7] & [9,-1] & [1,1] \\ [4,5] & [2,0] & [2,7] & [2,5] \end{bmatrix}$ be a 4 × 4 interval matrix.

(i) Find |S| (determinant of S).
(ii) Is S invertible?
(iii) Can S have 4^2 eigen values in $N_c(R)$?
(iv) Find the interval characteristic polynomial?

78. Solve the equation $\begin{bmatrix} [0,2] & [1,3] \\ [4,0] & [5,2] \end{bmatrix}(\ddot{x}) + \begin{bmatrix} [1,1] & [1,0] \\ [0,1] & [3,5] \end{bmatrix}(\dot{x})$

$+\begin{bmatrix} [1,0] & [0,1] \\ [5,2] & [7,3] \end{bmatrix}x = \begin{bmatrix} [0,2] & [1,0] \\ [7,2] & [1,1] \end{bmatrix}$

(x some unknown displacement vector)
(i) Can this equation be solved?

79. Solve $\begin{bmatrix} (3,0) & (7,0) \\ (48,1) & (8,1) \end{bmatrix}(\dot{x}) = \begin{bmatrix} (9,1) & (1,-4) \\ (2,3) & (5,-2) \end{bmatrix}x$.

(x is unknown)

80. Prove if equations are given in interval matrix converting them into matrix interval solves all problems.

81. Prove this interval polynomial
$p(x) = [7,8)x^4 + [-8,2)x^2+[2,0)$ can be solved as polynomial intervals without any difficulty.
Show this equation has 16 roots which are intervals of the form [a, b); a, b ∈ R.

82. Let $p(x) = (8, 2]x^2 + (9,3]x + (7,2]$ find all roots of p(x).

83. Prove an interval polynomial can be solved in the same time as usual polynomial once an interval polynomial is written in its equivalent form as polynomial intervals.

84. Invent a method for solving fuzzy equations.

85. Solve $p(x) = 0.89x^3 + 0.75x^2 + 0.2 = 0$

86. Solve $f(x) = 0.43x^2 + 0.7x + 0.3 = 0$. Does f(x) have roots in [0,1]?

87. Can one guarantee all roots of a fuzzy polynomial (i.e., a polynomial which has its coefficients from [0,1]) have their roots in [0,1]?

88. Can a fuzzy polynomial have its roots as a complex number? Justify.

89. Solve $p(x) = 0.2x^2 + 0.3x + 0.7$.

90. Solve $p(x) = 0.4x^4 + 0.3x^2 + 0.2$.

91. Find the determinant value of

$$A = \begin{bmatrix} [0,1] & [1,1] & [0,0] \\ [0.3,0] & [1,0] & [1,0.3] \\ [0.1,1] & [0.2,0.4] & [0.7,0.2] \end{bmatrix}.$$

(i) Is A invertible?
(ii) Find eigen values of A.

92. Let $M = \begin{bmatrix} [1,1] & [0,0.3] & [0.7,0] \\ [0,1] & [0.2,0.1] & [1,0.7] \\ [1,0] & [0.4,0.2] & [0.1,1] \end{bmatrix}$ be a fuzzy interval matrix.
(i) Find eigen values of M.

(ii) Is M invertible?
(iii) Find the determinant value of M.

93. Let $P(x) = [0,1]x^3 + [0.7, 0.2]x^2 + [0.1, 0.5]x + [0,1]$ be a interval fuzzy polynomial.

(i) Is P(x) solvable?
(ii) Find all interval fuzzy roots of P(x).

94. Find some interesting applications of interval fuzzy polynomials.

95. Apply the concept of interval matrices in fuzzy element analysis method.

96. Solve the equation $p(x) = [0,3] + [6,0]x^2 +[7,2]x^4$.

97. Sketch the interval graph of $f(x) = [x^2+1, x]$.

98. Sketch the graph of $f(x) =[0,3]x^2 + [1,2]x + [3,2]$.

99. Draw the graph of f(x) = [sinx, cosx].

100. Find the 2^{nd} derivative of $p(x) = [6,9]x^3 + [7,3]x^2 + [0,2]x + [1,1]$.

101. Find the integral p(x) in problem (100).

102. Solve the equation p(x) in problem (100).

103. Draw the graph of p(x) in problem (100).

FURTHER READING

1. ABRAHAM, R., *Linear and Multilinear Algebra*, W. A. Benjamin Inc., 1966.
2. ALBERT, A., *Structure of Algebras*, Colloq. Pub., 24, Amer. Math. Soc., 1939.
3. BIRKHOFF, G., and MACLANE, S., *A Survey of Modern Algebra*, Macmillan Publ. Company, 1977.
4. BIRKHOFF, G., *On the structure of abstract algebras,* Proc. Cambridge Philos. Soc., 31, 433-435, 1995.
5. CHARLES W. CURTIS, *Linear Algebra – An introductory Approach*, Springer, 1984.
6. HALMOS, P.R., *Finite dimensional vector spaces*, D Van Nostrand Co, Princeton, 1958.
7. PADILLA, R., Smarandache algebraic structures, *Smarandache Notions Journal*, 9, 36-38, 1998.
8. SMARANDACHE, FLORENTIN (editor), Proceedings of the First International Conference on Neutrosophy, Neutrosophic Logic, Neutrosophic set, Neutrosophic probability and Statistics, December 1-3, 2001 held at the University of New Mexico, published by Xiquan, Phoenix, 2002.
9. SMARANDACHE, Florentin, *Special Algebraic Structures*, in Collected Papers III, Abaddaba, Oradea, 78-81, 2000.

10. VASANTHA KANDASAMY, W.B., and SMARANDACHE, Florentin, *Basic Neutrosophic Algebraic Structures and their Applications to Fuzzy and Neutrosophic Models,* Hexis, Church Rock, 2005.

11. VASANTHA KANDASAMY, W.B., and SMARANDACHE, Florentin, *Fuzzy Interval Matrices, Neutrosophic Interval Matrices and their Application,* Hexis, Phoenix, 2005.

12. VASANTHA KANDASAMY, W.B., and SMARANDACHE, Florentin, *Algebraic structures using natural class of intervals,* The Educational publisher Inc., Ohio, 2011.*

13. VASANTHA KANDASAMY, W.B., *Linear Algebra and Smarandache Linear Algebra*, Bookman Publishing, 2003.

14. VASANTHA KANDASAMY, W.B., Semivector spaces over semifields, *Zeszyty Nauwoke Politechniki*, 17, 43-51, 1993.

15. VASANTHA KANDASAMY, W.B., *Smarandache rings,* American Research Press, Rehoboth, 2002.

16. VASANTHA KANDASAMY, W.B., *Smarandache Semigroups*, American Research Press, Rehoboth, 2002.

17. VASANTHA KANDASAMY, W.B., Smarandache semirings and semifields, *Smarandache Notions Journal*, 7, 88-91, 2001.

18. VASANTHA KANDASAMY, W.B., *Smarandache Semirings, Semifields and Semivector spaces*, American Research Press, Rehoboth, 2002.

19. ZADEH, L.A., Fuzzy Sets, Inform. and control, 8, 338-353, 1965.

* This book won the 2011 New Mexico award in the category of Science and Maths.

INDEX

P

R

S

T

V

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In this book the authors introduce and study the properties of natural class of intervals bulit using $(-\infty, \infty)$ and $(\infty, -\infty)$. The operations on these matrices with entries from natural class of intervals behave like usual reals. So working with these interval matrices take the same time as usual matrices. Hence when applying them to fuzzy finite element methods or finite element methods the determination of solutions is simple and time saving.

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