Prime Distribution in Pythagorean Triples (1)

(the greatest problem in mathematics)

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Abstract. Using Jiang function we study the prime distribution in Pythagorean triples.

Pythagorean triples

$$a^2 + b^2 = c^2, \qquad (1)$$

in comprime integers must be of the form

$$a = x^2 - y^2, \quad b = xy, \quad c = x^2 + y^2, \qquad (2)$$

where x and y ae coprime integers.

Theorem 1. From (2) we have

$$a = (x+y)(x-y) \qquad (3)$$

Let $x - y = 1$ and $a = x + y = P_1$, we have,

$$P_1^2 = (x+y)^2 = x^2 + y^2 + 2xy = c + b, \qquad (4)$$

$$1 = (x-y)^2 = x^2 + y^2 - 2xy = c - b \qquad (5)$$

From (4) and (5) we have

$$a = P_1, \quad b = \frac{P_1^2 - 1}{2}, \quad c = \frac{P_1^2 + 1}{2} = P_2 \qquad (6)$$

There are infinitely many primes P_1 such that P_2 is a prime.

Proof. We have Jiang function [1]

$$J_2(\omega) = \prod_{P>2} (P - 1 - \chi(P)), \qquad (7)$$

where $\omega = \prod_{P \ge 2} P$, $\chi(P)$ is the number of solutions of congruence

$$q^2 + 1 \equiv 0 (\bmod P), \; q = 1, \cdots, P-1. \qquad (8)$$

From (8) we have

$$\chi(P) = 1 + (-1)^{\frac{P-1}{2}} \qquad (9)$$

Substituting (9) into (7) we have

$$J_2(\omega)=\prod_{P>2}\left(P-2-(-1)^{\frac{P-1}{2}}\right)\neq 0 \qquad (10)$$

Since $J_2(\omega)\neq 0$, we prove that there are infinitely many prime P_1 such that P_2 is a prime.

We have the best asymptotic formula [1]

$$\pi_2(N,2)=\left|\{P_1\leq N: P_2=prime\}\right|\sim\frac{J_2(\omega)\omega}{2\phi^2(\omega)}\frac{N}{\log^2 N}=\left(1-\frac{1+P(-1)^{\frac{P-1}{2}}}{(P-1)^2}\right)\frac{N}{\log^2 N}, \qquad (11)$$

where $\phi(\omega)=\prod_{P\geq 2}(P-1)$.

Theorem 2. Let $x+y=P_1$ and $x-y=P_1-2$, we have $a=P_1(P_1-2)$ and

$$P_1^2=(x+y)^2=c+b, \qquad (12)$$

$$(P_1-2)^2=(x-y)^2=c-b \qquad (13)$$

From (12) and (13) we have

$$a=P_1(P_1-2),\ b=\frac{P_1^2-(P_1-2)^2}{2},\ c=\frac{P_1^2+(P_1-2)^2}{2}=P_2 \qquad (14)$$

There are infinitely many primes P_1 such that P_2 is a prime.

Proof. We have Jiang function [1]

$$J_2(\omega)=\prod_{P>2}(P-1-\chi(P)), \qquad (15)$$

where $\chi(P)$ is the number of solutions of congruence

$$q^2+(q-2)^2\equiv 0(\bmod P),\ q=1,\cdots,P-1. \qquad (16)$$

From (16) we have

$$\chi(P)=1+(-1)^{\frac{P-1}{2}} \qquad (17)$$

Substituting (17) into (15) we have

$$J_2(\omega)=\prod_{P>2}\left(P-2-(-1)^{\frac{P-1}{2}}\right)\neq 0 \qquad (18)$$

Since $J_2(\omega)\neq 0$, we prove that there are infinitely many prime P_1 such that P_2 is a prime.

We have the best asymptotic formula [1]

$$\pi_2(N,2)=\left|\{P_1\le N: P_2=prime\}\right|\sim\left(1-\frac{1+P(-1)^{\frac{P-1}{2}}}{(P-1)^2}\right)\frac{N}{\log^2 N}, \quad (19)$$

Theorem 3. Let $x-y=1$ and $a=x+y=P_1^2$, we have,

$$a=P_1^2,\ b=\frac{P_1^4-1}{2},\quad c=\frac{P_1^4+1}{2}=P_2 \ . \qquad (20)$$

There are infinitely many primes P_1 such that P_2 is a prime.

Proof. We have Jiang function [1]

$$J_2(\omega)=\prod_{P>2}(P-1-\chi(P)), \qquad (21)$$

where $\chi(P)$ is the number of solutions of congruence

$$q^4+1\equiv 0(\bmod P),\ q=1,\cdots,P-1. \qquad (22)$$

From (22) we have

$$\chi(P)=4 \quad \text{if } 8|P-1, \quad \chi(P)=0 \text{ otherwise.} \qquad (23)$$

Since $J_2(\omega)\ne 0$, we prove that there are infinitely many prime P_1 such that P_2 is a prime.

We have the best asymptotic formula [1]

$$\pi_2(N,2)=\left|\{P_1\le N: P_2=prime\}\right|\sim\frac{J_2(\omega)\omega}{4\phi^2(\omega)}\frac{N}{\log^2 N}, \qquad (24)$$

These results are in wide use in biological, physical and chemical fields.

Reference

[1] Chun-Xuan Jiang, Jiang function $J_{n+1}(\omega)$ in prime distribution,
http: //vixra.org/pdf/0812.0004v2.pdf;
http: //www.wbabin.net/math/xuan2.pdf