A COMPLETE TREATISE ON THE GENERAL THEORY OF RELATIVITY

WITH **THE SPECIAL THEORY OF RELATIVITY** AND **GRAVITY FROM ELECTROMAGNETISM** AS APPENDIXES

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Abstract:

In this paper you can find a complete treatise on the General Theory of Relativity, starting from the basic geometry, through the Einstein's field equations, to the calculation of the deflection of light by the Sun and of the precession of the perihelion of planets.

Moreover, as appendixes, you will also find the Restricted Theory of Relativity and an explanation on how I see the Gravity (coming from) the Electromagnetism!

Contents:

Simplicity is the closest thing to intelligence.

Introduction.

The General Theory of Relativity (GTR) is an extension of the Special Theory of Relativity (or Restricted) (STR) shown in App. 1; it was necessary for Einstein to explain the Gravitation. The word gravity reminds the word acceleration; in fact, we will see in Par. 2.1 that where there is an acceleration, rotation and gravity with a reference system, the metric is not simple anymore as in STR.

Moreover, the GTR explains gravity as a curvature of the space, or better of the space-time (mathematic space-time, in the opinion of the writer) caused by matter (and by the energy!) which is in such space-time. It's like when, for instance, you put a ball of lead on a mattress: around the sphere you have a funnel-like hollow and there, the mattress is curved. Then, we can say that in such an area where the mattress is curved is the gravitational field of the ball of lead. If now we throw a small ball over the mattress, and neglecting frictions, it will move uniformly on a straight line, over the flat side of the mattress until, as it approaches the curved hollow, it will fall towards the ball of lead.

Matter, in GTR, sees the space-time as a railway over which it can move; therefore, if this railway is curved, the trajectories followed by the matter will be curved.

Then, if the ball of lead is so heavy that it completely sinks into the mattress, then the funnel will become like a closed bag and we would call it a black hole, and from it nothing would come out, not even light.

In the GTR the Equivalence Principle holds, according to which a gravitational field can be cancelled by an acceleration and so it is not possibile to absolutely tell an acceleration from a gravitational field. In fact, let us consider the Einstein Elevator, in which a guy, standing stopped at a floor, rests with his weight on the floor of the elevator; if now we cut the cable holding the elevator, it will start a free falling in the terrestrial gravitational field and the guy inside will float as if in a space ship where there is no gravity, as he is falling with the elevator and with its floor and this floor will always fall under his feet. Therefore, an acceleration, that of the free falling, has cancelled the gravitational effect; and, at the same time, when the elevator is stopped, the guy inside it, instead of thinking that he was standing in a gravitational field (as he is resting on the floor of the elevator) could have thought that there weren't any gravitational fields, but that the elevator was accelerating upwards, so pushing the soles of his shoes.

Through the example of the mattress we have just introduced the concept of the (mathematical) space-time curvature, caused by the matter/energy. The tensor equation, which will be here proved, and which shows the correspondence between the matter/energy and the curvature indeed, is the Einstein Gravitational Field tensor Equation:

$$R_{mn} - \frac{1}{2} g_{mn} R = -8pGT_{mn}$$

Its left side, all in R, shows the "curvature radius" and the geometric characteristics of the space in which the matter/energy is, and the measure of such a matter/energy is, on the contrary, given by the right side, through the momentum-energy tensor T_{mn} , that, as we will see, in some of its components, is proportional to the density r etc.

Perhaps, only in the opinion of the writer (as the thought which follows, as well as many others, hasn't been ever read on any books by me) the Newton classic gravitational equation shows a correspondence between geometrical characteristics and the presence of matter:

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} \ , \ m\vec{a} = -G\frac{Mm}{r^2}\hat{r} \ , \ \rightarrow\rightarrow\rightarrow$$

$$\rightarrow\rightarrow\rightarrow \frac{d^2\vec{r}}{dt^2} = -G\frac{M}{r^2}\hat{r}$$

in fact, the left side of the last equation, that is $\frac{d^2\vec{r}}{dt^2}$, that is the second derivative of the spatial position, over the time, is a geometric characteristic of the space indeed, while the right side $-G\frac{M}{r^2}\hat{r}$ tells us about M!

Since the time when it was born, officially in 1916, the GTR has been always seen by many people with suspects, as it's full of complexity, mathematical as well as conceptual; so, thinking that so many hypothesis and relevant calculations can lead to equations which stick to reality, sometimes led some critics to hold it as a weird theory.

Classic tests in Chapt. 4 are encouraging in the opposite direction, even though also there the preambles, the suppositions and calculations are a lot, and then, for instance, in the calculation of the deflection of the light of stars by the Sun, during an eclipse in 1919, the accuracy of the measurement was very close to the result. Moreover, there are also alternative explanations and in competition with the GTR, to explain the deflection of light and the precession of the perihelion of planets.

In the opinion of the writer, GTR is for sure a beautiful physical-mathematical theory, mathematical more than physical, maybe the most beautiful, but it's also true that it has somewhat weird concepts inside. I think that the GTR is the typical falsifiable Popper-like theory, like if it were an interpretative model which works to explain many phenomena, but that it's not the real essence of the phenomena just explained. And then, provided that the geometric interpretation of the curvature is real, we should still explain why the matter causes it; ok, it causes that, but why? To see is not the same as to explain and justify.

In the GTR, the gravity is just attractive and Einstein, in his Theory of Unified Fields (let's sum up a bit, out of brevity) after having used the concept of curvature in the GTR to explain the gravitational pull, also used the concept of torsion to try to explain also the repulsive forces of the electricity. All this unfortunately without success, that is, his unitary field equations (maybe 33) couldn't be proved in the real Universe. Therefore, Einstein work didn't finish with the GTR; in fact, he died in 1955 in a bed in a Hospital, with paper and pen in his hands!

I personally think the force of gravity is a macroscopic force which is made of microscopic and electric forces among particles, positive and negative, which make the Universe, and that can be considered as randomly spread (see App. 2). In fact, I prove in Appendix 2 that the electric energy of an electron in an electron-positron pair, which is:

$$\frac{1}{4pe_0} \cdot \frac{e^2}{r_e}$$

Is exactly the gravitational energy given to an electron by all the mass of the Universe M_{Univ} at a distance R_{Univ}, that is:

$$\frac{GM_{Univ}m_e}{R_{Univ}}$$

Therefore, we have:

$$\frac{1}{4\pi\varepsilon_0}\cdot\frac{e^2}{r_e}=\frac{GM_{Univ}m_e}{R_{Univ}}\ !!$$

And it really doesn’t seem to be just by chance the fact that if we see the Universe as if made just by electrons and positrons (fundamental harmonics, whose mass is m_e), and whose number is N, we easily have:

$$N=\frac{M_{Univ}}{m_e}\cong 1{,}75\cdot 10^{85}$$

and nothing is strange, so far, but we realize that if we multiply the square root of N ($\sqrt{N}\cong 4{,}13\cdot 10^{42}$) by the classic radius of the electron r_e, we get exactly $\sqrt{N}r_e\cong R_{Univ}\cong 1{,}18\cdot 10^{28}m$, that is, the radius of the Universe! And an explanation of all this, in a perfect harmony with the equivalence of electricity and gravity just shown, has been put in App. 2/at Par.4.1.

Therefore, the attraction (/repulsion) particle-antiparticle, that is, the fast oscillations of the particle-antiparticle pairs, in composing together, generate the slow oscillation of the Universe (Big Bang, expansion, contraction, Big Crunch). Now, we are in the era of the contraction, that is, the matter is contracting all towards the centre of mass of the Universe, and that’s why we see the actractive force of gravity every day, but hundreds of billion of years ago, when the Universe was expanding, the gravity was (as a consequence) repulsive-like (see still App. 2, as a support of all this), from which the similarity between the electricity (attractive and repulsive) and the gravity (also attractive and repulsive); “unfortunately”, when the Earth was born, the gravity already stopped to be repulsive a very long time before!

Chapter 1: Preamble on Geometry.

Par. 1.1: Formalism, lengths of arcs and areas of curved surfaces.

<u>the sphere and the circle:</u>
with reference to figure 1, we want to represent through a formula the surface of a sphere Σ:

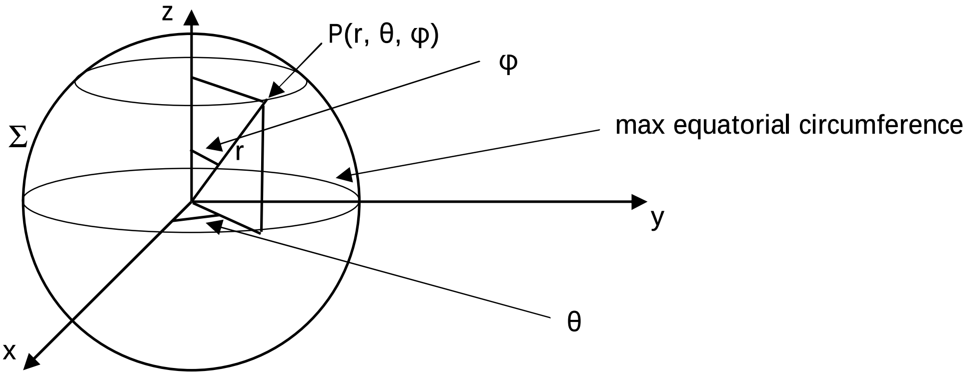


Fig. 1.1: The Sphere.
In Cartesian coordinates, we just use the Pythagorean Theorem to get such a formula:

$$x^2+y^2+z^2=r^2 \tag{1.1}$$

On the contrary, with the more friendly spherical coordinates, we have, very easily and intuitively:

$$\vec{t}_\Sigma=(r\cos\theta\sin\varphi)\hat{x}+(r\sin\theta\sin\varphi)\hat{y}+(r\cos\varphi)\hat{z} \tag{1.2}$$

where, of course, $\vec{t}_\Sigma$ is the vector which describes (by moving) all the surface of the sphere. We see that the components are a function of two parameters (θ e φ).
Of course, in the simpler case of a circle γ , we would have:

$$\vec{t}_\gamma = (r\cos\theta)\hat{x} + (r\sin\theta)\hat{y} \qquad (1.3)$$

and here we see that the components are a function of just one parameter (θ), if r=const.

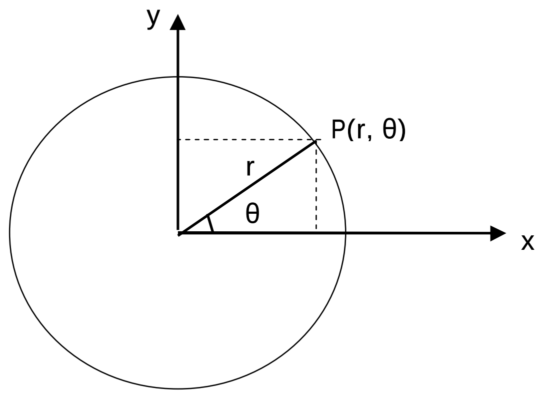


Fig. 1.2: The Circle.

(1.1) and (1.2) can therefore be written in a more general form, as functions of respectively 2 and 1 parameters:

$$\vec{t}_\Sigma = t_1(u,v)\hat{x} + t_2(u,v)\hat{y} + t_3(u,v)\hat{z} \quad (u,v=\theta,\varphi \text{ and } r=r(u,v)) \qquad (1.4)$$

$$\vec{t}_\gamma = t_1(u)\hat{x} + t_2(u)\hat{y} \quad (u=\theta \text{ and } r=r(u)) \qquad (1.5)$$

Of course, if in (1.4) and (1.5) all the t_i don't have the expressions they have in (1.2) and (1.3), but they have other ones, generic ones, then they (still (1.4) and (1.5)) can represent not anymore the sphere and the circle, buth other generic surfaces and curves.
Now, if we get a bit closer to the Cartesian coordinates, (x,y)=(x,f(x)), we make in (1.5) a change of parameter (u >>> x) so that we then have:

$$\vec{t}_\gamma = t_1(u)\hat{x} + t_2(u)\hat{y} = x\hat{x} + f(x)\hat{y} \qquad (1.6)$$

We will so consider (1.4) as the general expression for a surface and (1.6) that of a curve.
<u>length of an arc on a curve:</u>

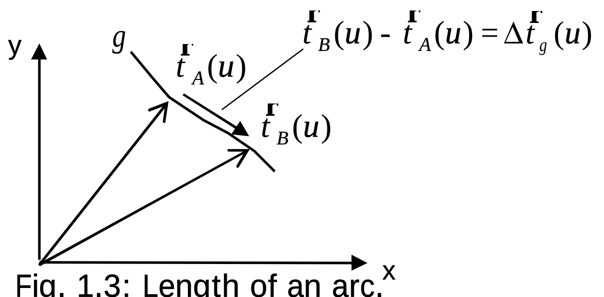


Fig. 1.3: Length of an arc.
With reference to figure 1.3, if $\vec{t}_B(u)$ tends to $\vec{t}_A(u)$, then $\Delta\vec{t}_\gamma(u)$ will become a $d\vec{t}_\gamma(u)$; not only; it will correspond to the infinitesimal arc $d\vec{l}$ on γ . We can so write that: $d\vec{l} = d\vec{t}_\gamma(u)$, from which:

$$l = \int_{l(A-B)} \left|d\vec{l}\right| = \int_{l(A-B)} \left|d\vec{t}_\gamma(u)\right| = \int_{l(A-B)} \left|\vec{t}\,'_\gamma(u)\right| du = \int_{l(A-B)} \sqrt{[(\frac{dt_1(u)}{du})^2 + (\frac{dt_2(u)}{du})^2]}\,du = \int_{l(A-B)} \sqrt{(1+f'^2(x))}\,dx \qquad (1.7)$$

and we also see from the vectorial composition in Fig. 1.3, that $d\vec{t}_g(u)$ is tangent to g , as it meets it just in one point, and therefore the vector

$$\vec{t} = \frac{d\vec{t}_g(u)}{du} \qquad (1.8)$$

is tangent to g .

<u>area of a curved surface:</u>

Let's consider again (1.4), and we report it here again: $\vec{t}_\Sigma = t_1(u,v)\hat{x} + t_2(u,v)\hat{y} + t_3(u,v)\hat{z}$.

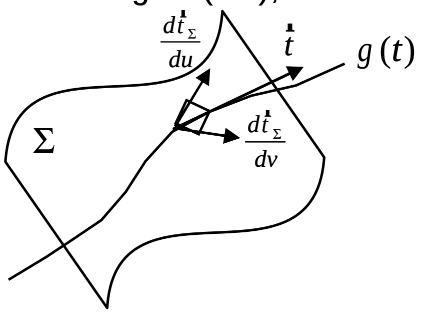


Fig. 1.4: Small area on a curved surface.

Now, with reference to Figure 1.4, we have a curved surface Σ, indeed, and a curve $g(t)$ on it.
Of course, if I want to say that the curve $g(t)$ is really on Σ, then both parameters u and v of Σ must be a function of the only parameter t of $g(t)$:
Now, we know from (1.8) that the derivative of the vector $\vec{t}_g$ which represents a curve, over its parameter t, yields the tangent vector $\vec{t}$, to the curve. By the same token, then the derivative of the vector $\vec{t}_\Sigma$ which represents the surface Σ yields a vector $\vec{t}$ tangent to the surface itself, and if the derivative is calculated on the parameter t of the curve $g(t)$ which lies on Σ, then, of course, such a tangent vector will be also the tangent of the curve $g(t)$. Such a relationship can be analytically shown by seeing the two parameters u and v as functions of the parameter t of the curve: $u = u(t)$ and $v = v(t)$.
Then, the tangent to the curve $g(t)$ will be:

$$\vec{t} = \frac{d\vec{t}_\Sigma}{dt} = \frac{d\vec{t}_\Sigma}{du}\frac{du}{dt} + \frac{d\vec{t}_\Sigma}{dv}\frac{dv}{dt} \qquad (1.9)$$

So, still with reference to Figure 1.4, for the (1.9) we see that the tangent vector $\vec{t}$ and vectors $\frac{d\vec{t}_\Sigma}{du}$ and $\frac{d\vec{t}_\Sigma}{dv}$ lie on the same plane, as they are a three element composition, as well as in the figure, indeed. Now, as the vectorial product of two vectors (modulus = product of moduli by the sine of the angle between them) yields a vector again, which is normal to the original vectors, then, in order to obtain the vector normal to the surface Σ we carry out the vectorial product between $\frac{d\vec{t}_\Sigma}{du}$ and $\frac{d\vec{t}_\Sigma}{dv}$ and if, then, I also want the versor $\hat{n}$ (unitary modulus) I will divide by the modulus of the normal vector:

$$\hat{n} = \frac{\frac{d\vec{t}_\Sigma}{du} \times \frac{d\vec{t}_\Sigma}{dv}}{\left| \frac{d\vec{t}_\Sigma}{du} \times \frac{d\vec{t}_\Sigma}{dv} \right|}$$ (normal versor) (1.10)

For what the area of a surface (in general) is concerned, we know from the elementary geometry that the area of a trapezium is given by the product of both sides by the sine of the angle formed by them, and if we also remind the definition of vectorial product above reported, we can then say that the small area $d\Sigma$ delimited by the two small vectors $\frac{d\vec{t}_\Sigma}{du}$ and $\frac{d\vec{t}_\Sigma}{dv}$ (Fig. 1.4) is:

$d\Sigma = \left| \frac{d\vec{t}_\Sigma}{du} \times \frac{d\vec{t}_\Sigma}{dv} \right|$, from which, by integration over all u and v:

$$\Sigma = \int_\Sigma d\Sigma = \iint_{u-v} \left| \frac{d\vec{t}_\Sigma}{du} \times \frac{d\vec{t}_\Sigma}{dv} \right| dudv \qquad (1.11)$$

Par. 1.2: Base differential Geometry.

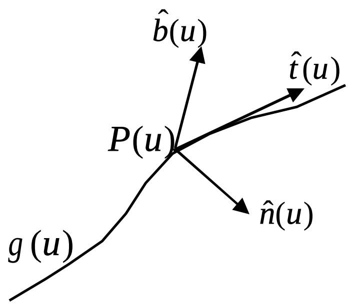


Fig. 1.5: Fundamental Trihedron.

We saw with (1.8) that the vector $\vec{t}(u) = \frac{d\vec{t}_g(u)}{du} = \vec{t}'_g(u)$ is tangent to the curve $g(u)$. As we want a versor $\hat{t}$ (unitary modulus), we'll have:

$$\hat{t}(u) = \frac{\vec{t}'_g(u)}{\left|\vec{t}'_g(u)\right|}$$ (tangent versor) (1.12)

So, still with (1.8) we saw that the derivation operation yields a normal vector.
Now, we derive the (1.12), so getting the normal versor $\hat{n}(u)$:

$$\hat{n}(u) = \frac{d}{du}\hat{t}(u)$$ (normal versor) (1.13)

At last, we define the binormal versor $\hat{b}(u)$, of course in the following way, by using the vectorial product:

$\hat{b}(u) = \hat{t}(u) \times \hat{n}(u)$ (binormal versor) (1.14)

$$\begin{cases} \hat{t}(u) = \frac{\vec{t}'_g(u)}{\left|\vec{t}'_g(u)\right|} \\ \hat{n}(u) = \frac{d}{du}\hat{t}(u) \\ \hat{b}(u) = \hat{t}(u) \times \hat{n}(u) \end{cases}$$ (fundamental trihedron with generic parameter u) (1.15)

We saw through (1.7) that the length of an arc is:

$$s(u) = \int_{s(A-B)} |d\vec{s}| = \int_{s(A-B)} |d\vec{t}_g(u)| = \int_{s(A-B)} |\vec{t}\,'_g(u)|du = \int_{u_0}^{u} |\vec{t}\,'_g(u)|du \qquad (1.16)$$

from which: $s'(u) = \frac{ds}{du} = |\vec{t}\,'_g(u)|$.

If now, in the trihedron (1.15), we make a change of parameter (u >>> s, with s as an intrinsic parameter), we'll have:

$\vec{t}\,'_g(s) = \vec{t}\,'_g[u(s)] = \frac{d\vec{t}_g(u)}{du}\frac{du}{ds} = \vec{t}\,'_g(u)\frac{1}{s'(u)}$, from which:

$|\vec{t}\,'_g(s)| = \frac{|\vec{t}\,'_g(u)|}{|s'(u)|} = 1$. Then:

$$\vec{t}\,''_g(s) = \frac{\vec{t}\,''_g(u)\frac{du}{ds}s'(u) - \vec{t}\,'_g(u)s''(u)\frac{du}{ds}}{(s'(u))^2} = \frac{\vec{t}\,''_g(u) - \vec{t}\,'_g(u)s''(u)\frac{du}{ds}}{(s'(u))^2}$$

and so we get the fundamental trihedron in the intrinsic parameterization:

$$\begin{cases} \hat{t}(s) = \vec{t}\,'_g(s) \\ \hat{n}(s) = \dfrac{\vec{t}\,''_g(s)}{|\vec{t}\,''_g(s)|} \\ \hat{b}(s) = \hat{t}(s)\times\hat{n}(s) = \dfrac{\vec{t}\,'_g(s)\times\vec{t}\,''_g(s)}{|\vec{t}\,''_g(s)|} \end{cases}$$ (fundamental trihedron with the intrinsic parameter s) (1.17)

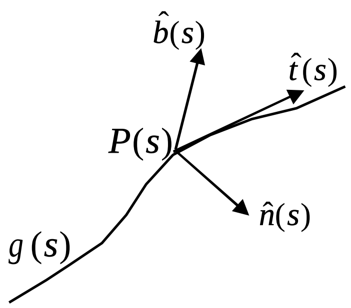


Fig. 1.6: Fundamental trihedron in the intrinsic parameterization.

<u>Curvature and radius of curvature:</u>

$|\vec{t}\,''_g(s)|$ is zero for lines, while it is $\neq 0$ on circles etc.

$|\vec{t}\,''_g(s)|$ is defined as CURVATURE of g in $\vec{t}_g(s)$.

$$r(s) = \frac{1}{|\vec{t}\,''_g(s)|} \qquad (1.18)$$

is the RADIUS OF CURVATURE.

Example on a circle (on the plane x-y):

$x^2 + y^2 = r^2$ (z=0)

$x = r\cos a = r\cos(\frac{s}{r})$, $y = r\sin a = r\sin(\frac{s}{r})$, z=0 (where $s = ar$ is the arc on the circle),

therefore: $\vec{t}_g(s) = (x, y, z) = (r\cos(\frac{s}{r}), r\sin(\frac{s}{r}), 0)$ from which:

$\vec{t}'_g(s) = \frac{d\vec{t}_g(s)}{ds} = (x', y', z') = (-\sin(\frac{s}{r}), \cos(\frac{s}{r}), 0)$ and so:

$\vec{t}''_g(s) = \frac{d\vec{t}'_g(s)}{ds} = (x'', y'', z'') = (-\frac{1}{r}\cos(\frac{s}{r}), -\frac{1}{r}\sin(\frac{s}{r}), 0)$, from which, again:

$\left|\vec{t}''_g(s)\right| = \sqrt{x''^2 + y''^2 + z''^2} = \frac{1}{r}$ (curvature)!!

the torsion:

$\hat{b}'(s)$ is // to $\hat{n}(s)$; in fact:

$$\frac{d}{ds}\hat{b}(s) = \frac{d}{ds}(\hat{t}(s) \times \hat{n}(s)) = \frac{d}{ds}\hat{t}(s) \times \hat{n}(s) + \hat{t}(s) \times \frac{d}{ds}\hat{n}(s) = \vec{t}''_g(s) \times \hat{n}(s) + \hat{t}(s) \times \frac{d}{ds}\hat{n}(s) = \hat{t}(s) \times \frac{d}{ds}\hat{n}(s)$$

Now, we notice that as $\hat{n}(s)$ is a versor (modulus 1 constant), $\frac{d}{ds}\hat{n}(s)$ does not represent a variation of the modulus of $\hat{n}(s)$, that is, along its extension as a vector, but, as a consequence, it is just a normal to $\hat{n}(s)$ variation, so we can say that: $\frac{d}{ds}\hat{n}(s) \perp \hat{n}(s)$, and as we are talking about an orthogonal trihedron, we also have that (of course) $[\hat{t}(s) \times \frac{d}{ds}\hat{n}(s)]$ // $\hat{n}(s)$ and so it shows to be proportional to $\hat{n}(s)$:

$$\hat{b}'(s) = -\frac{1}{t(s)}\hat{n}(s) ; \qquad (1.19)$$

$\frac{1}{t(s)}$ is the TORSION of g in $\vec{t}_g(s)$.

Now, there is a link between curvature and torsion and it’s expressed by the following:

Frénet formulas:

from the first two of (1.17) and from (1.18) we have the following: $\hat{t}'(s) = \frac{1}{r(s)}\hat{n}(s)$. Then, we have the (1.19):

$\hat{b}'(s) = -\frac{1}{t(s)}\hat{n}(s)$. Moreover, from the last of the (1.17) we get: $\hat{n}(s) = \hat{b}(s) \times \hat{t}(s)$ and so:

$$\frac{d}{ds}\hat{n}(s)=\frac{d}{ds}(\hat{b}(s)\times\hat{t}(s))=\hat{b}'(s)\times\hat{t}(s)+\hat{b}(s)\times\hat{t}'(s)=-\frac{1}{t(s)}\hat{n}(s)\times\hat{t}(s)+\hat{b}(s)\times\frac{1}{r(s)}\hat{n}(s)=$$

$$=\frac{1}{t(s)}\hat{b}(s)-\frac{1}{r(s)}\hat{t}(s)$$

from which we have the Frénet formulas:

$$\begin{cases}\hat{t}'(s)=\dfrac{1}{r(s)}\hat{n}(s)\\ \hat{b}'(s)=-\dfrac{1}{t(s)}\hat{n}(s)\\ \hat{n}'(s)=\dfrac{1}{t(s)}\hat{b}(s)-\dfrac{1}{r(s)}\hat{t}(s)\end{cases}\qquad\text{(Frénet formulas)}\qquad(1.20)$$

Curio: a body moving along a curve can have a tangential acceleration a_t and a centrifugal one a_c , of course, and from physics we know it's v^2 /r. Now, let's see if all the equations and all the formalism presented so far show this. We have:

$$\frac{d\vec{t}_g(s)}{dt}=\frac{d\vec{t}_g(s)}{ds}\frac{ds}{dt}=\frac{ds}{dt}\hat{t}(s)$$

$$\frac{d^2\vec{t}_g(s)}{dt^2}=\frac{d^2s}{dt^2}\hat{t}(s)+(\frac{ds}{dt})^2\frac{d\hat{t}(s)}{ds}=\frac{d^2s}{dt^2}\hat{t}(s)+(\frac{ds}{dt})^2\frac{1}{r(s)}\hat{n}(s)=\boxed{\vec{a}=\vec{a}_t+\vec{a}_c=a_t\hat{t}(s)+\frac{v^2}{r}\hat{n}(s)}.$$

Par. 1.3: Space differential Geometry.

<u>first fundamental form:</u>

we saw through (1.4) that a surface can be represented as follows:

$\vec{t}_\Sigma=t_1(u,v)\hat{x}+t_2(u,v)\hat{y}+t_3(u,v)\hat{z}$, that is: $\vec{t}_\Sigma=\vec{t}_\Sigma(u,v)$ and in a differential form:

$$d\vec{t}_\Sigma=\frac{d\vec{t}_\Sigma}{du}du+\frac{d\vec{t}_\Sigma}{dv}dv=\vec{t}_u du+\vec{t}_v dv$$

So, let's define the 1st fundamental form for $\vec{t}_\Sigma(u,v)$ as follows:

$$I=d\vec{t}_\Sigma\cdot d\vec{t}_\Sigma=(\vec{t}_u\cdot\vec{t}_u)du^2+2(\vec{t}_u\cdot\vec{t}_v)dudv+(\vec{t}_v\cdot\vec{t}_v)dv^2=Edu^2+2Fdudv+Gdv^2$$

where $E=(\vec{t}_u\cdot\vec{t}_u)$, $F=(\vec{t}_u\cdot\vec{t}_v)$, $G=(\vec{t}_v\cdot\vec{t}_v)$

If now we make a change of parameters ($\vec{t}(u,v)>>>>\vec{t}*(q,f)$), we'll have a change in the coefficients E, F and G, but I=I*:

$$I*(dq,df)=d\vec{t}\cdot d\vec{t}=\left|d\vec{t}^2\right|=\left|\vec{t}_q^*dq+\vec{t}_f^*df\right|^2=\left|\vec{t}_q^*(q_u du+q_v dv)+\vec{t}_f^*(f_u du+f_v dv)\right|^2=$$

$$=\left|(\vec{t}_q^*q_u+\vec{t}_f^*f_u)du+(\vec{t}_q^*q_v+\vec{t}_f^*f_v)dv\right|^2=\left|\vec{t}_u^*du+\vec{t}_v^*dv\right|^2=\left|d\vec{t}^2\right|=I(du,dv)$$

<u>length of an arc:</u>

we have an arc on $\vec{t}=\vec{t}(u(t),v(t))$; (1.21) here, we still see the two parameters u and v, typical for surfaces, but as we pointed out their common dependance from one single parameter t, then (1.21) is also the expression for a curve (on which the arc is). About the length s between a and b, we then have, of course:

$$s=\int_a^b\left|\frac{d\vec{t}}{dt}\right|dt=\int_a^b\left(\frac{d\vec{t}}{dt}\cdot\frac{d\vec{t}}{dt}\right)^{1/2}dt=\int_a^b[(\vec{t}_u\frac{du}{dt}+\vec{t}_v\frac{dv}{dt})(\vec{t}_u\frac{du}{dt}+\vec{t}_v\frac{dv}{dt})]^{1/2}dt=$$

$$=\int_a^b[E(\frac{du}{dt})^2+2F\frac{du}{dt}\frac{dv}{dt}+G(\frac{dv}{dt})^2]^{1/2}dt \qquad (1.22)$$

<u>area A of the surface:</u>

by the (1.11), we saw that $A=\int_A dA=\iint_{u-v}|d\vec{t}_u\times d\vec{t}_v|dudv$

Now, as the following vectorial identity holds: $|\vec{a}\times\vec{b}|^2=|\vec{a}|^2|\vec{b}|^2-|\vec{a}\cdot\vec{b}|^2$, then we have: $|d\vec{t}_u\times d\vec{t}_v|^2=EG-F^2$ and so:

$$A=\int_A dA=\iint_{u-v}|d\vec{t}_u\times d\vec{t}_v|dudv=\iint_{u-v}\sqrt{EG-F^2}\,dudv \qquad (1.23)$$

-<u>example 1: length of a circle:</u>

we already saw through (1.2) that the sphere (Fig. 1.1) is represented by the following equation:

$\vec{t}=(r\cos q\sin j)\hat{x}+(r\sin q\sin j)\hat{y}+(r\cos j)\hat{z}$.

If now we consider the maximum equatorial circle, (see Fig. 1.1), you can get it by putting φ=90°, from which:

$\vec{t}=(r\cos q)\hat{x}+(r\sin q)\hat{y}.....(+0\hat{z})$ and then we have a $\vec{t}=\vec{t}(u(t),v(t))=\vec{t}(q(t),r(t))$, just like in (1.21), from which:

$\vec{t}_q=-(r\sin q)\hat{x}+(r\cos q)\hat{y}$

$\vec{t}_r=-\cos q\hat{x}+\sin q\hat{y}$,

$E=\vec{t}_q\cdot\vec{t}_q=r^2\sin^2 q+r^2\cos^2 q=r^2$, $F=\vec{t}_q\cdot\vec{t}_r=-r\cos q\sin q+r\cos q\sin q=0$,

$G=\vec{t}_r\cdot\vec{t}_r=\cos^2 q+\sin^2 q=1$, and so (1.22) yields (r=const $\rightarrow\frac{dr}{dt}=0$):

$$s=\int_a^b[E(\frac{du}{dt})^2+2F\frac{du}{dt}\frac{dv}{dt}+G(\frac{dv}{dt})^2]^{1/2}dt=\int_a^b[E(\frac{dq}{dt})^2+2F\frac{dq}{dt}\frac{dr}{dt}+G(\frac{dr}{dt})^2]^{1/2}dt=$$

$$=\int_a^b[E(\frac{dq}{dt})^2]^{1/2}dt=\int_a^b\sqrt{E}\frac{dq}{dt}dt=\int_a^b rwdt=rwT=r\frac{2p}{T}T=2pr=C$$

that is really the length of a circle!!!

-<u>example 2: the surface of the sphere:</u>

we still consider the same sphere (Fig. 1.1), which, through (1.2), is shown by the following equation:

$\vec{t}=(r\cos q\sin j)\hat{x}+(r\sin q\sin j)\hat{y}+(r\cos j)\hat{z}$, from which:

$\vec{t}_q=-(r\sin q\sin j)\hat{x}+(r\cos q\sin j)\hat{y}$

$\vec{t}_\varphi = (r\cos\theta\cos\varphi)\hat{x} + (r\sin\theta\cos\varphi)\hat{y} - (r\sin\varphi)\hat{z}$

$E = \vec{t}_\theta \cdot \vec{t}_\theta = r^2\sin^2\varphi$, $F = \vec{t}_\theta \cdot \vec{t}_\varphi = 0$, $G = \vec{t}_\varphi \cdot \vec{t}_\varphi = r^2$ and for the (1.23), we get:

$$A = \int_A dA = \iint_{\theta-\varphi} \sqrt{EG - F^2}\, d\theta d\varphi = \iint_{\theta-\varphi} \sqrt{r^4 \sin^2\varphi - 0}\, d\theta d\varphi = \iint_{\theta-\varphi} r^2 \sin\varphi\, d\theta d\varphi =$$

$$= r^2 \int_\theta d\theta \int_\varphi \sin\varphi\, d\varphi = r^2 2\pi \left| -\cos\varphi \right|_0^\pi = r^2 2\pi \cdot 2 = 4\pi r^2$$

That is really the surface of the sphere we all know!!!

<u>second fundamental form:</u>

let's write again the (1.10), which supplies a vector/versor $\vec{N}$ normal to the surface:

$\vec{N} = \frac{\vec{t}_u \times \vec{t}_v}{|\vec{t}_u \times \vec{t}_v|}$; we then have, of course: $|\vec{N}| = 1$ and $0 = d(1) = d(\vec{N}\cdot\vec{N}) = 2d\vec{N}\cdot\vec{N}$, from which: $d\vec{N} \perp \vec{N}$. (1.24)

If it's so, then $d\vec{N}$ lies on the surface, and so it can be expressed as follows (u,v):

$$d\vec{N} = \frac{\partial \vec{N}}{\partial u} du + \frac{\partial \vec{N}}{\partial v} dv = \vec{N}_u du + \vec{N}_v dv \qquad (1.25)$$

Now we can define the second fundamental form II:

$$II = -d\vec{t} \cdot d\vec{N} = -(\vec{t}_u du + \vec{t}_v dv)(\vec{N}_u du + \vec{N}_v dv) = -\vec{t}_u \vec{N}_u du^2 - (\vec{t}_u \vec{N}_v + \vec{t}_v \vec{N}_u) du dv - \vec{t}_v \vec{N}_v dv^2 =$$
$$= L du^2 + 2M du dv + N dv^2$$

<u>properties of II:</u>

as we have: $(\vec{t}_u, \vec{t}_v) \perp \vec{N}$, then $0 = (\vec{t}_u \cdot \vec{N})_u = \vec{t}_{uu} \cdot \vec{N} + \vec{t}_u \cdot \vec{N}_u$,

$0 = (\vec{t}_u \cdot \vec{N})_v = \vec{t}_{uv} \cdot \vec{N} + \vec{t}_u \cdot \vec{N}_v$, $0 = (\vec{t}_v \cdot \vec{N})_u = \vec{t}_{vu} \cdot \vec{N} + \vec{t}_v \cdot \vec{N}_u$, $0 = (\vec{t}_v \cdot \vec{N})_v = \vec{t}_{vv} \cdot \vec{N} + \vec{t}_v \cdot \vec{N}_v$,

therefore: $\vec{t}_{uu} \cdot \vec{N} = -\vec{t}_u \cdot \vec{N}_u$, $\vec{t}_{vv} \cdot \vec{N} = -\vec{t}_v \cdot \vec{N}_v$, $\vec{t}_{uv} \cdot \vec{N} = -\vec{t}_u \cdot \vec{N}_v$, $\vec{t}_{vu} \cdot \vec{N} = -\vec{t}_v \cdot \vec{N}_u$ and so:

$L = \vec{t}_{uu} \cdot \vec{N}$, $M = \vec{t}_{uv} \cdot \vec{N}$, $N = \vec{t}_{vv} \cdot \vec{N}$ from which:

$II = L du^2 + 2M du dv + N dv^2 = \vec{t}_{uu} \cdot \vec{N} du^2 + 2\vec{t}_{uv} \cdot \vec{N} du dv + \vec{t}_{vv} \cdot \vec{N} dv^2 = d^2\vec{t} \cdot \vec{N}$

<u>normal curvature:</u>

if we have a surface S which contains a curve C and if P is a point on C, the (1.20-1) supplies the vector curvature, while we define the curvature $\hat{t}'_n(s)$ normal to C in P the projection of the curvature vector $\hat{t}'(s)$ on the normal $\vec{N}$ (and $\vec{N}$ is the versor $\hat{n}(s)$ of (1.10)):

$\hat{t}'_n(s) = (\hat{t}'(s) \cdot \vec{N})\vec{N}$ and the component along $\vec{N}$ is: $t_n = \hat{t}'(s) \cdot \vec{N}$.

Now, as $\hat{t}(s) \perp \vec{N}$, we have:

$\frac{d}{ds}(\hat{t}(s)\cdot\vec{N})=0=\hat{t}'(s)\cdot\vec{N}+\hat{t}(s)\cdot\frac{d}{ds}\vec{N}$, from which: $\hat{t}'(s)\cdot\vec{N}=-\hat{t}(s)\cdot\frac{d}{ds}\vec{N}$ and so:

$t_n=\hat{t}'(s)\cdot\vec{N}=-\hat{t}(s)\cdot\frac{d}{ds}\vec{N}=-\frac{d\vec{t}_g(s)}{ds}\frac{d\vec{N}}{ds}=-\frac{d\vec{t}_g\cdot d\vec{N}}{ds^2}=\frac{II}{I}$, as the numerator $d\vec{t}_g\cdot d\vec{N}$ is really the definition of II, while ds^2 can be figured out by deriving (1.22) and then squaring, and we'll really have I.

<u>example: normal curvature of the sphere:</u>

we already saw with (1.2) that the sphere (Fig. 1.1) is represented by the following equation:

$\vec{t}=(r\cos\theta\sin\varphi)\hat{x}+(r\sin\theta\sin\varphi)\hat{y}+(r\cos\varphi)\hat{z}$, from which:

$\vec{t}_\theta=-(r\sin\theta\sin\varphi)\hat{x}+(r\cos\theta\sin\varphi)\hat{y}$

$\vec{t}_\varphi=(r\cos\theta\cos\varphi)\hat{x}+(r\sin\theta\cos\varphi)\hat{y}-(r\sin\varphi)\hat{z}$

$\vec{t}_{\theta\theta}=-(r\cos\theta\sin\varphi)\hat{x}-(r\sin\theta\sin\varphi)\hat{y}$

$\vec{t}_{\theta\varphi}=-(r\sin\theta\cos\varphi)\hat{x}+(r\cos\theta\cos\varphi)\hat{y}$

$\vec{t}_{\varphi\varphi}=-(r\cos\theta\sin\varphi)\hat{x}-(r\sin\theta\sin\varphi)\hat{y}-(r\cos\varphi)\hat{z}$

$\vec{N}=-(r\cos\theta\sin\varphi)\hat{x}-(r\sin\theta\sin\varphi)\hat{y}-(r\cos\varphi)\hat{z}$

$E=\vec{t}_\theta\cdot\vec{t}_\theta=r^2\sin^2\varphi$, $F=\vec{t}_\theta\cdot\vec{t}_\varphi=0$, $G=\vec{t}_\varphi\cdot\vec{t}_\varphi=r^2$, $L=\vec{t}_{\theta\theta}\cdot\vec{N}=r\sin^2\varphi$, $M=\vec{t}_{\theta\varphi}\cdot\vec{N}=0$

$N=\vec{t}_{\varphi\varphi}\cdot\vec{N}=r$, from which:

$$t_n=\frac{Ld\theta^2+2Md\theta d\varphi+Nd\varphi^2}{Ed\theta^2+2Fd\theta d\varphi+Gd\varphi^2}=\boxed{\frac{1}{r}}\;!!!$$

<u>curvatures and main directions:</u>

the two orthogonal directions where t_n has its maximum and minimum values, are known as main directions and the relevant normal curvatures t_1 and t_2 are the main curvatures.

Theorem: t_0 is main and with main direction du_0,dv_0 if and only if du_0,dv_0 and t_0 satisfy the conditions:

$$\begin{cases}(L-t_0E)du_0+(M-t_0F)dv_0=0\\(M-t_0F)du_0+(N-t_0G)dv_0=0\end{cases}\qquad(1.26)$$

proof:

t_n is a bound if (t_n=II/I)

$\left.\frac{dt_n}{du}\right|_{(du_0,dv_0)}=0$ and $\left.\frac{dt_n}{dv}\right|_{(du_0,dv_0)}=0$, that is: $\left.\frac{I\cdot II_{du}-II\cdot I_{du}}{I^2}\right|_{(du_0,dv_0)}=0$ and:

$\left.\frac{I\cdot II_{dv}-II\cdot I_{dv}}{I^2}\right|_{(du_0,dv_0)}=0$. Now, if we multiply by I, we have:

$\left.II_{du}-\frac{II}{I}I_{du}\right|_{(du_0,dv_0)}=0$, and: $\left.II_{dv}-\frac{II}{I}I_{dv}\right|_{(du_0,dv_0)}=0$ but: $\frac{II}{I}(du_0,dv_0)=t_0$, so:

$II_{du} - t_0 I_{du}\Big|_{(du_0,dv_0)} = 0$ and $II_{dv} - t_0 I_{dv}\Big|_{(du_0,dv_0)} = 0$. Now, as:
$II_{du} = 2Ldu + 2Mdv$ and $I_{du} = 2Edu + 2Fdv$ and so:

$$\begin{cases}(Ldu_0 + Mdv_0) - t_0(Edu_0 + Fdv_0) = 0 \\ (Mdu_0 + Ndv_0) - t_0(Fdu_0 + Gdv_0) = 0\end{cases}$$

that is, what we wanted to prove.
Now, we rewrite the (1.26) in the following way:

$$\begin{cases}(L - tE)du + (M - tF)dv = 0 \\ (M - tF)du + (N - tG)dv = 0\end{cases}$$

and we multiply side to side:

$$(EG - F^2)t^2 - (EN + GL - 2FM)t + (LN - M^2) = 0 . \qquad (1.27)$$

The two solutions are the main curvatures.

Gauss curvature and mean curvature:

by dividing the previous equation (1.27) by $(EG - F^2)$, we get: $t^2 - 2Ht + K = 0$, where: $H = \frac{1}{2}(t_1 + t_2)$ (mean curvature) and $K = t_1 t_2$ (Gauss curvature). ($H = \frac{LN - M^2}{EG - F^2}$)

Gauss-Weingarten equations:

we saw $\vec{t}_u$, $\vec{t}_v$, and $\vec{N}$ are linearly independent (orthogonal) and so we can use them as a base to write their derivatives:

$$\begin{cases}\left.\begin{aligned}\vec{t}_{uu} &= \Gamma_{11}^1 \vec{t}_u + \Gamma_{11}^2 \vec{t}_v + b_{11}\vec{N} \\ \vec{t}_{uv} &= \Gamma_{12}^1 \vec{t}_u + \Gamma_{12}^2 \vec{t}_v + b_{12}\vec{N} \\ \vec{t}_{vv} &= \Gamma_{22}^1 \vec{t}_u + \Gamma_{22}^2 \vec{t}_v + b_{22}\vec{N}\end{aligned}\right\} \text{Gauss} \\ \left.\begin{aligned}\vec{N}_u &= b_1^1 \vec{t}_u + b_1^2 \vec{t}_v + g_1\vec{N} \\ \vec{N}_v &= b_2^1 \vec{t}_u + b_2^2 \vec{t}_v + g_2\vec{N}\end{aligned}\right\} \text{Weingarten}\end{cases} \qquad (1.28)$$

Where the Γ_{ij}^k are the 2nd kind Christoffel simbols.

We saw by (1.24) that: $d\vec{N} \perp \vec{N}$ and the (1.25) tells us that $d\vec{N}$ can be espressed in terms of $\vec{N}_u, \vec{N}_v$, from which we have that $\vec{N} \perp (\vec{N}_u, \vec{N}_v)$ and, according to the Weingarten equations, we can write that:

$$0 = \vec{N}_u \cdot \vec{N} = b_1^1 \vec{t}_u \cdot \vec{N} + b_1^2 \vec{t}_v \cdot \vec{N} + g_1 \vec{N} \cdot \vec{N}$$
$$0 = \vec{N}_v \cdot \vec{N} = b_2^1 \vec{t}_u \cdot \vec{N} + b_2^2 \vec{t}_v \cdot \vec{N} + g_2 \vec{N} \cdot \vec{N}$$

but we also know that: $\vec{t}_u \cdot \vec{N} = \vec{t}_v \cdot \vec{N} = 0$ and $\vec{N} \cdot \vec{N} = 1$, from which: $g_1 = g_2 = 0$ and so (1.28) get easier, as follows:

$$\begin{cases}\left.\begin{aligned}\vec{t}_{uu} &= \Gamma_{11}^1 \vec{t}_u + \Gamma_{11}^2 \vec{t}_v + b_{11}\vec{N} \\ \vec{t}_{uv} &= \Gamma_{12}^1 \vec{t}_u + \Gamma_{12}^2 \vec{t}_v + b_{12}\vec{N} \\ \vec{t}_{vv} &= \Gamma_{22}^1 \vec{t}_u + \Gamma_{22}^2 \vec{t}_v + b_{22}\vec{N}\end{aligned}\right\} \text{Gauss} \\ \left.\begin{aligned}\vec{N}_u &= b_1^1 \vec{t}_u + b_1^2 \vec{t}_v \\ \vec{N}_v &= b_2^1 \vec{t}_u + b_2^2 \vec{t}_v\end{aligned}\right\} \text{Weingarten}\end{cases} \qquad (1.29)$$

Let's write (1.29), more simply, in a TENSOR form, more completely:

$$\vec{t}_{ij} = \Gamma^a_{ij}\vec{t}_a + b_{ij}\vec{N} \quad \text{(i,j=1,2)} \tag{1.30}$$

and let's not forget that by (1.30) we have just started to use the EINSTEIN CONVENTION, according to which if in a term an index is repeated, then on it we have to sum up. In fact, in the term $\Gamma^a_{ij}\vec{t}_a$ in (1.30), a is repeated and so this term will yield two values, as well as happens in the Gauss equations (1.29).

<u>THE METRIC TENSOR</u> g_{ij}:

let's review our terminology used so far, using more compendious forms:

$u = u^1$, $v = u^2$, $u^i = (u,v)$, $\vec{t}_i = \dfrac{\partial \vec{t}}{\partial u^i}$, $\vec{t}_{ij} = \dfrac{\partial^2 \vec{t}}{\partial u^i \partial u^j}$;

$I = d\vec{t}\cdot d\vec{t} = \vec{t}_1\cdot\vec{t}_1 du^1du^1 + 2\vec{t}_1\cdot\vec{t}_2 du^1du^2 + \vec{t}_2\cdot\vec{t}_2 du^2du^2 = g_{11}du^1du^1 + g_{12}du^1du^2 + g_{21}du^2du^1 +$
$+ g_{22}du^2du^2 = \sum_{i,k} g_{ik}du^idu^k = g_{ik}du^idu^k$, with: $g_{11} = E$, $g_{12} = g_{21} = F$, $g_{22} = G$ and:

$$g = \det\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = g_{11}g_{22} - g_{21}g_{22} = EG - F^2 = g \tag{1.31}$$

Moreover: $d\vec{N} = \vec{N}_1 du^1 + \vec{N}_2 du^2$ and:

$II = -d\vec{t}\cdot d\vec{N} = -\vec{t}_1\vec{N}_1 du^1du^1 - \vec{t}_1\vec{N}_2 du^1du^2 - \vec{t}_2\vec{N}_1 du^2du^1 - \vec{t}_2\vec{N}_2 du^2du^2 =$
$= b_{11}du^1du^1 + b_{12}du^1du^2 + b_{21}du^2du^1 + b_{22}du^2du^2 = \sum_{i,k} b_{ik}du^idu^k$

with: $b_{11} = L$, $b_{12} = b_{21} = M$, $b_{22} = N$ and: $b = \det\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = b_{11}b_{22} - b_{21}b_{22} = LN - M^2 = b$.

By scalarly multiplying Gauss equations by $\vec{t}_k$, we have:

$\vec{t}_{ij}\cdot\vec{t}_k = \Gamma^a_{ij}\vec{t}_a\vec{t}_k + b_{ij}\vec{N}\cdot\vec{t}_k = \Gamma^a_{ij}\vec{t}_a\vec{t}_k + 0 = \Gamma^a_{ij}\vec{t}_a\vec{t}_k = \Gamma^a_{ij}g_{ak} = \Gamma_{ijk} \quad (a, i, j = 1,2)$

Γ_{ijk} are the 1st kind Christoffel symbols.

Then, remember that: $g_{ia}g^{aj} = d^j_i$ (by definition of g^{aj}), with d^j_i which is the Kronecker's Delta, and is 0 if $i \neq j$ and 1 if $i = j$; in fact:

$g_{ia}g^{aj} = \vec{t}_i\cdot\vec{t}_a\cdot\vec{t}^a\cdot\vec{t}^j = \vec{t}_i\cdot\vec{t}^j\cdot\vec{t}_a\cdot\vec{t}^a = \vec{t}_i\cdot\vec{t}^j\cdot 1 = d^j_i$, as $\vec{t}_i$ and $\vec{t}^j$ are, by definition, normal, if $i \neq j$ (definition of $\vec{t}^j$).

From this, we have: $\Gamma_{ijb}g^{bk} = \Gamma^a_{ij}g_{ab}g^{bk} = \Gamma^a_{ij}d^k_a = \Gamma^k_{ij}$ and so: $\Gamma_{ijk} = g_{ka}\Gamma^a_{ij}$ and

$$\Gamma^k_{ij} = g^{ka}\Gamma_{ija} . \tag{1.32}$$

We have: $$\frac{\partial g_{ij}}{\partial u^k} = \Gamma_{ikj} + \Gamma_{jki} \tag{1.33}$$

<u>proof:</u>

we have, by definition of g_{ij}, that: $g_{ij} = \vec{t}_i\cdot\vec{t}_j$, from which:

$$\frac{\partial g_{ij}}{\partial u^k} = \vec{t}_{ik}\cdot\vec{t}_j + \vec{t}_i\cdot\vec{t}_{jk} = \Gamma_{ikj} + \Gamma_{jki}$$

Then, we also have: $$\Gamma_{ikj} = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial u^i} + \frac{\partial g_{ki}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^k}\right) \tag{1.34}$$

proof:

we have, according to (1.33), that: $\frac{\partial g_{ik}}{\partial u^i} = \Gamma_{jik} + \Gamma_{kij}$, $\frac{\partial g_{ki}}{\partial u^j} = \Gamma_{kji} + \Gamma_{ijk}$ and $\frac{\partial g_{ij}}{\partial u^k} = \Gamma_{ikj} + \Gamma_{jki}$, (it's still about (1.33), but with indexes every time different, but, all in all, indexes have values 1 and 2, whatever their name is), from which we have what we wanted to prove.

It follows that:

$$\Gamma^k_{ij} = \frac{1}{2} g^{ka} \left(\frac{\partial g_{ja}}{\partial u^i} + \frac{\partial g_{ai}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^a}\right) \quad (1.35)$$

and moreover, by multiplying (1.32) $\Gamma^k_{ij} = g^{ka}\Gamma_{ija}$ in both sides by g_{ma} (the reciprocal of g^{ka}) , where, by definition of reciprocal: $g_{ma} g^{ka} = d^k_m$, we have: $g_{ma}\Gamma^k_{ij} = g_{ma} g^{ka}\Gamma_{ija} = d^k_m \Gamma_{ija}$, that is, by removing d^k_m and provided that $m = k$, in the left side, we have:

$\Gamma_{ija} = g_{ma}\Gamma^m_{ij}$ and by using the last equation, the (1.33) $\frac{\partial g_{ij}}{\partial u^k} = \Gamma_{ikj} + \Gamma_{jki}$ becomes:

$$\frac{\partial g_{ij}}{\partial u^k} = \Gamma_{ikj} + \Gamma_{jki} = g_{mj}\Gamma^m_{ik} + g_{mi}\Gamma^m_{ak} \quad (1.36)$$

By scalarly multiplying by $\vec{t}_j$, the Weingarten equations (1.29) $\vec{N}_i = b_i^a \vec{t}_a$, we get:

$-b_{ij} = \vec{N}_i \cdot \vec{t}_j = b_i^a \vec{t}_a \cdot \vec{t}_j = b_i^a g_{aj}$; if now we put: $b_i^j = b_{ia} g^{aj}$, we have:

$b_i^j = b_{ig} g^{gj} = -b_i^a g_{ag} g^{gj} = -b_i^a d_a^j = -b_i^j$; therefore $\vec{N}_i = -b_i^a \vec{t}_a$, with:

$b_i^j = g^{aj} b_{ia}$ e $b_{ij} = g_{aj} b_i^a$

<u>the symbols (tensors) of Riemann (1st and 2nd kind):</u>

$R_{mijk} = b_{ik}b_{jm} - b_{ij}b_{km}$ (2nd kind, rank 4 tensor) (1.37)

$R^r_{ijk} = g^{ar} R_{aijk}$ (1st kind, rank 4 tensor) (1.38)

R_{mijk} is the covariant Riemann curvature tensor

R^r_{ijk} is the combined Riemann curvature tensor

Of course: $R^r_{ijk} = g^{ar} R_{aijk} = g^{ar}(b_{ik}b_{ja} - b_{ij}b_{ka}) = b_{ik}b^r_j - b_{ij}b^r_k$ (1.39)

According to (1.37), we have: $R_{imjk} = -R_{mijk}$, $R_{mikj} = -R_{mijk}$; moreover $R_{mijk} = 0$ if the first two indexes or the last two are the same; therefore, just four components are not zero, and are:

$R_{1212} = R_{2121} = b_{22}b_{11} - b_{12}b_{21} = LN - M^2 = b$ and $R_{1221} = R_{2112} = b_{12}b_{21} - b_{22}b_{11} = -(LN - M^2) = -b$

We notice that: $\frac{LN - M^2}{EG - F^2} = \frac{b}{g} = \frac{R_{1212}}{g} = K(Gauss)$. (1.40)

We have: $R^a_{ijk} = (\Gamma^a_{ik})_j - (\Gamma^a_{ij})_k + \Gamma^b_{ik}\Gamma^a_{bj} - \Gamma^b_{ij}\Gamma^a_{bk}$ (1.41) proof:

$\vec{t}_{ij} = \Gamma^a_{ij}\vec{t}_a + b_{ij}\vec{N}$ >>>

$$\frac{\partial \vec{t}_{ij}}{\partial u^k} = \vec{t}_{ijk} = (\Gamma^a_{ij})_k \vec{t}_a + \Gamma^a_{ij}\vec{t}_{ak} + (b_{ij})_k\vec{N} + b_{ij}\vec{N}_k = (\Gamma^a_{ij})_k\vec{t}_a + \Gamma^a_{ij}(\Gamma^b_{ak}\vec{t}_b + b_{ak}\vec{N}) +$$

$$+ (b_{ij})_k\vec{N} + b_{ij}(-b^a_k\vec{t}_a) = [(\Gamma^a_{ij})_k + \Gamma^b_{ij}\Gamma^a_{bk} - b_{ij}b^a_k]\vec{t}_a + [\Gamma^a_{ij}b_{ak} + (b_{ij})_k]\vec{N}$$

where we used the Weingarten equation $\vec{N}_i = -b^a_i\vec{t}_a$.

Similarly: $\vec{t}_{ikj} = [(\Gamma^a_{ik})_j + \Gamma^b_{ik}\Gamma^a_{bj} - b_{ik}b^a_j]\vec{t}_a + [\Gamma^a_{ik}b_{aj} + (b_{ik})_j]\vec{N}$

Now, the third order derivatives are not depending on the order of derivation if and only if:

$\vec{t}_{ijk} = \vec{t}_{ikj}$, that is:

$$\vec{t}_{ijk} - \vec{t}_{ikj} = [(\Gamma^a_{ij})_k - (\Gamma^a_{ik})_j + \Gamma^b_{ij}\Gamma^a_{bk} - \Gamma^b_{ik}\Gamma^a_{bj} - b_{ij}b^a_k + b_{ik}b^a_j]\vec{t}_a + [\Gamma^a_{ij}b_{ak} + (b_{ij})_k - \Gamma^a_{ik}b_{aj} - (b_{ik})_j]\vec{N} = 0$$

and as $\vec{t}_1$, $\vec{t}_2$ and $\vec{N}$ are linearly independent, the last equation means that:

$$[(\Gamma^a_{ij})_k - (\Gamma^a_{ik})_j + \Gamma^b_{ij}\Gamma^a_{bk} - \Gamma^b_{ik}\Gamma^a_{bj} - b_{ij}b^a_k + b_{ik}b^a_j] = 0 \qquad (1.42)$$

$$[\Gamma^a_{ij}b_{ak} + (b_{ij})_k - \Gamma^a_{ik}b_{aj} - (b_{ik})_j] = 0$$

The (1.42), through the (1.39), yields: $R^a_{ijk} = (\Gamma^a_{ik})_j - (\Gamma^a_{ij})_k + \Gamma^b_{ik}\Gamma^a_{bj} - \Gamma^b_{ij}\Gamma^a_{bk}$, that is what we wanted to show.

Chapter 2: The main quantities in the Theory of General Relativity.

Par. 2.1: Introductory concepts on General Relativity.

First of all, please read again the Introduction on page 2.

Moreover, we know from STR (in App. 1) that the Lorentz contraction happens just in the direction of the movement, so, if we have a rotating system or a point which rotates, for instance, around a circle, the movement will be sometimes along x, then along x and y, then also along z; therefore, the Lorentz contraction is not acting still on just one coordinate and so, the run circle will appear as squashed, when seen by a rotating reference system, therefore, not inertial somehow, and therefore geometrically modified.

As a matter of fact, if:

$dt^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$, ($dt^2 = -h_{ik}dx^idx^k$,see after) (2.1)

then, in another system I’ makes which is accelerating along x with respect to the former one, we’ll have:

$$\begin{cases} x = x' + \frac{1}{2}at^2 \\ y = y' \\ z = z' \\ t = t' \end{cases}$$

and:

$$\begin{cases} dx = dx' + at'dt' \\ dy = dy' \\ dz = dz' \\ dt = dt' \end{cases}$$

from which:

$$dt^2 = c^2 dt'^2 - (dx' + at')^2 - dy'^2 - dz'^2 \quad \text{or} \qquad (2.2)$$

$$dt^2 = (c^2 - a^2 t'^2) dt'^2 - 2at' dx' dt' - - dx'^2 - dy'^2 - dz'^2 \qquad (2.3)$$

If, then, we also have another system I' whose plane x-y is rotating (with angular velocity w) with respect to that of the former system, as we then have the following transformation system:

$$\begin{cases} x = x'\cos wt - y'\sin wt \\ y = x'\sin wt + y'\cos wt \end{cases}$$

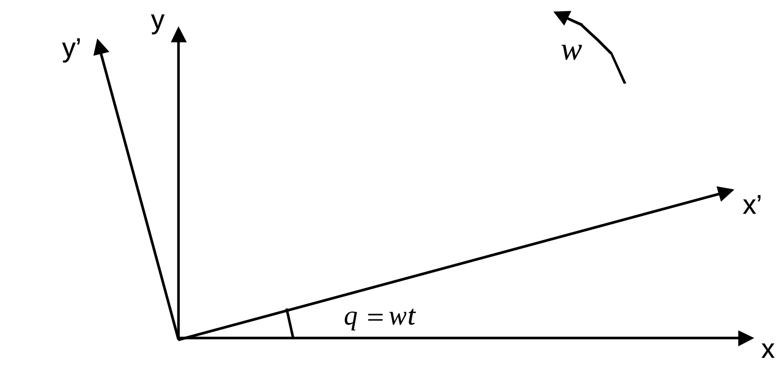


Fig. 2.1: Two reference systems, one rotating with respect to the other.

and remembering that, easily, for instance, $d(\sin wt) = w\cos wt \cdot dt$ etc, we have for dt^2:

$$dt^2 = [c^2 - w^2(x'^2 + y'^2)]dt^2 + 2wy' dx' dt' - 2wx' dy' dt' - dx'^2 - dy'^2 - dz'^2 \qquad (2.4)$$

and we can see that in no cases ((2.2), (2.3), and (2.4), which are of the kind $dt^2 = -g_{mn} dx^m dx^n$; see after) we can reduce dt^2, by means of time transformations, to the algebraic summation of the squares of the differentials of the four coordinates, as in (2.1) and as would be, on the contrary, for another inertial reference system.
Therefore, the presence of linear accelerations of reference systems (that can cancel gravitational fields) and also centrifugal/centripetal ones, that is, central ones, such as for the gravity (for instance, after rotations), introduce combined terms which change the metric, and so the geometry of the space-time. From this comes the need to formulate a relativistic theory for gravitation (GTR).

Par. 2.2: On the metric tensor and other main quantities.

When we have dealt with the Gauss-Weingarten equations, just before, we saw that $\vec{t}_u$, $\vec{t}_v$, and $\vec{N}$ are linearly independent (orthogonal) and so they really are a reference system, but curvilinear, and local, as they lie on a point of a surface, and when we move on it, such a tern moves and they also change their direction. That's a valid example of a curvilinear reference system, in the opinion of the writer, of course.
In all the equations introduced in the last chapter on geometry, indexes i, j, k etc changed from 1 to 2 or also 3. Now, getting a bit deeper in the Universe, and so in the General Relativity, we first of all notice that our Universe looks tridimensional, therefore, on

indexes, we'll have a variability which reaches at least three, and then, as also shown in App. 1 on Special Relativity, there exists a mathematically four-dimensional Universe (for the standard physics it's also really four-dimensional; to me, it's not!), in which there is covariance, and so conservation, when passing from an inertial system to another, then, with Einstein, we start once and for all to consider the Universe on a four-dimensional basis and that's it; therefore, the indexes of all the geometrical equations introduced in the last chapter, will have, from now on, all indexes with a variability on four values and the fourth value is the time one (ct). Then, the Einstein's convention will hold, according to which if in a term of an equation an index is present twice, then the summation over it is understood.
We report here the mains, which will be needed by us:

$\vec{t}_{ij} = \Gamma^a_{ij}\vec{t}_a$ (Gauss' equations in a more compendious form; all in Γ^a_{ij}) (i,j=1,2,3,4) (2.5)
(this gives us also the derivative of a versor)

$dt^2 = -g_{ik}du^i du^k$ (i,j=1,2,3,4) (<u>metric tensor</u> g_{ik} /de-square four-distance dt^2) (2.6)

$\Gamma^k_{ij} = \frac{1}{2}g^{ka}(\frac{\partial g_{ja}}{\partial u^i} + \frac{\partial g_{ai}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^a})$ (i,j=1,2,3,4) (1st kind Christoffel symbol) (2.7)

$R^a_{ijk} = (\Gamma^a_{ik})_j - (\Gamma^a_{ij})_k + \Gamma^b_{ik}\Gamma^a_{bj} - \Gamma^b_{ij}\Gamma^a_{bk}$ (i,j=1,2,3,4) (Riemann combined curvature tensor) (2.8)

<u>The metric tensor</u> g_{ik}, in case we are dealing with Euclidean spaces, reduces to Minkowski's tensor h_{ik} (it's 1 for i=k=1,2,3 and it's -1 for i=k=4; it's 0 when i is different from k) and without combined terms, that is, if i,k are not the same, then $\eta_{ik}=0$; in fact, we should then have ($u^i = x, y, z, ct$):

$dt^2 = -h_{ik}dx^i dx^k = -x^2 - y^2 - z^2 + (ct)^2$, just like in Special Relativity (App. 1) and the x^i would represent the Euclidean coordinate system. Then, when passing to curvilinear systems ($dx^i >>>> dx^i$), we'll have:

$$dt^2 = -h_{ik}dx^i dx^k = -h_{ik}\frac{\partial x^i}{\partial x^m}\frac{\partial x^k}{\partial x^n}dx^m dx^n = -g_{mn}dx^m dx^n \text{ , with} \qquad (2.9)$$

$$g_{mn} = h_{ik}\frac{\partial x^i}{\partial x^m}\frac{\partial x^k}{\partial x^n} \qquad (2.10)$$

which is the link equation from one system to another.

In the future we'll keep for the indexes the letters of the common alphabete (i,j,k etc) in case of Euclidean spaces (h_{ik}) and those of the Greek alphabete (m,n etc) for curvilinear spaces g_{mn}, with strong gravity.

We saw with (1.1) and (1.2) that a sphere can be represented through an equation in which there are the three classic Cartesian coordinates (x,y,z) or also by fixing a radius and making two angles change (r,q,j):

$x^2 + y^2 + z^2 = r^2$ (x,y,z)

$\overset{r}{t}_\Sigma = (r\cos q \sin j)\hat{x} + (r\sin q \sin j)\hat{y} + (r\cos j)\hat{z}$ (r, q, j)

Now, in case we make a change of coordinate system (x,y,z) >>>> (x’,y’,z’) where the latter is, for instance, shifted and rotated with respect to the former, there will be classic equations to go from one system to another, but both system will still have the same graphical representation by axes stretching from the origin to infinite, in positive as well as in negative.

$x'^2 + y'^2 + z'^2 = r^2$ (x’,y’,z’)

but all this in the Euclidean geometry, or non curvilinear, if we like.

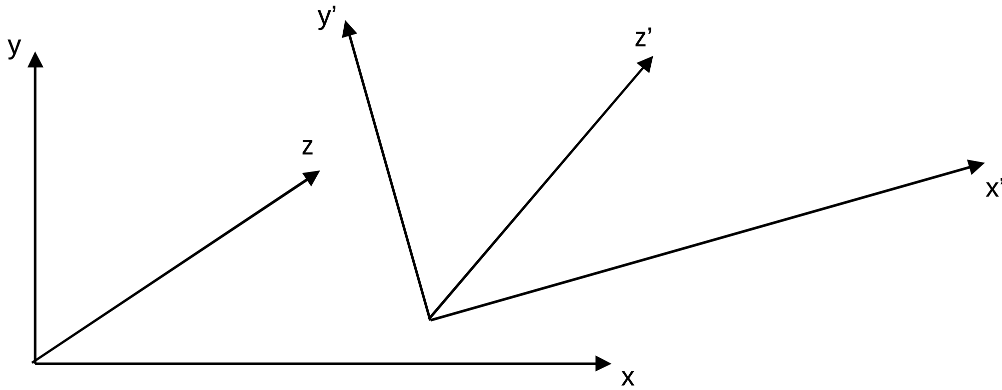


Fig. 2.2: Two different “Euclidean” reference systems.

If now we suppose to go from a system as that in Fig. 2.2 to another in which the space is, for any reason, curved, for instance by the gravity of matter and energy, as supposed in the General Relativity, then the Euclidean geometry isn’t enough anymore and the curvilinear one, the non Euclidean Riemann-like is more helpful. In fact, in the opinion of the writer, when passing from a standard system to a curvilinear one, you cannot have the representation of Fig. 2.2, but the curvilinear one will look like a tern of straight Cartesian axes only in the infinitesimal range (“d” = de), as per Fig. 2.3; in fact, as it’s curved, as long as you get farther from the origin “0”, every single axis bends and loses any linearity and proportionality.

Fig. 2.3: Case of curvilinear coordinate systems.

Therefore, we’ll have a system of link equations from an Euclidean system (x^i) to a curvilinear one (x^i), in the infinitesimal range, for all what just said so far, and that will be, in general, like this:

$$\begin{cases} x^1 = x^1(x^1, x^2, x^3, x^4) \\ x^2 = x^2(x^1, x^2, x^3, x^4) \\ x^3 = x^3(x^1, x^2, x^3, x^4) \\ x^4 = x^4(x^1, x^2, x^3, x^4) \end{cases} \qquad (2.11)$$

and vice versa:

$$\begin{cases} x^1 = x^1(x^1,x^2,x^3,x^4) \\ x^2 = x^2(x^1,x^2,x^3,x^4) \\ x^3 = x^3(x^1,x^2,x^3,x^4) \\ x^4 = x^4(x^1,x^2,x^3,x^4) \end{cases} \qquad (2.12)$$

and for the conversion equations for the expressions for the surfaces and for geometrical object ($\vec{t}$), we obviously have ($\vec{t} = x^1\hat{i} + x^2\hat{j} + x^3\hat{k} + x^4\hat{t} = (x^1,x^2,x^3,x^4)$ and ($\vec{t} = x^1\vec{t}_1 + x^2\vec{t}_2 + x^3\vec{t}_3 + x^4\vec{t}_4 = (x^1,x^2,x^3,x^4)$):

$$\begin{cases} \vec{t}_1 = \dfrac{\partial \vec{t}}{\partial x^1} = \dfrac{\partial x^1}{\partial x^1}\hat{i} + \dfrac{\partial x^2}{\partial x^1}\hat{j} + \dfrac{\partial x^3}{\partial x^1}\hat{k} + \dfrac{\partial x^4}{\partial x^1}\hat{t} \\ \vec{t}_2 = \dfrac{\partial \vec{t}}{\partial x^2} = \dfrac{\partial x^1}{\partial x^2}\hat{i} + \dfrac{\partial x^2}{\partial x^2}\hat{j} + \dfrac{\partial x^3}{\partial x^2}\hat{k} + \dfrac{\partial x^4}{\partial x^2}\hat{t} \\ \vec{t}_3 = \dfrac{\partial \vec{t}}{\partial x^3} = \dfrac{\partial x^1}{\partial x^3}\hat{i} + \dfrac{\partial x^2}{\partial x^3}\hat{j} + \dfrac{\partial x^3}{\partial x^3}\hat{k} + \dfrac{\partial x^4}{\partial x^3}\hat{t} \\ \vec{t}_4 = \dfrac{\partial \vec{t}}{\partial x^4} = \dfrac{\partial x^1}{\partial x^4}\hat{i} + \dfrac{\partial x^2}{\partial x^4}\hat{j} + \dfrac{\partial x^3}{\partial x^4}\hat{k} + \dfrac{\partial x^4}{\partial x^4}\hat{t} \end{cases} \qquad (2.13)$$

and moreover, of course:

$$\begin{cases} dx^1 = \dfrac{\partial x^1}{\partial x^1}dx^1 + \dfrac{\partial x^1}{\partial x^2}dx^2 + \dfrac{\partial x^1}{\partial x^3}dx^3 + \dfrac{\partial x^1}{\partial x^4}dx^4 \\ dx^2 = \dfrac{\partial x^2}{\partial x^1}dx^1 + \dfrac{\partial x^2}{\partial x^2}dx^2 + \dfrac{\partial x^2}{\partial x^3}dx^3 + \dfrac{\partial x^2}{\partial x^4}dx^4 \\ dx^3 = \dfrac{\partial x^3}{\partial x^1}dx^1 + \dfrac{\partial x^3}{\partial x^2}dx^2 + \dfrac{\partial x^3}{\partial x^3}dx^3 + \dfrac{\partial x^3}{\partial x^4}dx^4 \\ dx^4 = \dfrac{\partial x^4}{\partial x^1}dx^1 + \dfrac{\partial x^4}{\partial x^2}dx^2 + \dfrac{\partial x^4}{\partial x^3}dx^3 + \dfrac{\partial x^4}{\partial x^4}dx^4 \end{cases} \qquad (2.14)$$

and so: $d\vec{t} = dx^1\hat{i} + dx^2\hat{j} + dx^3\hat{k} + dx^4\hat{t} = (dx^1,dx^2,dx^3,dx^4) = (dx^1,dx^2,dx^3,dx^4) =$

$$= d\vec{t} = dx^1\vec{t}_1 + dx^2\vec{t}_2 + dx^3\vec{t}_3 + dx^4\vec{t}_4 = \boxed{dx^i\vec{t}_i = d\vec{t}} \qquad (2.15)$$

and therefore (dx^1,dx^2,dx^3,dx^4) are the components of $\vec{t}$ in the curvilinear base $(\vec{t}_1,\vec{t}_2,\vec{t}_3,\vec{t}_4)$.

There exist a reciprocal set of four numbers $(\vec{t}^1,\vec{t}^2,\vec{t}^3,\vec{t}^4)$ that, by definition: $\vec{t}_i \cdot \vec{t}^j = d_i^j$.

Now, if we go back for a while to the (2.15), where we plainly used the Einstein's convention, we have: $d\vec{t} = dx^i\vec{t}_i$

If now we want to make a change of curvilinear base, with coordinates from dx^i to $dx^{'l}$, we will obviously write: $dx^i\vec{t}_i = dx^{'l}\,\vec{t}'_l$ and, by multiplying both sides by $\vec{t}^i$, we'll have:

$dx^i = dx'^l \vec{t}'_l \cdot \vec{t}^i$, but it's also true that (as well as for (2.14)): $dx^i = \frac{\partial x^i}{\partial x'^l} dx'^l$; therefore: $\frac{\partial x^i}{\partial x'^l} = \vec{t}'_l \cdot \vec{t}^i$, that is:

$$\vec{t}'_l = \vec{t}_i \frac{\partial x^i}{\partial x'^l} \qquad (2.16)$$

and (2.16) is the equation for the base change.

Law of transformation for the components of a 4-vector:

let $\vec{V} = V^i \vec{t}_i$ be a generic vector expressed by curvilinear coordinates in the base $\vec{t}_i$; in another base $\vec{t}'_l$, we will have: $\vec{V} = V'^l \vec{t}'_l$ and, according to the (2.16):

$\vec{V} = V'^l \vec{t}'_l = V'^l \frac{\partial x^i}{\partial x'^l} \vec{t}_i = V^i \vec{t}_i$, with:

$$V^i = V'^l \frac{\partial x^i}{\partial x'^l} \qquad (2.17)$$

which is the transformation equation for the components of a 4-vector after a base change. Very simply, its inverse is: $V'^i = V^l \frac{\partial x'^i}{\partial x^l}$

Law of transformation for the components of a 4-tensor:

in App. 1 on Special Relativity we said that we can get a tensor T with rank n when we multiply the components of n vectors. So, if we have two vectors V and S: (where we are simultaneously reminding how their components transformate)

$V'^m = V^n \frac{\partial x'^m}{\partial x^n}$ e $S'^l = S^s \frac{\partial x'^l}{\partial x^s}$ >>>>

$T'^{ml} = V'^m S'^l = V^n \frac{\partial x'^m}{\partial x^n} S^s \frac{\partial x'^l}{\partial x^s} = V^n S^s \frac{\partial x'^m}{\partial x^n} \frac{\partial x'^l}{\partial x^s} = \boxed{T^{ns} \frac{\partial x'^m}{\partial x^n} \frac{\partial x'^l}{\partial x^s} = T'^{ml}}$ and this is how the components of a rank 2 tensor transformate. Then, you can proceed similarly for higher rank tensors.

Derivation of a 4-vector:

we have a 4-vector: $\vec{V} = V^m \vec{t}_m$; let's derive it:

$$\frac{d\vec{V}}{dx^m} = \frac{dV^m}{dx^m} \vec{t}_m + V^s \frac{d\vec{t}_s}{dx^n}, \qquad (2.18)$$

Indexes changes from one term to another, as they change on four values and must not necessarily be simultaneously the same in all terms.
Now, multiply the right side of (2.18) by the unitary (=1) quantity $\vec{t}^m \cdot \vec{t}_m$:

$\frac{d\vec{V}}{dx^m} = [\frac{dV^m}{dx^m} + V^s \frac{d\vec{t}_s}{dx^n}\vec{t}^m]\vec{t}_m$; well, the quantity between the brackets is the covariant derivative and is a tensor, for all that has been said so far:

$$V^m_{;n} = \frac{dV^m}{dx^m} + V^s \frac{d\vec{t}_s}{dx^n}\vec{t}^m = \frac{dV^m}{dx^m} + V^s\Gamma^m_{ns} \quad \text{(covariant derivative)} \qquad (2.19)$$

where Γ^m_{ns} are said the Christoffel's symbols (affine connection) and already introduced by (1.30).

Moreover, for the system (2.13), we could write, in a more compendious vectorial form:

$\vec{t}_m = (\sum_a)h_a \frac{\partial x^a}{\partial x^m}$, and, from this, we also have a dual form for the reciprocal set of four $\vec{t}^m$ (so that, by definition of reciprocity: $\vec{t}^m\vec{t}_n = d^n_m$): $\vec{t}^m = (\sum_m)h_m \frac{\partial x^m}{\partial x^a}$ and so the coefficient Γ^m_{ns} in (2.19) can be espressed also in the following way:

$\Gamma^m_{ns} = \frac{d\vec{t}_s}{dx^n}\vec{t}^m = \frac{d}{dx^n}(h_a\frac{\partial x^a}{\partial x^s})h_m\frac{\partial x^m}{\partial x^a} = h_a h_m \frac{\partial^2 x^a}{\partial x^s\partial x^n}\frac{\partial x^m}{\partial x^a}$, that, if inserted in (2.19), can be also written in a simpler way, without the unitary coefficients η:

$$\Gamma^m_{ns} = \frac{\partial^2 x^a}{\partial x^s\partial x^n}\frac{\partial x^m}{\partial x^a} \qquad (2.20)$$

Then, already in App. 1 on Special Relativity, we reminded that such a derivative (of a vector) gives a tensor. We also notice that such a derivative is a tensor (rank 2) just because the terms which make it, have two indexes, just like a tensor 2.

Moreover, (1.30) in the last chapter on Geometry is an example of a derivative of a vector (versor) which looks like a tensor 2, indeed.

<u>derivation of a tensor:</u>

we have: $T^{ms}_{l;r} = \frac{\partial}{\partial x^r}T^{ms}_l + \Gamma^m_{rn}T^{ns}_l + \Gamma^s_{rn}T^{ms}_l - \Gamma^k_{lr}T^{ms}_k$ (2.21)

in fact (T^{ms}_l are just the components, without "versors" and, moreover $\vec{t}_m\vec{t}^m = d^m_m = 1$):

T(tensor)= $T^{ms}_l\vec{t}_m\vec{t}_s\vec{t}^l$, from which:

$$\frac{\partial}{\partial x^r}(T^{ms}_l\vec{t}_m\vec{t}_s\vec{t}^l) = (\frac{\partial}{\partial x^r}T^{ms}_l)(\vec{t}_m\vec{t}_s\vec{t}^l) + T^{ns}_l(\frac{\partial}{\partial x^r}\vec{t}_n)\vec{t}_s\vec{t}^l(\vec{t}_m\vec{t}^m) + T^{mn}_l(\frac{\partial}{\partial x^r}\vec{t}_n)\vec{t}_m\vec{t}^l(\vec{t}_s\vec{t}^s) +$$

$$-T^{ms}_k(\frac{\partial}{\partial x^r}\vec{t}_m)\vec{t}^k\vec{t}_s(\vec{t}_l\vec{t}^l) = \frac{\partial}{\partial x^r}T^{ms}_l + \Gamma^m_{rn}T^{ns}_l + \Gamma^s_{rn}T^{ms}_l - \Gamma^k_{lr}T^{ms}_k = T^{ms}_{l;r} \quad \text{cvd.}$$

(as: $\frac{\partial}{\partial x^n}\vec{t}^m\vec{t}_k = \frac{\partial}{\partial x^n}d^m_k = 0 = \frac{\partial\vec{t}^m}{\partial x^n}\vec{t}_k + \vec{t}^m\frac{\partial\vec{t}_k}{\partial x^n}$, da cui: $\frac{\partial\vec{t}^m}{\partial x^n}\vec{t}_k = -\vec{t}^m\frac{\partial\vec{t}_k}{\partial x^n}$)

Par. 2.3: On the Lorentz Transformation in the General Relativity.

Let's go back to (2.9), and we know from App. 1 on Special Relativity that dt^2 is Lorentz invariant:

$$dt^2 = -h_{ab}\,dx'^a\,dx'^b = -h_{ab}\,\frac{\partial x'^a}{\partial x^g}\frac{\partial x'^b}{\partial x^d}\,dx^g\,dx^d = -h_{gd}\,dx^g\,dx^d$$, with:

$$h_{gd} = h_{ab}\,\frac{\partial x'^a}{\partial x^g}\frac{\partial x'^b}{\partial x^d} \qquad (2.22)$$

Let's differentiate (2.22) on x^e :

$0 = h_{ab}\,\frac{\partial^2 x'^a}{\partial x^g \partial x^e}\frac{\partial x'^b}{\partial x^d} + h_{ab}\,\frac{\partial x'^a}{\partial x^g}\frac{\partial^2 x'^b}{\partial x^d \partial x^e}$; now, we sum to this the same, but with g and e swapped and then we subtract the same equation, but with e and d swapped, so getting:

$0 = 2h_{ab}\,\frac{\partial^2 x'^a}{\partial x^g \partial x^e}\frac{\partial x'^b}{\partial x^d}$ >>>>> $\frac{\partial^2 x'^a}{\partial x^g \partial x^e} = 0$, whose solution is:

$$x'^a = \Lambda^a_b x^b + a^a \qquad \text{(Lorentz Transformation)} \qquad (2.23)$$

This one, together with (2.22), yields: $h_{gd} = h_{ab}\,\frac{\partial x'^a}{\partial x^g}\frac{\partial x'^b}{\partial x^d} = h_{ab}\Lambda^a_g\Lambda^b_d$.

Moreover, (2.23) in a differential form is: $dx'^a = \Lambda^a_g dx^g$. (2.24)

Let's figure out the elements of the Lorentz matrix (or of the Lorentz tensor) Λ^a_b :

from the Lorentz Transformations (A1.8) in App. 1, we have that (// and $\perp$ refers to the direction of the movement):

$\vec{x}'_{//} = g(\vec{x}_{//} - \vec{V}t)$, $\vec{x}'_{\perp} = \vec{x}_{\perp}$, $t' = g(t - \frac{\vec{V}\cdot\vec{x}}{c^2})$; moreover, of course, we have:

$\vec{x}_{//} = (\vec{x}\cdot\vec{V})\frac{\vec{V}}{V^2}$ e $\vec{x}_{\perp} = \vec{x} - \vec{x}_{//}$, and so:

$$\vec{x}' = \vec{x} + (g-1)(\vec{x}\cdot\vec{V})\frac{\vec{V}}{V^2} - g\vec{V}t \quad \text{and} \qquad (2.25)$$

$$t' = g(t - \frac{\vec{V}\cdot\vec{x}}{c^2})$$

While, for (2.24), we have: $dx'^i = \Lambda^i_b dx^b$, from which, using the common letters (i,j etc) for the three spatial components, 0 as the fourth time coefficient and the Greek letters for all of them:

$$dx'^i = \Lambda^i_j dx^j + \Lambda^i_0 dx^0 \qquad \text{(for (2.24))} \qquad (2.26)$$

$$dx'^i = dx^i + (g-1)(d\vec{x}\cdot\vec{V})\frac{V^i}{V^2} + g\,\frac{1}{c}V^i dx^0 \qquad \text{(for (2.25))} \qquad (2.27)$$

(here, the last term has got a + and not a – because $dx^0 = -cdt$)

By comparing (2.26) and (2.27), we have:

$\Lambda^i_0 = +g\,\frac{1}{c}V^i$ and $\Lambda^i_j = d^i_j + \frac{(g-1)}{V^2}V^iV_j$, and if we use the normalization c=1, out of simplicity:

$$\Lambda^i_0 = +gV^i \quad \text{e} \quad \Lambda^i_j = d^i_j + \frac{(g-1)}{V^2} V^i V_j$$ (Lorentz Tensor)

where the product $d\vec{x} \cdot \vec{V}$ in (2.27) yielded just the component $dx^j V_j$ in (2.26), as in (2.26) there was just dx^j.

If the direct Lorentz T. is given by (2.24): $dx'^a = \Lambda^a_g dx^g$, then, the inverse one will be represented as follows: $dx^g = \Lambda'^l_a \, dx'^a$.

Therefore, as also seen in App. 1 on Special Relativity, not only the spatial components (x) and temporal (ct) can be Lorentz transformed, but also 4-vectors and 4-tensors can:

$V'^a = \Lambda^a_b V^b$

$T'^g_{ab} = \Lambda^g_d \Lambda^e_a \Lambda^x_b T^d_{ex}$ (remembering that the components of a tensor are obtained by multiplying those of vectors)

Par. 2.4: The 4-vector Momentum-Energy and the Tensor Momentum-Energy.

Preamble on the Delta of Dirac:

By definition, the Delta of Dirac must satisfy the following equation:

$f(\vec{y}) = \int f(\vec{x}) d^3(\vec{x} - \vec{y}) d^3 x$

In practice, if you put it in the integral (which is a summation) it yields the same integrated function f, but of a different variable. See some good books on the Fourier Transform to have useful versions of the Delta of Dirac.

Preamble on currents and densities (of matter/energy):

if in various points $\vec{x}_n(t)$ we have energy (and so also matter) with a volume density $e_n[J/m^3]$, in order to have the total one, we obviously have to sum on n:

$e(\vec{x},t) = \sum_n e_n d^3(\vec{x} - \vec{x}_n(t))$,

where the Delta of Dirac gives the right value for e_n in the summation for every position $\vec{x}_n(t)$.

About the relevant current density of matter/energy $\vec{J}(\vec{x},t)$, we obviously have:

$$\vec{J}(\vec{x},t) = \sum_n e_n d^3(\vec{x} - \vec{x}_n(t)) \frac{d\vec{x}_n(t)}{dt}$$

in fact, $e_n[J/m^3]$, multiplied by $\frac{d\vec{x}_n(t)}{dt}[m/s]$, really gives a current of joule per square meter.

In components: $J^a(x) = \sum_n e_n d^3(\vec{x} - \vec{x}_n(t)) \frac{dx^a_n(t)}{dt}$ and, of course: $J^0 = e$ and $x^0_n(t) = ct$.

Now, this summation is over the points n; in order to have the total value, one must integrate also over the time:

$J^a(x) = \int dt' d(x - x_n(t')) J^a(x) = \int dt' \sum_n e_n d^4(x - x_n(t')) \frac{dx^a_n(t')}{dt'}$; now, by multiplying numerator and denominator by c, as “de” proper time is $dt = cdt'$, we’ll have:

$$J^a(x) = \int dt \sum_n e_n d^4(x - x_n(t)) \frac{dx_n^a(t)}{dt}$$

$J^a(x)$ is a 4-vector, ad so $\frac{dx_n^a(t)}{dt}$ is.

Moreover, $\vec{\nabla} \cdot \vec{J}(x,t) = \sum_n e_n \frac{\partial}{\partial x^i} d^3(\vec{x} - \vec{x}_n(t)) \frac{dx_n^i(t)}{dt} = -\sum_n e_n \frac{\partial}{\partial x_n^i} d^3(\vec{x} - \vec{x}_n(t)) \frac{dx_n^i(t)}{dt} =$

$= -\sum_n e_n \frac{\partial}{\partial t} d^3(\vec{x} - \vec{x}_n(t)) = -\frac{\partial}{\partial t} e(\vec{x},t)$. If now we bring $-\frac{\partial}{\partial t} e(\vec{x},t)$ together with $\vec{\nabla} \cdot \vec{J}(\vec{x},t)$, we will have the 4-divergence (see also App.1):

$\frac{\partial}{\partial x^a} J^a(x) = 0$ (the invariance on Lorentz is clear). (2.28)

Moreover, $Q(mat - energ) = \int d^3x J^0(x) = \int d^3x \, e(\vec{x},t)$

----

the 4-vector momentum-energy:

we already dealt with it in Special Relativity (App. 1).

Of course, $p^a = m_0 \frac{dx^a}{dt}$; then $\frac{dp^a}{dt} = m_0 \frac{d^2x^a}{dt^2} = f^a$ and

$dt = (c^2dt^2 - d\vec{x}^2)^{1/2} = (1 - \vec{V}^2/c^2)^{1/2} dt = dt/g$

from which, for the tridimensional component and for the temporal one:

$\vec{p} = m_0 g \vec{V}$ and $p^0 = E_0/c = m_0 g c$ and so: (2.29)

$p^b = (E_0/c^2) \frac{dx^b}{dt}$. (2.30)

Of course, for Lorentz: $p'^a = \Lambda_b^a p^b$.

(*): In reality, we will consider the mass m as a mass referred to the unity of volume [kg/m^3]

the TENSOR momentum-energy:

$T^{a0}(\vec{x},t) = \sum_n p_n^a(t) d^3(\vec{x} - \vec{x}_n(t))$ ("0" is the temporal component) (2.31)

$T^{ai}(\vec{x},t) = \sum_n p_n^a(t) \frac{dx_n^i(t)}{dt} d^3(\vec{x} - \vec{x}_n(t))$ ("i" , on the contrary, refers to the three spatial components) and, all together, in a more compendious form:

$T^{ab}(x) = \sum_n p_n^a(t) \frac{dx_n^b(t)}{dt} d^3(\vec{x} - \vec{x}_n(t))$ ($x_n^0(t) = ct$) (2.32)

The first one is a moment (p), actually, and the second is an energy, indeed (p x v). As before, we summed over the particles n.

(Then, by summing also over the time, we'd have, here too, after having multiplied numerator and denominator by c: $T^{ab} = \int dt \sum_n p_n^a \frac{dx_n^b}{dt} d^4(x - x_n(t))$)

Now, (2.32), through (2.30), becomes:

$$T^{ab}(x)=\sum_n \frac{p_n^a p_n^b}{(E_n/c^2)} d^3(\vec{x}-\vec{x}_n(t))$$

We notice the simmetry $T^{ab}(x)=T^{ba}(x)$; moreover, $T^{ab}(x)$ is a tensor and, being so, according to Lorentz, transforms, as follows:

$$T'^{ab}=\Lambda_g^a \Lambda_d^b T^{gd} \quad . \qquad (2.33)$$

Moreover, just like previously done to get (2.28), through the 4-divergence, we have:

$$\frac{\partial}{\partial x^i} T^{ai}(\vec{x},t)=\sum_n p_n^a(t)\frac{dx_n^i(t)}{dt}\frac{\partial}{\partial x^i} d^3(\vec{x}-\vec{x}_n(t))=-\sum_n p_n^a(t)\frac{dx_n^i(t)}{dt}\frac{\partial}{\partial x_n^i} d^3(\vec{x}-\vec{x}_n(t))=$$

$$=-\sum_n p_n^a(t)\frac{\partial}{\partial t} d^3(x-\vec{x}_n(t))=-\frac{\partial}{\partial t}T^{a0}(\vec{x},t)+\sum_n \frac{dp_n^a(t)}{dt} d^3(\vec{x}-\vec{x}_n(t))$$

as $\frac{\partial}{\partial t}T^{a0}(\vec{x},t)$ in the last equation is made of those two terms it has on its sides, for the rule of the derivation.

Therefore, $\frac{\partial}{\partial x^i}T^{ai}+\frac{\partial}{\partial t}T^{a0}=\frac{\partial}{\partial x^b}T^{ab}=\sum_n \frac{dp_n^a(t)}{dt} d^3(\vec{x}-\vec{x}_n(t))=G^a(x,t)$, that is:

$\frac{\partial}{\partial x^b}T^{ab}=G^a$, with:

$$G^a(\vec{x},t)=\sum_n \frac{dp_n^a(t)}{dt} d^3(\vec{x}-\vec{x}_n(t))=\sum_n \frac{dt}{dt}\frac{dp_n^a(t)}{dt} d^3(\vec{x}-\vec{x}_n(t))=\sum_n \frac{dt}{dt} f_n^a(t) d^3(\vec{x}-\vec{x}_n(t))$$

where $f_n^a=\frac{dp_n^a}{dt}$ is obviously a force.

If particles are free, then $p_n^a=const$ and so >>>> $\frac{\partial}{\partial x^b}T^{ab}=0$!!!

Par. 2.5: Relativistic Hydrodynamics.

Now, it's very important to find a form for the tensor momentum-energy T^{ab}, as we'll see that in the Newtonian limit of the relativistic gravitation, a component of it appears (T^{00}), so suggesting to involve, once we are out of the limit situation, all T^{ab} indeed. Out of simplicity, now we consider that c=1 (normalization).

Now we see that $T^{ab}=ph^{ab}+(p+r)U^aU^b$ and for fields of any intensity:

$$T^{mn}=pg^{mn}+(p+r)U^mU^n \quad \text{(c=1)} \qquad (2.34)$$

proof:

let's put the symbol ~ over the quantities which refer to a system at rest; moreover, from (2.32), we have that, in the right side, there is a product of a moment [(kg/m^3)(m/s)] by a velocity [(m/s)] (remind the note (*) on page 26) and so:

mass x velocity x velocity (=J) divided by m^3, that is a pressure p. (2.35)

Then, when the quantity dx/dt=d(ct)/dt=c is that which corresponds to the index zero, as per (2.31), then we'll have on the right side the product p^0 (mc, see (2.29)) by c, but according to the note (*) on page 26, m is a ρ and so we have ρc^2. Let's sum up:

$\tilde{T}^{ij}=pd^{ij}\,[Pa]$, $\tilde{T}^{i0}=\tilde{T}^{0i}=0$, $\tilde{T}^{00}=r$, that is: (rc^2 with c=1 $[Pa]$) (2.36)

As T is a tensor, let's transform it according to Lorentz, as per (2.33), to get its values for a generic system, that is, not at rest:

$T^{ab} = \Lambda^a_g \Lambda^b_d \tilde{T}^{gd}$; we'll have: $\left(g = \frac{1}{\sqrt{1-(V^2/c^2)}} = \frac{1}{\sqrt{1-V^2}}\right.$, with c=1)

$$\begin{cases} T^{ij} = pd^{ij} + (p+r)g^2 V^i V^j \\ T^{i0} = (p+r)g^2 V^i \\ T^{00} = g^2(pV^2 + r) \end{cases}$$

in fact, as, according to the Lorentz Tensor, it is:

$\Lambda^0_0 = g$, $\Lambda^i_0 = gV^i$ ($g\frac{1}{c}V^i$ with c=1), $\Lambda^0_i = gV_i$ ($g\frac{1}{c}V_i$ with c=1), $\Lambda^i_j = d^i_j + \frac{(g-1)}{V^2}V^i V_j$, it follows that: $T^{00} = \Lambda^0_g \Lambda^0_d \tilde{T}^{gd} = \Lambda^0_0 \Lambda^0_0 \tilde{T}^{00} + \sum_i (\Lambda^0_i \Lambda^0_i \tilde{T}^{ii}) = g^2 r + \sum_i [g^2 (V^i)^2] p = g^2(r + pV^2)$

$T^{i0} = \Lambda^i_g \Lambda^0_d \tilde{T}^{gd} = \Lambda^i_0 \Lambda^0_0 \tilde{T}^{00} + \Lambda^i_j \Lambda^0_j \tilde{T}^{jj} = g^2 V^i r + p\sum_j (\Lambda^i_j \Lambda^0_j) = g^2 V^i r + p(\Lambda^i_i \Lambda^0_i + \Lambda^i_k \Lambda^0_k + \Lambda^i_l \Lambda^0_l) =$

(k,l$\neq$0,i and j can be i, or k,l)

$= g^2 V^i r + p[(1+(V^i)^2 \frac{(g-1)}{V^2})gV^i + V^i V^k \frac{(g-1)}{V^2})gV^k +$

$+ V^i V^l \frac{(g-1)}{V^2})gV^l] = g^2 V^i r + p[gV^i + gV^i (V^i)^2 \frac{(g-1)}{V^2}) + gV^i (V^k)^2 \frac{(g-1)}{V^2}) + gV^i (V^l)^2 \frac{(g-1)}{V^2})] =$

$= g^2 V^i r + pgV^i + pgV^i \frac{(g-1)}{V^2}[(V^i)^2 + (V^k)^2 + (V^l)^2] = g^2 V^i r + pgV^i + pgV^i (g-1) =$

$= g^2 V^i (p+r)$

On the contrary, for the calculation of T^{ij}, let's split this in two cases:
(i=j e i$\neq$j)

$T^{ii} = \Lambda^i_g \Lambda^i_d \tilde{T}^{gd} = \Lambda^i_0 \Lambda^i_0 \tilde{T}^{00} + \Lambda^i_i \Lambda^i_i \tilde{T}^{ii} + \Lambda^i_k \Lambda^i_k \tilde{T}^{kk} + \Lambda^i_l \Lambda^i_l \tilde{T}^{ll} = (k, l \neq i) = g^2 (V^i)^2 r + [1 + (V^i)^2 \frac{(g-1)}{V^2}]^2 p +$

$+ [V^i V^k \frac{(g-1)}{V^2}]^2 p + [V^i V^l \frac{(g-1)}{V^2}]^2 p = g^2 (V^i)^2 r + p + p(V^i)^2 \frac{(g-1)^2}{V^4}[(V^i)^2 + (V^k)^2 + (V^l)^2] +$

$+ 2p(V^i)^2 \frac{(g-1)}{V^2} = g^2 (V^i)^2 r + p + p(V^i)^2 \frac{(g-1)^2}{V^2} + 2p(V^i)^2 \frac{(g-1)}{V^2} = p + g^2 (V^i)^2 r +$

$+ p(V^i)^2 \frac{(g-1)}{V^2}(g - 1 + 2) = p + g^2 (V^i)^2 r + p\frac{(V^i)^2}{V^2}(g^2 - 1) = p + g^2 (V^i)^2 r + p(V^i)^2 g^2 =$

$= p + g^2 (V^i)^2 (p+r)$. (as $\frac{(g^2-1)}{V^2} = \frac{g^2}{c^2} = g^2$ if c=1)

On the contrary, if i$\neq$j:

$T^{ij} = \Lambda^i_g \Lambda^j_d \tilde{T}^{gd} = \Lambda^i_0 \Lambda^j_0 \tilde{T}^{00} + \Lambda^i_i \Lambda^j_i \tilde{T}^{ii} + \Lambda^i_j \Lambda^j_j \tilde{T}^{jj} + \Lambda^i_k \Lambda^j_k \tilde{T}^{kk} = (k \neq i, j) = g^2 V^i V^j r +$

$+ p[1 + (V^i)^2 \frac{(g-1)}{V^2}][V^i V^j \frac{(g-1)}{V^2}] + p[V^i V^j \frac{(g-1)}{V^2}][1 + (V^j)^2 \frac{(g-1)}{V^2}] +$

$p[d^{ik}_{(=0)} + V^i V^k \frac{(g-1)}{V^2}][d^{jk}_{(=0)} + V^j V^k \frac{(g-1)}{V^2}] = g^2 V^i V^j r + pV^i V^j \frac{(g-1)}{V^2} + p(V^i)^2 V^i V^j \frac{(g-1)^2}{V^4} +$

$$+pV^iV^j\frac{(g-1)}{V^2}+p(V^j)^2V^iV^j\frac{(g-1)^2}{V^4}+p(V^k)^2V^iV^j\frac{(g-1)^2}{V^4}=g^2V^iV^jr+2pV^iV^j\frac{(g-1)}{V^2}+$$

$$+pV^iV^j\frac{(g-1)^2}{V^4}[(V^i)^2+(V^j)^2+(V^k)^2]=g^2V^iV^jr+2pV^iV^j\frac{(g-1)}{V^2}+pV^iV^j\frac{(g-1)^2}{V^2}=$$

$$=g^2V^iV^jr+pV^iV^j\frac{(g-1)}{V^2}(2+g-1)=g^2V^iV^j(p+r)$$ (as $\frac{(g^2-1)}{V^2}=\frac{g^2}{c^2}=g^2$ if c=1).

Totally:

$T^{ij}=pd^{ij}+(p+r)V^iV^jg^2$ (for the spatial components)

$T^{ab}=ph^{ab}+(p+r)U^aU^b$ (for all 4 components, but in weak fields (h^{ab}))

where $U^i=\frac{dx^i}{dt}=g\frac{dx^i}{dt}=gV^i$ e $U^0=gc$.

At last, for any gravitational fields (h^{ab} >>>> g^{ab}):

$\boxed{T^{mn}=pg^{mn}+(p+r)U^mU^n}$ (c=1) .

Par. 2.6: The geodetic Equation.

A free falling parachutist does not have any floor over which his body can rest; therefore, he does not detect any gravitational acceleration and he feels as if floating in the vacuum. He realizes he is falling only if he looks at the moving objects around. Therefore, for a free falling particle, there is a reference system in which $\dot{a}=0$, that is:

$$\frac{\partial^2x^a}{\partial t^2}=0 \quad \text{(free falling in Euclidean coordinates)} \qquad (2.37)$$

with $dt^2=-h_{ik}dx^idx^k$.

But we can see (2.37) in the following way:

$$\frac{\partial^2x^a}{\partial t^2}=0=\frac{d}{dt}(\frac{\partial x^a}{\partial x^m}\frac{dx^m}{dt})=\frac{\partial x^a}{\partial x^m}\frac{d^2x^m}{dt^2}+\frac{\partial^2x^a}{\partial x^m\partial x^n}\frac{dx^m}{dt}\frac{dx^n}{dt}$$

If we multiply both sides by $\frac{\partial x^l}{\partial x^a}$ and consider that:

$\frac{\partial x^a}{\partial x^m}\frac{\partial x^l}{\partial x^a}=d^l_m$, we'll have:

$$\frac{d^2x^l}{dt^2}+\Gamma^l_{mn}\frac{dx^m}{dt}\frac{dx^n}{dt}=0 \quad \text{(free falling in curvilinear coordinates)} \qquad (2.38)$$

(equation of the geodetic, where geodetic, on the Earth, is the shortest path between two places)

with $\Gamma^l_{mn}=\frac{\partial^2x^a}{\partial x^m\partial x^n}\frac{\partial x^l}{\partial x^a}$.

Par. 2.7: The relation between g_{mn} and Γ^l_{mn}.

We already got such a relation in a context all geometric (see (1.35)). Now we get the same relation starting from a direct calculation:

We already know that: $g_{mn} = h_{ab} \dfrac{\partial x^a}{\partial x^m} \dfrac{\partial x^b}{\partial x^n}$; now we apply $\dfrac{\partial}{\partial x^l}$ to it:

$\dfrac{\partial g_{mn}}{\partial x^l} = \dfrac{\partial^2 x^a}{\partial x^l \partial x^m} \dfrac{\partial x^b}{\partial x^n} h_{ab} + \dfrac{\partial x^a}{\partial x^m} \dfrac{\partial^2 x^b}{\partial x^l \partial x^n} h_{ab}$; if we now remember that $\dfrac{\partial^2 x^a}{\partial x^m \partial x^n} = \Gamma^l_{mn} \dfrac{\partial x^a}{\partial x^l}$, we have:

$$\frac{\partial g_{mn}}{\partial x^l} = \Gamma^r_{lm} \frac{\partial x^a}{\partial x^r} \frac{\partial x^b}{\partial x^n} h_{ab} + \Gamma^r_{ln} \frac{\partial x^a}{\partial x^m} \frac{\partial x^a}{\partial x^r} h_{ab} = \Gamma^r_{lm} g_{rn} + \Gamma^r_{ln} g_{rm} \qquad (2.39)$$

Similarly, we also calculate $\dfrac{\partial g_{ln}}{\partial x^m}$ and $\dfrac{\partial g_{ml}}{\partial x^n}$, from which:

$\dfrac{\partial g_{mn}}{\partial x^l} + \dfrac{\partial g_{ln}}{\partial x^m} - \dfrac{\partial g_{ml}}{\partial x^n} = 2 g_{kn} \Gamma^k_{lm}$; now, by defining a reciprocal matrix (or tensor) g^{ns} so that: $g^{ns} g_{kn} = d^s_k$, we'll have, therefore:

$$\Gamma^s_{lm} = \frac{1}{2} g^{ns} \left(\frac{\partial g_{mn}}{\partial x^l} + \frac{\partial g_{ln}}{\partial x^m} - \frac{\partial g_{ml}}{\partial x^n}\right), \qquad (2.40)$$

which is exactly what we got in a purely geometrical context in Chapter 1.

Par. 2.8: The Newtonian limit.

We know that in the Newtonian limit (V<<c): $dt^2 = c^2 dt^2 - d\vec{x}^2 = c^2 dt^2 - \vec{V}^2 dt^2 \cong c^2 dt^2$

This is to say that $V^i = dx^i / dt \propto dx^i / dt << dx^0 / dt \propto dx^0 / dt = d(ct)/dt = c$, that is, indeed: $dx^i / dt << dx^0 / dt$.

So, (2.38) becomes: $\dfrac{d^2 x^m}{dt^2} + \Gamma^m_{00} (\dfrac{dt}{dt})^2 = 0$ (with the convention c=1); moreover, as, in this limit, the field is stationary, g_{mn} tends to h_{mn} (const) and so the temporal derivatives of g_{mn} are zero, and so, according to (2.40):

$\Gamma^m_{00} = -\dfrac{1}{2} g^{mn} \dfrac{\partial g_{00}}{\partial x^n}$ e $g_{ab} = h_{ab} + h_{ab}$, with $|h_{ab}| << 1$ and so: $\Gamma^a_{00} = -\dfrac{1}{2} h^{ab} \dfrac{\partial h_{00}}{\partial x^b}$ and:

$\dfrac{d^2 \vec{x}}{dt^2} = \dfrac{1}{2} (\dfrac{dt}{dt})^2 \nabla h_{00} = 0$ from which:

$$\frac{d^2 \vec{x}}{dt^2} = \frac{1}{2} \nabla h_{00} = 0 \quad \text{and} \qquad (2.41)$$

$\dfrac{d^2 t}{dt^2} = 0$ from which: $\dfrac{dt}{dt} = C$.

Now, we know from Newton's gravitation that: $m\vec{a} = -G \dfrac{mM}{r^2} \hat{r}$, from which: $\vec{a} = -\dfrac{GM}{r^2} \hat{r}$,

but: $\dfrac{GM}{r^2} \hat{r} = \nabla(-\dfrac{GM}{r}) = \nabla f$ (con $f = -\dfrac{GM}{r}$) and so: $\vec{a} = \dfrac{d^2 \vec{x}}{dt^2} = -\nabla f$ and from both this one and (2.41), we have: $h_{00} = -2f + const$; as at infinite: $h_{00} = f = 0$, then const=0 and:

$$g_{00} = h_{00} + h_{00} = -1 - 2f = -(1 + 2f) \qquad (2.42)$$

Poisson’s equation:

now, we define, with simplicity, the flux of the acceleration vector $\vec{a}$:

$\Phi(\vec{a})$; we have: $d\Phi(\vec{a}) = \vec{a}\cdot d\vec{S} = -\dfrac{GM}{r^2}\hat{r}\cdot\hat{u}dS = -\dfrac{GM}{r^2}\cos q dS = -GM\dfrac{dS_n}{r^2} = -GMd\Omega$, from which:

$\Phi(\vec{a}) = \int_S \vec{a}\cdot d\vec{S} = -GM\int d\Omega = -4pGM$, but: $M = \int_V r(x,y,z)dV$, from which:

$\Phi(\vec{a}) = -4pG\int_V r(x,y,z)dV = \int_S \vec{a}\cdot d\vec{S} = for\_Theorem\_of\_Div. = \int_V \vec{\nabla}\cdot\vec{a}dV$, that is:

$\vec{\nabla}\cdot\vec{a} = -4pGr(x,y,z)$

Now, as we know from mathematics that $\vec{\nabla}(\frac{1}{r}) = -\frac{\hat{r}}{r^2}$, then: $\nabla f = -\vec{a}$ and so:

$\Delta f = \nabla^2 f = 4pGr$ (Poisson’s Equation) (2.43)

Par. 2.9: The Riemann-Christoffel Curvature Tensor.

At Par. 2.7 we proved that $\Gamma^s_{lm} = \frac{1}{2}g^{ns}(\frac{\partial g_{mn}}{\partial x^l} + \frac{\partial g_{ln}}{\partial x^m} - \frac{\partial g_{ml}}{\partial x^n})$, that is, exactly what we got at Chapter 1, through (1.35), in a purely geometrical basis. Now, it could be possible, but a bit boring, to deduct the form of the Riemamm-Christoffel curvature tensor through direct calculations, purely mathematical, exactly like in Par. 2.7, but we would get exactly what already obtained at Chapter. 1 with (1.41), here reported again, and obtained in a more suitable geomertical basis and where we will use Greek letters for the indexes, to show that here we are talking about gravitational fields with any intensity and also reminding here that those indexes have four values each (space-time), three for space and one for time:

$$R^l_{mnk} = (\Gamma^l_{mn})_k - (\Gamma^l_{mk})_n + \Gamma^h_{mn}\Gamma^l_{kh} - \Gamma^h_{mk}\Gamma^l_{nh} = \frac{\partial\Gamma^l_{mn}}{\partial x^k} - \frac{\partial\Gamma^l_{mk}}{\partial x^n} + \Gamma^h_{mn}\Gamma^l_{kh} - \Gamma^h_{mk}\Gamma^l_{nh} \qquad (2.43)$$

and also $R^s_{mnk} = g^{ls}R_{lmnk}$, from which, also: $R_{lmnk} = g_{ls}R^s_{mnk}$ and reminding the expression (2.40) for Γ^c_{ab} :

$$R_{lmnk} = g_{ls}R^s_{mnk} = \frac{1}{2}g_{ls}\frac{\partial}{\partial x^k}[g^{sr}(\frac{\partial g_{rm}}{\partial x^n} + \frac{\partial g_{rn}}{\partial x^m} - \frac{\partial g_{mn}}{\partial x^r})] - \frac{1}{2}g_{ls}\frac{\partial}{\partial x^n}[g^{sr}(\frac{\partial g_{rm}}{\partial x^k} + \frac{\partial g_{rk}}{\partial x^m} - \frac{\partial g_{mk}}{\partial x^r})] +$$
$$+ g_{ls}[\Gamma^h_{mn}\Gamma^s_{kh} - \Gamma^h_{mk}\Gamma^s_{nh}] \qquad (2.44)$$

Now, we know that $g_{ls}g^{sr} = d^r_l$, from which: $\frac{\partial}{\partial x^k}(g_{ls}g^{sr}) = 0$ and so, also through (2.39), that is, the following: $\frac{\partial g_{ij}}{\partial u^k} = g_{mj}\Gamma^m_{ik} + g_{mi}\Gamma^m_{ak}$, we have:

$$g_{ls}\frac{\partial}{\partial x^k}g^{sr} = -g^{sr}\frac{\partial}{\partial x^k}g_{ls} = -g^{sr}(\Gamma^h_{kl}g_{hs} + \Gamma^h_{ks}g_{hl}) \qquad (2.45)$$

where we have made a small rearranging of indexes, which have (those indexes) generic values, that is, we don’t care if we use j or k; what’s important is that they can have all four values, as we already know, and that their repetition is consistent.

If we go back to (2.44), it becomes, through (2.45):

$$R_{lmnk} = \frac{1}{2}[\frac{\partial^2 g_{ln}}{\partial x^k \partial x^m} - \frac{\partial^2 g_{mn}}{\partial x^k \partial x^l} - \frac{\partial^2 g_{lk}}{\partial x^n \partial x^m} + \frac{\partial^2 g_{mk}}{\partial x^n \partial x^l})] - [\Gamma^h_{kl} g_{hs} + \Gamma^h_{ks} g_{hl}]\Gamma^s_{mn} + [\Gamma^h_{nl} g_{hs} + \Gamma^h_{ns} g_{hl}]\Gamma^s_{mk} +$$

$+ g_{ls}[\Gamma^h_{mn}\Gamma^s_{kh} - \Gamma^h_{mk}\Gamma^s_{nh}]$ that is:

$$R_{lmnk} = \frac{1}{2}[\frac{\partial^2 g_{ln}}{\partial x^k \partial x^m} - \frac{\partial^2 g_{mn}}{\partial x^k \partial x^l} - \frac{\partial^2 g_{lk}}{\partial x^n \partial x^m} + \frac{\partial^2 g_{mk}}{\partial x^n \partial x^l})] + g_{hs}[\Gamma^h_{nl}\Gamma^s_{mk} - \Gamma^h_{kl}\Gamma^s_{mn}] \qquad (2.46)$$

As already made at the end of Par. 1.3, we deduce three properties of the tensor R_{lmnk}, which are directly verifiable:

-Simmetry $R_{lmnk} = R_{nklm}$

-Antisimmetry $R_{lmnk} = -R_{mlnk} = -R_{lmkn} = R_{mlkn}$

-Cyclicity $R_{lmnk} + R_{lkmn} + R_{lnkm} = 0$

<u>The Ricci Tensor:</u>

$$R_{mk} = g^{ln} R_{lmnk} \qquad (R_{mk} = R_{km}) \qquad (2.47)$$

and one can directly verify that : $R_{mk} = -g^{ln} R_{mlnk} = -g^{ln} R_{lmkn} = +g^{ln} R_{mlkn}$

It's then clear the strong relationship of such a tensor with the Gauss curvature (see (1.40), bidimensional), from which its name curvature tensor, indeed.

<u>The Bianchi's Identity:</u>

if we put ourselves in a locally inertial reference system (not strong gravitational field) all Γ^c_{ab} are zero; in fact, the difference between the geodetic equation of an Euclidean space (2.37) and that of a space strongly curved up by gravity, that is, the (2.38), is really the presence of a Γ^c_{ab}. In a locally inertial coordinate system, therefore, (2.46) yields (derivation ;η expressed by the presence of $\frac{\partial}{\partial x^h}$):

$$R_{lmnk;h} = \frac{1}{2}\frac{\partial}{\partial x^h}[\frac{\partial^2 g_{ln}}{\partial x^k \partial x^m} - \frac{\partial^2 g_{mn}}{\partial x^k \partial x^l} - \frac{\partial^2 g_{lk}}{\partial x^n \partial x^m} + \frac{\partial^2 g_{mk}}{\partial x^n \partial x^l})] \qquad (2.48)$$

and so, by direct verification, we have:

$$R_{lmnk;h} + R_{lmhn;k} + R_{lmkh;n} = 0$$

By contracting (by multiplying with), in the above equation, l,n with g^{ln}, we have:

$R_{mk;h} - R_{mh;k} + R^n_{mkh;n} = 0$ and by contracting again:

$R_{;h} - R^m_{h;m} - R^n_{h;n} = 0$ which is the same as to say also that (the last two terms are similar: 2x >>> 1/2):

$$(R^m_h - \frac{1}{2}d^m_h R)_{;m} = 0 \quad \text{and} \quad (R^{mn} - \frac{1}{2}g^{mn} R)_{;m} = 0 \qquad (2.49)$$

Chapter 3: The Einstein's Equations of the Gravitational Field.

Par. 3.1: The ten Einstein's Equations of the Gravitational Field.

They are 16 equations, actually, as they contain rank 2 tensors, that is, with two indexes each, and everyone of them can have 4 values, and so 4x4=16, but such equations are not all linearly independent among them, that is, there are doubles, and the independent ones are ten, indeed.

We know from (2.42) that $g_{00} = -(1+2f)$ (contact point with Newton's theory and starting base), while from (2.36) we know that, for non relativistic matter: $T_{00} = rc^2 = r$ (with normalization c=1); we also have that:
$\Delta g_{00} = \nabla^2 g_{00} = \nabla^2[-(1+2f)] = -2\nabla^2 f = -8pGr = -8pGT_{00}$, that is:
$\nabla^2 g_{00} = -8pGT_{00}$; so we can suppose, out of extension, that the following equality holds:
$G_{ab} = -8pGT_{ab}$ and for gravitational fields of any intensity:

$$G_{mn} = -8pGT_{mn}$$

Let's deduce, now, five peculiarities G_{mn} must have:
A)by definition, G_{mn} is a tensor, as the momentum-energy tensor T_{mn} is
B) G_{mn} is consisting of terms with second derivatives of the metric tensor (just look at $\nabla^2 g_{00}$)
C) G_{mn} is symmetric, as well as T_{mn}
D)as T_{mn} is conserved ($T_{mn\,;m} = 0$), then G_{mn} and similars are, as well ($G^m_{n\,;m} = 0$)
E)for weak stationary fields, non relativistic ones, we have $G_{mn} = \nabla^2 g_{00}$

A and B say that G_{mn} is proportional to the curvature tensor (2.46), or better to (2.48), clearly made of second derivatives of the metric tensor.
Moreover, the symmetry of indexes wants that the curvature tensor is represented by the Ricci tensor $R_{mk} = R_{km}$ and by the symmetric, as well, $R = R^m_m$ (see the paragraph on the Ricci tensor):
$G_{mn} = C_1 R_{mn} + C_2 g_{mn} R$ (3.1)
but, through (2.49), we have also seen that:
$(R^m_n - \frac{1}{2} d_n^{\ m} R)_{;m} = 0$, from which: $R^m_{n\,;m} = \frac{1}{2} d_n^{\ m} R_{,m} = \frac{1}{2} R_{,n}$; now, multiply (3.1) by g^{mm} :
$G^m_n = C_1 R^m_n + C_2 R g^m_n$ and $G^m_{n\,;m} = C_1 R^m_{n\,;m} + C_2 R_{,n} = \frac{C_1}{2} R_{,n} + C_2 R_{,n} = (\frac{C_1}{2} + C_2) R_{,n}$
For the peculiarity D, $C_2 = -\frac{C_1}{2}$ or $R_{,n} = 0$; the second one must be rejected, as:
$G^m_m = (C_1 + g_{mn} g^{mn} C_2) R = (C_1 + 4C_2) R = -8pGT^m_m$, and so, if $R_{,n} = \partial R / \partial x^n$ becomes zero, the same must happen to $\partial T^m_m / \partial x^n$, but now we aren't yet in the case of non relativistic matter.

Therefore: $G_{mn} = C_1(R_{mn} - \frac{1}{2} g_{mn} R)$ (3.2)

Now, because of the peculiarity E, we figure out C_1: for non relativistic systems, we always have:

$|T_{ij}| << T_{00}$, that is: $|G_{ij}| << G_{00}$, from which, for the above (3.2): $R_{ij} \cong \frac{1}{2} g_{ij} R$; moreover, $g_{ab} \cong h_{ab}$ (Minkowski's tensor) and so: $R_{kk} \cong \frac{3}{2} R$ and $R_{00} \cong \frac{R}{2}$.

(3.2) with $m = n = 0$ ($g_{00} \cong h_{00}$) yields:

$G_{00} = C_1(R_{00} - \frac{1}{2}(-1)2R_{00}) = 2C_1 R_{00}$; moreover, in case of weak fields, we can say (see (2.46) with Γ^c_{ab} =0 or directly (2.48)):

$R_{lmnk} = \frac{1}{2}[\frac{\partial^2 g_{ln}}{\partial x^k \partial x^m} - \frac{\partial^2 g_{mn}}{\partial x^k \partial x^l} - \frac{\partial^2 g_{lk}}{\partial x^n \partial x^m} + \frac{\partial^2 g_{mk}}{\partial x^n \partial x^l})]$ and $R_{mk} = h^{ln} R_{lmnk}$; being the field static, the temporal derivatives are zero:

$R_{00} = h^{ln} R_{l0n0} = \frac{1}{2} h^{ln} (\frac{\partial^2 g_{00}}{\partial x^n \partial x^l}) = \frac{1}{2} h^{ij} (\frac{\partial^2 g_{00}}{\partial x^i \partial x^j}) - \frac{1}{2} h^{00} 0 = \frac{1}{2} \nabla^2 g_{00}$, from which:

$G_{00} = 2C_1 \frac{1}{2} \nabla^2 g_{00} = C_1 \nabla^2 g_{00}$, from which ($C_1 = 1$) $G_{mn} = R_{mn} - \frac{1}{2} g_{mn} R$ and so:

$R_{mn} - \frac{1}{2} g_{mn} R = -8pGT_{mn}$ (3.3)

which are the **Einstein's equations of the gravitational field**, and we rewrite them:

$$R_{mn} - \frac{1}{2} g_{mn} R = -8pGT_{mn}$$

which tell us that the curvature ($\propto R_{mn}, R$) of the space-time is equal to the presence of matter-energy ($\propto T_{mn}$) in it!!!

Now, by contracting with g^{mn} , we have: $R - 2R = -8pGT^m_m$, that is: $R = 8pGT^m_m$ and so:

$R_{mn} = -8pG(T_{mn} - \frac{1}{2} g_{mn} T^l_l)$ (another form for the Einstein's equations).

In the vacuum, f(T)=0 and so $R_{mn} = 0$.

Chapter 4: Classic tests of Einstein's theory.

Par. 4.1: The metric.

We still have c=1, as a simplifying convention. Now, we define a general metric tensor through which a gravitational field is static and isotropic; static means that the metric tensor does not depend on time, about its form and its characteristics, with a clear reference to its coefficients. Isotropic means that there is a dependance from the irrotational invariants; in fact, we know that the norm of a vector and the scalar product between two vectors are invariant for rotations:

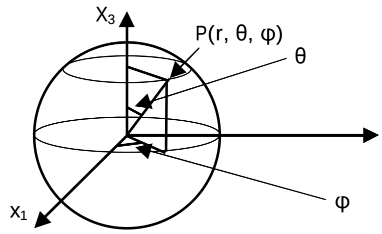
$\vec{x}^2 = |\vec{x}|^2$, $(d\vec{x})^2$, $\dot{x} \cdot d\dot{x}$

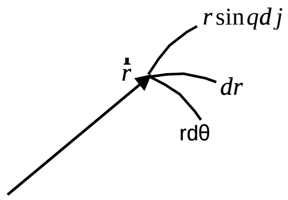
All this for orthogonal coordinates almost Minkowskian (almost h_{ab})

$$dt^2 = -g_{mn} dx^m dx^n \ , \ dt^2 = F(r)dt^2 - 2E(r)dtx \cdot dx - D(r)(x \cdot dx)^2 - C(r)dx^2 \qquad (4.1)$$

$r = (\vec{x} \cdot \vec{x})^{1/2}$ and in spherical coordinates:

$$\begin{cases} x_1 = r \sin q \cos j \\ x_2 = r \sin q \sin j \\ x_3 = r \cos q \end{cases}$$





$$dt^2 = F(r)dt^2 - 2rE(r)dtdr - r^2 D(r)(dr)^2 - C(r)(dr^2 + r^2 dq^2 + r^2 \sin^2 q dj^2)$$

Now, we define the linear application $t' = t + f(r)$ and get rid of the non diagonal elements (combined) by putting:

$\frac{df}{dr} = -\frac{rE(r)}{F(r)}$; there exists a linear rototranslation/application which reduces to a canonical form the above quadratic form:

$dt^2 = F(r)dt'^2 - G(r)dr^2 - C(r)(dr^2 + r^2 dq^2 + r^2 \sin^2 q dj^2)$, with:

$$G(r) = r^2 (D(r) + \frac{E^2(r)}{F(r)})$$

Let's define $r'^2 = C(r)r^2$; then, we get the standard form:

$$dt^2 = B(r')dt'^2 - A(r')dr'^2 - r'^2 (dq^2 + \sin^2 q dj^2) \qquad (4.2)$$

with $B(r') = F(r)$ $A(r') = (1 + \frac{C(r)}{G(r)})(1 + \frac{r}{2C(r)} \frac{dC(r)}{dr})^{-2}$

We are now interested in the standard form (4.2):

$dt^2 = B(r)dt^2 - A(r)dr^2 - r^2 (dq^2 + \sin^2 q dj^2)$ and through a comparison with the general expression of the metric (2.9), we get: $g_{rr} = A(r)$, $g_{qq} = r^2$, $g_{jj} = r^2 \sin^2 q$, $g_{tt} = -B(r)$ and as g_{mn} is orthogonal, we have that: $g^{rr} = A^{-1}(r)$, $g^{qq} = r^{-2}$, $g^{jj} = r^{-2} \sin^{-2} q$, $g^{tt} = -B^{-1}(r)$.

Then, as we know that according to (2.40): $\Gamma^l_{mn} = \frac{1}{2} g^{lr} (\frac{\partial g_{rm}}{\partial x^n} + \frac{\partial g_{rn}}{\partial x^m} - \frac{\partial g_{mn}}{\partial x^r})$, it comes out that the only non zero quantities are:

$\Gamma^r_{rr} = \frac{1}{2A(r)} \frac{dA(r)}{dr}$, $\Gamma^r_{qq} = -\frac{r}{A(r)}$, $\Gamma^r_{jj} = -\frac{r \sin^2 q}{A(r)}$, $\Gamma^r_{tt} = \frac{1}{2A(r)} \frac{dB(r)}{dr}$, $\Gamma^q_{rq} = \Gamma^q_{qr} = \frac{1}{r}$,

$\Gamma^q_{jj} = -\sin q \cos q$, $\Gamma^j_{rj} = \Gamma^j_{jr} = \frac{1}{r}$, $\Gamma^j_{qj} = \Gamma^j_{jq} = \cot q$, $\Gamma^t_{tr} = \Gamma^t_{rt} = \frac{1}{2B(r)} \frac{dB(r)}{dr}$.

Now, we calculate the Ricci tensor ($l = n$ in R^l_{mnk} of the (2.43)):

$R_{mk} = \frac{\partial \Gamma^l_{ml}}{\partial x^k} - \frac{\partial \Gamma^l_{mk}}{\partial x^l} + \Gamma^h_{ml} \Gamma^l_{kh} - \Gamma^h_{mk} \Gamma^l_{lh}$ and we remind that $\frac{\partial \Gamma^l_{ml}}{\partial x^k}$ is symmetric over m and k, also by a direct verification; therefore, we totally have:

$$R_{rr} = \frac{B''(r)}{2B(r)} - \frac{1}{4}(\frac{B'(r)}{B(r)})(\frac{A'(r)}{A(r)} + \frac{B'(r)}{B(r)}) - \frac{1}{r}\frac{A'(r)}{A(r)}$$

$$R_{qq} = -1 + \frac{r}{2A(r)}(-\frac{A'(r)}{A(r)} + \frac{B'(r)}{B(r)}) + \frac{1}{A(r)}$$

$R_{jj} = \sin^2 q \cdot R_{qq}$, $R_{tt} = -\frac{B''(r)}{2A(r)} + \frac{1}{4}(\frac{B'(r)}{A(r)})(\frac{A'(r)}{A(r)} + \frac{B'(r)}{B(r)}) - \frac{1}{r}\frac{B'(r)}{A(r)}$, $R_{mn} = 0$ for $m \neq n$.

Par. 4.2: The Schwarschild's Solution.

We already know that: $dt^2 = B(r)dt^2 - A(r)dr^2 - r^2dq^2 - r^2\sin^2 qdj^2$
Moreover, in the vacuum $R_{mn} = 0$, so:

$R_{rr} = R_{qq} = R_{tt} = 0$; (4.3)

moreover, we notice that:
$\frac{R_{rr}}{A} + \frac{R_{tt}}{B} = -\frac{1}{rA}(\frac{A'}{A} + \frac{B'}{B})$ and for the (4.3), we have: $\frac{A'}{A} = -\frac{B'}{B}$, that is:

$A(r)B(r) = const$. (4.4)

Then, for $r \to \infty$, the metric tensor g_{mn} must get close to the Minkowski's tensor h_{mn} in spherical coordinates, that is: $\lim_{r\to\infty} A(r) = \lim_{r\to\infty} B(r) = 1$, from which, for the (4.4):
$A(r) = \frac{1}{B(r)}$ and by using this one in the expressions for R_{qq} and R_{rr}, we have:

$R_{qq} = -1 + B'(r)r + B(r)$ and $R_{rr} = \frac{B''(r)}{2B(r)} + \frac{B'(r)}{rB(r)} = \frac{R'_{qq}(r)}{2rB(r)}$; by putting $R_{qq} = 0$, we have:

$\frac{d}{dr}rB(r) = rB'(r) + B(r) = 1$, from which: $rB(r) = r + const$.

Moreover, for $r \to \infty$: $g_{tt} = g_{00} = -B = -1 - 2f \to (f = -\frac{GM}{r}) \to$

$B(r) = [1 - \frac{2GM}{r}]$ and $A(r) = [1 - \frac{2GM}{r}]^{-1}$ and so, finally:

$$dt^2 = [1 - \frac{2GM}{r}]dt^2 - [1 - \frac{2GM}{r}]^{-1}dr^2 - r^2dq^2 - r^2\sin^2 qdj^2 \qquad (4.5)$$

(Schwarschild's solution)

Par. 4.3: The general equations of motion.

We know that $dt^2 = B(r)dt^2 - A(r)dr^2 - r^2dq^2 - r^2\sin^2 qdj^2$ and we also consider the geodetic equation (2.38): $\frac{d^2x^m}{dp^2} + \Gamma^n_{nl}\frac{dx^n}{dp}\frac{dx^l}{dp} = 0$ (in p generic, for the moment): we have, by making m change:

$$0 = \frac{d^2r}{dp^2} + \frac{A'(r)}{2A(r)}(\frac{dr}{dp})^2 - \frac{r}{A(r)}(\frac{dq}{dp})^2 - \frac{r\sin^2 q}{A(r)}(\frac{dj}{dp})^2 + \frac{B'(r)}{2A(r)}(\frac{dt}{dp})^2 \qquad (4.6)$$

$$0=\frac{d^2q}{dp^2}+\frac{2}{r}\frac{dq}{dp}\frac{dr}{dp}-\sin q\cos q(\frac{dj}{dp})^2 \quad (4.7)$$

$$0=\frac{d^2j}{dp^2}+\frac{2}{r}\frac{dj}{dp}\frac{dr}{dp}+2\cot q\frac{dj}{dp}\frac{dq}{dp} \quad (4.8)$$

$$0=\frac{d^2t}{dp^2}+\frac{B'(r)}{B(r)}\frac{dt}{dp}\frac{dr}{dp} \quad (4.9)$$

Now, as the field is isotropic, we put $q=p/2$ and so the last two equations, (4.8) and (4.9), becomes:

$\frac{d}{dp}[\ln\frac{dj}{dp}+\ln r^2]=0$ and $\frac{d}{dp}[\ln\frac{dt}{dp}+\ln B]=0$, from which:

$$r^2\frac{dj}{dp}=J \text{ (constant)} \quad (4.10)$$

$$\frac{dt}{dp}B=const \quad \text{(=1, by choice)} \quad (4.11)$$

from which: $\frac{dt}{dp}=\frac{1}{B}$. Now, by putting (4.10), (4.11) and the condition ($q=p/2$) used before, in the (4.6), we'll have:

$0=\frac{d^2r}{dp^2}+\frac{A'(r)}{2A(r)}(\frac{dr}{dp})^2-\frac{J^2}{r^3A(r)}+\frac{B'(r)}{2A(r)B^2(r)}$ and by multiplying, now, by $2A(r)\frac{dr}{dp}$:

$\frac{d}{dp}[A(r)(\frac{dr}{dp})^2+\frac{J^2}{r^2}-\frac{1}{B(r)}]=0$ that is:

$$A(r)(\frac{dr}{dp})^2+\frac{J^2}{r^2}-\frac{1}{B(r)}=-E \text{ (constant)} \quad (4.12)$$

If now we make a system with the equations ($q=p/2$) just used, then the (4.10), (4.11) and (4.12), we get:

$$dt^2=Edp^2 \quad (4.13)$$

We know that E=0 for photons and E>0 for material particles.

As A(r) is always positive, we have that the particle can reach r only if (see (4.12)):

$\frac{J^2}{r^2}+E\leq-\frac{1}{B(r)}$. Then, by using (4.11) in (4.10), (4.12) and (4.13), we get:

$$r^2\frac{dj}{dt}=JB(r) \quad (4.14)$$

$$\frac{A(r)}{B^2(r)}(\frac{dr}{dt})^2+\frac{J^2}{r^2}-\frac{1}{B(r)}=-E \quad (4.15)$$

and $dt^2=EB^2(r)dt^2$ (4.16)

Now we know that, for weak fields: $B-1=2f \quad \rightarrow \quad r^2\frac{dj}{dt}=J$ and

$$\frac{1}{2}(\frac{dr}{dt})^2+\frac{J^2}{2r^2}+f\cong\frac{1-E}{2} \quad (4.17)$$

as: $-\frac{1}{B(r)}=-\frac{1}{1+2f}\cong(forTaylor)\cong-(1-2f)$

(4.17) has got a similar correspondance in Newton's classic mechanics.

For general orbits, r=r(φ); then, we know that:

$$\begin{cases} r^2 \dfrac{dj}{dp} = J \\ A(r)(\dfrac{dr}{dp})^2 + \dfrac{J^2}{r^2} - \dfrac{1}{B(r)} = -E \end{cases}$$

if we get rid of dp, we have: $\dfrac{A(r)}{r^4}(\dfrac{dr}{dj})^2 + \dfrac{1}{r^2} - \dfrac{1}{J^2 B(r)} = -\dfrac{E}{J^2}$, (4.18)

whose solution is:

$$j = \pm \int \frac{A^{1/2}(r)dr}{r^2(\frac{1}{J^2 B(r)} - \frac{E}{J^2} - \frac{1}{r^2})^{1/2}} \quad (4.19)$$

Par. 4.4: The deflection of light by the Sun.

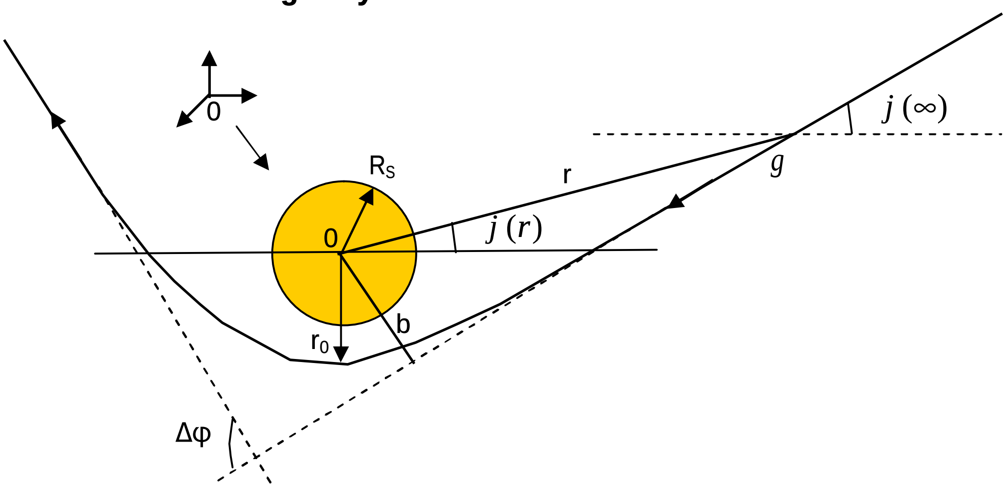


Fig. 4.1: The deflection of light by the Sun.

$R_S = R_{Sun}$; b is the collision parameter. At the infinite, $b = r\sin(j - j_{\infty}) \cong r(j - j_{\infty})$ and: $-V = \dfrac{d}{dt} r\cos(j - j_{\infty}) \cong \dfrac{dr}{dt}$; V is the motion speed (constant). As at the infinite A=B=1, if we put these two equations in (4.14) and (4.15), we have:

$J = bV$ and (4.20)

$E = 1 - V^2$ (4.21)

(4.21) is trivial; in order to get the (4.20), we see that:

$b \cong r(j - j_{\infty})$, from which: $0 = dr(j - j_{\infty}) + rdj$, and so: $r^2 \dfrac{dj}{dt} = -\dfrac{dr}{dt}(j - j_{\infty})r = bV = J$

For $r = r_0$ we have that $dr/dj = 0$ and (4.18) then becomes: $J = r_0(\dfrac{1}{B(r_0)} - 1 + V^2)^{1/2}$ and

(4.19) becomes, after an easy calculation:

$$j(r) = j(\infty) + \int_r^{\infty} \frac{A^{1/2}(r)dr}{r^2[\frac{1}{r_0^2}(\frac{1}{B(r)} - 1 + V^2)(\frac{1}{B(r_0)} - 1 + V^2)^{-1} - \frac{1}{r^2}]^{1/2}} \quad (4.22)$$

The total variation of φ is: $2|j(r_0) - j_\infty|$, while, if the ray of light walked unperturbed, we would have a variation of p, so, with reference to Fig. 4.1, we have: $\Delta j = 2|j(r_0) - j_\infty| - p$; for a photon, $V^2 = 1$ and (4.22) yields:

$$j(r) - j(\infty) = \int_r^\infty A^{1/2}(r)[(\frac{r}{r_0})^2(\frac{B(r_0)}{B(r)}) - 1]^{-\frac{1}{2}} \frac{dr}{r} \quad (4.23)$$

Now, by using the (Taylor's) developments due to Robertson:

$A(r) = 1 + 2\frac{GM}{r} + ...$, $B(r) = 1 - 2\frac{GM}{r} + ...$, we have:

$$(\frac{r}{r_0})^2(\frac{B(r_0)}{B(r)}) - 1 = (\frac{r}{r_0})^2[\frac{1 - 2\frac{GM}{r_0} + ...}{1 - 2\frac{GM}{r} + ...}] - 1 = (\frac{r}{r_0})^2[1 + 2GM(\frac{1}{r} - \frac{1}{r_0}) + ...] - 1 \cong$$

$$\cong [(\frac{r}{r_0})^2 + (\frac{r}{r_0})^2 2GM\frac{(r_0 - r)}{rr_0} - 1] = [(\frac{r}{r_0})^2 - 1][1 - \frac{2GMr}{r_0(r + r_0)} + ...]$$

The last equality can be directly verified.
(4.23) becomes, for Taylor:

(in order to solve the first two integrals, put $\frac{r_0}{r} = \cos x$, while for the third, put $\frac{r_0}{r} = t$ and the integral will be $\frac{\sqrt{1-t}}{\sqrt{1+t}}$)

$$j(r) - j(\infty) = \int_r^\infty A^{1/2}(r)\frac{1}{r}[(\frac{r}{r_0})^2 - 1]^{-\frac{1}{2}}[1 + \frac{GM}{r} + \frac{GMr}{r_0(r + r_0)} + ...]dr \quad \text{that is:}$$

1st Int.	2nd Int.	3rd Int.

$$j(r) - j(\infty) = \sin^{-1}(\frac{r_0}{r}) + \frac{GM}{r_0}[1 + 1 - \sqrt{1 - (\frac{r_0}{r})^2} - \sqrt{\frac{r - r_0}{r + r_0}}] + ... \quad \text{and so:}$$

$\Delta j = \frac{4GM}{r_0}$, and if we remind our normalization (c=1), we finally have: $\Delta j = \frac{4GM}{r_0 c^2}$; with $r_0 = R_S = 6{,}95 \cdot 10^8 m$ and $M_S = 1{,}97 \cdot 10^{30} kg$, we have:

$\Delta j = 1{,}75''$, in perfect agreement with experimental results. In reality, when in 1919 the deflection of star light by the Sun was measured in Brazil, during an eclipse, the accuracy of the measurement was as big as the measure itself.

Par. 4.5: An alternative calculation of the deflection, with profiles of antagonism to GTR.

This method (Firk) is based on the variation of velocity the light undergoes when it approaches a mass; for this reason I see profiles of antagonism to pure GTR. Then, there exsist also other methods, more or less similar, based on such a supposition (for instance Wåhlin) and it seems that, if we take into account the smallest cyphers after the point, those results are even more similar to experimental values.
First of all, we remind that, according to Schwarzschild (see, for instance, (4.5)):

$dt^2 = B(r)dt^2 - A(r)dr^2 - r^2 dq^2 - r^2 \sin^2 q dj^2$; (4.24)

Then, we know that, in general: $dt^2 = c^2(dt)^2 - (d\vec{x})^2$ and as, for a photon, $(d\vec{x})^2 = c^2(dt)^2$, we have: $dt^2 = 0$, from which, for the (4.24):

$0 = B(r)dt^2 - A(r)dr^2 - r^2 dq^2 - r^2 \sin^2 q dj^2$; moreover, if we consider light radially travelling towards the Sun, we can get rid of the components in dq and dj :

$0 = B(r)dt^2 - A(r)dr^2$; (4.25)

The speed of light is $c = dr/dt$ far from the mass of the Sun, while, close to it, the (4.25) holds, from which we get:

$V = dr/dt = c(B(r)/A(r))^{1/2} \neq c$;

From Par. 4.2, we have the values for A(r) and B(r), in which, during calculations, we do not forget that now c=1 does not hold anymore; therefore:

$B(r) = [1 - \frac{2GM}{rc^2}]$ e $A(r) = [1 - \frac{2GM}{rc^2}]^{-1} \cong [1 + \frac{2GM}{rc^2}]$

From the developments on the variable $\frac{2GM}{rc^2}$, we easily have that:

$V/c = ([1 - \frac{2GM}{rc^2}] / [1 + \frac{2GM}{rc^2}])^{1/2} \cong [1 - \frac{2GM}{2rc^2}][1 - \frac{2GM}{2rc^2}] \cong [1 - \frac{2GM}{rc^2}]$, that is: $V \cong c[1 - \frac{2GM}{rc^2}]$

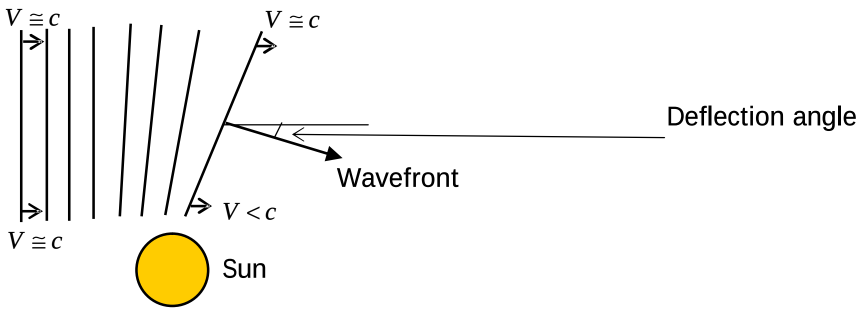


Fig. 4.2: The velocities of the wavefronts.

With reference to Figure 4.2, the part of wavefront wich is farther from the mass M has speed c, while the closer one has speed $V < c$.

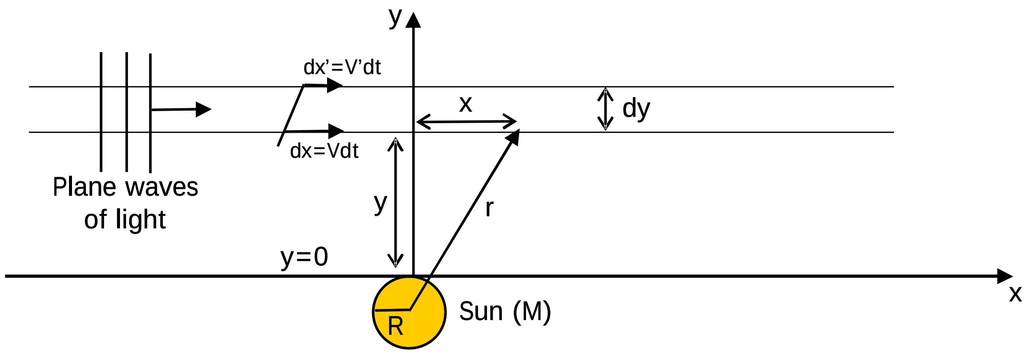


Fig. 4.3: Drawing for calculations.

Now, with reference to Figure 4.3, we have:

$r^2 = (y + R)^2 + x^2$ (eq. of a circle); (4.26)

now, we apply the operator $\partial/\partial y$ to (4.26), so having:

$2r(\partial r/\partial y) = 2(y + R)$, from which: $\partial r/\partial y = (y + R)/r$ and on the surface of the mass M:

$\partial r/\partial y|_{y\to 0} = R/r$

Now: $\frac{\partial V}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial V}{\partial r} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r}(c[1-\frac{2GM}{rc^2}]) = \frac{2GM}{r^2c}\frac{\partial r}{\partial y}$, from which:

$$\left.\frac{\partial V}{\partial y}\right|_{y\to 0} = \frac{2GM}{r^2c}\left.\frac{\partial r}{\partial y}\right|_{y\to 0} = \frac{2GM}{r^2c}\frac{R}{r} = \frac{2GMR}{r^3c} .$$

Now, we calculate the difference between the paths dx and dx' of wavefronts at a vertical distance y and y+dy, at which light has got velocities V and V' respectively:

dx'=V'dt and dx=Vdt, from which:
dx'- dx =V'dt-Vdt=dt(V'-V) ; (4.27)
moreover, we have, for Taylor: $V' = V + (\partial V/\partial y)dy$, that is: $V' - V = (\partial V/\partial y)dy$ and (4.27) becomes:

$$dx' - dx = (\partial V/\partial y)dydt \qquad (4.28)$$

Then, still from Figure 4.3 and from (4.28), we have:
$da = (dx' - dx)/dy = (\partial V/\partial y)dt = (\partial V/\partial y)dx/V$.
The total deflection Δa from $-\infty$ and $+\infty$ is, by considering that, in such a range, V is almost always equal to c (except for when it's right close to M):

$\Delta a = \int_{-\infty}^{+\infty} da = \int_{-\infty}^{+\infty}\frac{1}{V}(\partial V/\partial y)dx \cong \frac{1}{c}\int_{-\infty}^{+\infty}(\partial V/\partial y)dx$ and, close to the surface of M (y=0):

$\Delta a = \frac{1}{c}\int_{-\infty}^{+\infty}\frac{2GMR}{r^3c}dx = \frac{2GMR}{c^2}\int_{-\infty}^{+\infty}\frac{dx}{(R^2+x^2)^{3/2}} = \frac{2GMR}{c^2}\left|\frac{x}{R^2(R^2+x^2)^{1/2}}\right|_{-\infty}^{+\infty} = \frac{2GMR}{c^2}\cdot\frac{2}{R^2}$, that is:

$\boxed{\Delta a = \frac{4GM}{Rc^2} = 1{,}75''}$, right what we got in Par. 4.4!!!

Par. 4.6: The precession of the perihelion of planets.

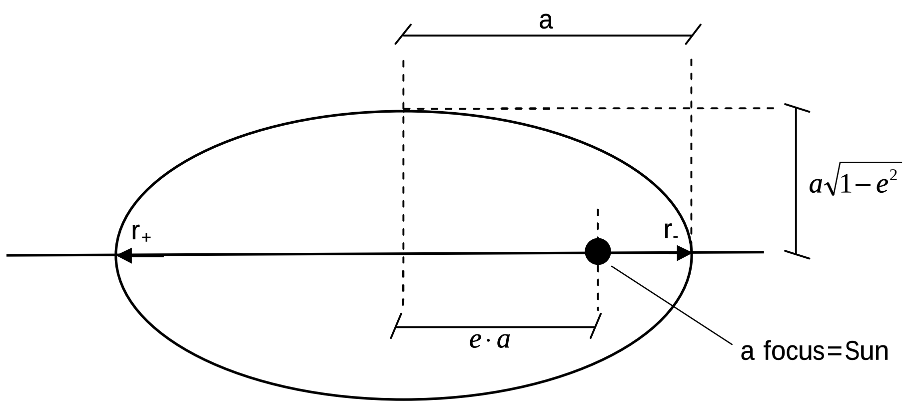


Fig. 4.4: The precession of the perihelion of Mercury.

For planets, and in particular for Mercury, through centuries, we notice that the perihelion moves.

In r_+ e r_- , we have that $\frac{dr}{dj}=0$. Then, we already know that (see (4.18)):

$\frac{A(r)}{r^4}(\frac{dr}{dj})^2+\frac{1}{r^2}-\frac{1}{J^2B(r)}=-\frac{E}{J^2}$; this one, in r_+ e r_- , becomes: $\frac{1}{r_\pm^2}-\frac{1}{J^2B(r_\pm)}=-\frac{E}{J^2}$,

from which: $E=\frac{\frac{r_+^2}{B(r_+)}-\frac{r_-^2}{B(r_-)}}{r_+^2-r_-^2}$ and $J^2=\frac{\frac{1}{B(r_+)}-\frac{1}{B(r_-)}}{\frac{1}{r_+^2}-\frac{1}{r_-^2}}$ (4.29)

We have: $j(r)=j(r_-)+\int_{r_-}^{r}\frac{A^{1/2}(r)dr}{r^2[\frac{1}{J^2B(r)}-\frac{E}{J^2}-\frac{1}{r^2}]^{1/2}}$, from which, for (4.29), we have:

$$j(r)-j(r_-)=\int_{r_-}^{r}[\frac{r_-^2[B^{-1}(r)-B^{-1}(r_-)]-r_+^2[B^{-1}(r)-B^{-1}(r_+)]}{r_+^2r_-^2[B^{-1}(r_+)-B^{-1}(r_-)]}-\frac{1}{r^2}]^{-1/2}A^{1/2}(r)r^{-2}dr \qquad (4.30)$$

The total variation of φ is: $\Delta j=2|j(r_+)-j(r_-)|$. If we did not have any precession, we would have:
$2|p|=2p$. The precession of the orbit is: $\Delta j=2|j(r_+)-j(r_-)|-2p$.
We remind the Robertson's developments on pag. 39:

$A(r)=1+2\frac{GM}{r}+....$, $B(r)=1-2\frac{GM}{r}+....$

We need a 2nd order development for B^{-1} , otherwise, in (4.30), B doesn't yield anything $\propto\frac{1}{r^2}$. For $B^{-1}(r)$ we then have:

$B^{-1}(r)\cong 1+2\frac{GM}{r}+4\frac{G^2M^2}{r^2}+....$.

Through such developments, the root in (4.30) can reduce to a quadratic form in $\frac{1}{r}$; anyway, we can notice that such a quantity cancels for $r=r_\pm$, therefore:

$$R=\frac{r_-^2[B^{-1}(r)-B^{-1}(r_-)]-r_+^2[B^{-1}(r)-B^{-1}(r_+)]}{r_+^2r_-^2[B^{-1}(r_+)-B^{-1}(r_-)]}-\frac{1}{r^2}=C(\frac{1}{r_-}-\frac{1}{r})(\frac{1}{r}-\frac{1}{r_+})$$

C can be calculated by executing the $\lim_{r\to\infty}$:

$C=\frac{r_+^2[1-B^{-1}(r_+)]-r_-^2[1-B^{-1}(r_-)]}{r_+r_-[B^{-1}(r_+)-B^{-1}(r_-)]}$; now, by factoring on both numerator and denominator:

$2(r_--r_+)MG$, we get: $C\cong 1-2MG(\frac{1}{r_+}+\frac{1}{r_-})$; with such results in (4.30), we get:

$j(r)-j(r_-)\cong ....\cong[1+MG(\frac{1}{r_+}+\frac{1}{r_-})]\int_{r_-}^{r}\frac{[1+\frac{GM}{r}]dr}{r^2[(\frac{1}{r_-}-\frac{1}{r})(\frac{1}{r}-\frac{1}{r_+})]^{1/2}}$; now, we define:

$$\frac{1}{r}=\frac{1}{2}(\frac{1}{r_+}+\frac{1}{r_-})+\frac{1}{2}(\frac{1}{r_+}-\frac{1}{r_-})\sin\Psi \qquad (4.31)$$

we get: ($r=r_-\to\Psi=-p/2$); moreover, we just get dr as a function of $d\Psi$ in (56.1) and make a replacement in the integral in dr:

$$j(r)-j(r_-)\cong ....\cong [1+\frac{3}{2}MG(\frac{1}{r_+}+\frac{1}{r_-})][\Psi+\frac{p}{2}]-\frac{1}{2}MG(\frac{1}{r_+}-\frac{1}{r_-})\cos\Psi \ .$$

For the (4.31), at the aphelion, $\Psi = p/2$, so: $\Delta j = 2|j(r_+)-j(r_-)|-2p = (\frac{6pMG}{L})$ [rad/rev] where $L=\frac{1}{2}(\frac{1}{r_+}+\frac{1}{r_-})$ (straight semiside).

Now, as we know that $r_\pm = (1\pm e)a$ and $L=(1-e^2)a$ (see note (**) below), we have: $\Delta j = (\frac{6pMG}{L})$ and, if we do not forget our initial normalization (c=1), we have: $\Delta j = (\frac{6pMG}{Lc^2})$; for Mercury, $L = 55{,}3\cdot 10^9 m$, from which $\Delta j = 0{,}1038''$. Now, as in a century Mercury makes 415 revolutions, we have $\Delta j_{century} = 43{,}03''$, in perfect agreement with experimental measurements, as the very first measurements on Mercury started in 1765, and Clemence, in 1943, calculated:
$\Delta j = 43{,}11'' \pm 0{,}45''$.

(**): some considerations on the ellipse:

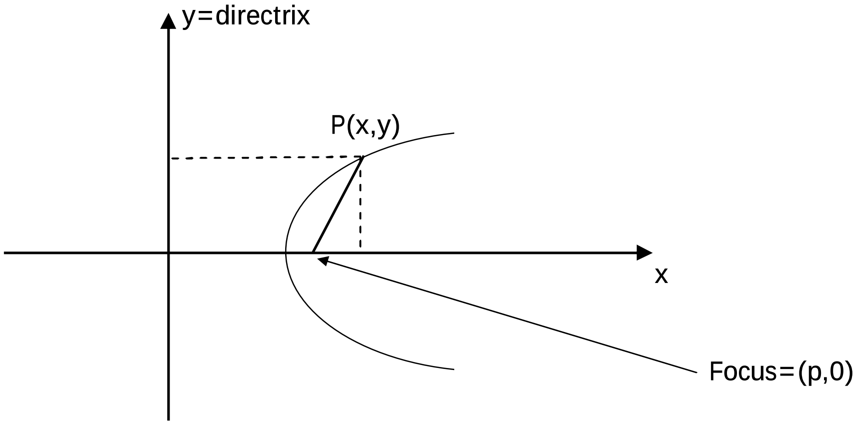


Fig. 4.5: The Ellipse.

we have, by the definition of ellipse, that: d(P,F)=ed(P,d), where e is the eccentricity and d is the directrix. Therefore: $(x-p)^2+y^2=e^2x^2$, from which $(1-e^2)x^2+y^2-2px+p^2=0$; through easy calculations, we get that, for 0<e<1,
$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, with $a=pe/(1-e^2)$ and $b=a\sqrt{1-e^2}$, but we also know that $b=\sqrt{a^2-c^2}$ (by the definition of ellipse), so, by comparison: $a\cdot e=c$.

Then, if we also take into consideration the other focus, we have: d(P,F)+ d(P,F')=const=2a; in fact: d(P,F)=ed(P,d) and d(P,F')=ed(P,d') and d(P,F)+ d(P,F')=e[d(P,d)+ d(P,d')]=constant, of course (by simmetry).

Appendixes:

App. 1: The Special Theory Of Relativity.

Contents of App. 1:

App.1-Introduction:

Time is just the name which has been assigned to a mathematical ratio relation between two different spaces; when I say that in order to go from home to my job place it takes half an hour, I just say that the space from home to my job place corresponds to the space of half a clock circumference run by the hand of minutes. In my own opinion, no mysterious or spatially four-dimensional stuff, as proposed by the STR (Special Theory of Relativity). On the contrary, on a mathematical basis, time can be considered as the fourth dimension, as well as temperature can be the fifth and so on. The speed of light (c=299.792,458 km/s) is an upper speed limit, but neither by an unexplainable mystery, nor by a principle, as asserted in the STR and also by Einstein himself, but rather because (and still in my opinion) a body cannot move randomly in the Universe where it's free falling with speed c, as it's linked to all the Universe around, as if the Universe were a spider's web that when the trapped fly tries to move, the web affects that movement and as much as those movements are wide (v~c), that is, just to stick to the web example, if the trapped fly just wants to move a wing, it can do that almost freely (v<<c), while, on the contrary, if it really wants to fly widely from one side to the other on the web (v~c), the spider's web resistance becomes high (mass which tends to infinite etc). On this purpose, see Appendix 2.

Anyway, Einstein's theory is formally founded on two principles:
-Principle of di Relativity: *laws of phisics have the same form in all inertial systems* (i.e. at relative movement with a constant speed); as it doesn't make any sense an absolute movement with respect to a standing ether which does not exist (see Par. 3.7) all reference systems are equal laboratories to verify in all laws of physics; so there aren't any privileged reference sysyems (except for, in my opinion, that of the center of mass of the Universe).
Anyway, the Michelson and Morley experiment (App.1-Par. 3.7) represented the end of the ether and opened the doors to STR.
-Principle of Constancy of the Speed of Light: *the speed of light in vacuum has always the same value c=300.000 km/s.* Therefore, no matter if you chase it at 299.000 km/s of if you run away from it still with that speed; light in vacuum will run away or chase you still at 300.000 km/s! (c=299.792,458 km/s)
In the opinion of the writer, there is something like a contradiction in the STR; the speed of light seems to be an "absolute" object, indeed, and not "relative", as we are here talking about "relativity". The point here is that speeds among objects in the Universe are relative with respect to themselves, but there is an absolute (or almost) speed c with which all objects in the Universe fall towards the centre of mass of it; from this the absolute essence of c. And here there is also an explanation of the reason why objects at rest have energy m_0c^2 (App.1-Par. 2.4), energy given to matter at rest by Einstein, unfortunately without telling us that such a matter is never at rest, as it's free falling with speed c towards the center of mass of the Universe, as chance would have it. On this purpose, see my complete personal opinion in Appendix 2.
If a common man hears the speed of light is the same everywhere and for everybody (all inertial observers), even when they have relative movements at constant speed, nothing happens. On the contrary, if it's heard by a particular man like Einstein, what he can understand from that can be surprising. The following simple experiment, made by a light clock on a space ship, shows that the fact that the speed of light is c for "everybody" implies that time is relative, from which the Twin Paradox comes (App.1-Par. 1.4) etc:

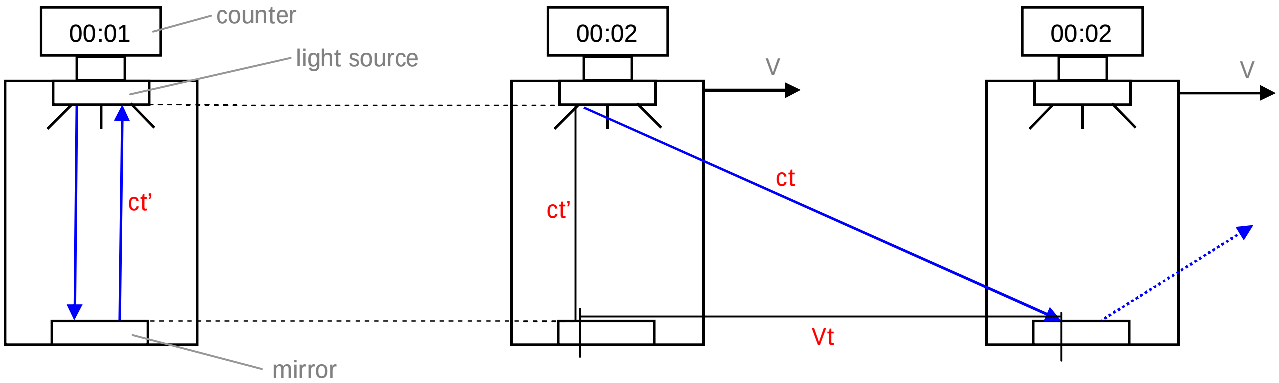


Fig. A1: Light clock not moving.   Fig. A2: Light clock moving at speed V.

As you can see in Fig. A1, every time light (blue arrows) goes from source to mirror and back, the light clock says, for instance, one second, or a certain time t'. The path in blue shown in Fig. A1 is seen by those who are not moving with respect to the clock, i.e. by who is on the space ship with the clock itself. On the contrary, those who are on the Earth and see the clock moving on the ship, will see the light travelling diagonally longer paths, as shown in Fig. A2, as the mirror moves while the light goes downwards, and the source moves, too, when the light goes back upwards. Now, as we are talking about light, the behaviour of it during its going downwards is not like that of a suitcase falling from the luggage compartment of a railway wagon, which is seen to fall vertically by the passenger on it and by a parabolic trajectory by the observers not moving, on the platform at the station, so taking the same time for both of them, as in the latter case (parabolic) the falling speed is higher; we are here talking about light, therefore its speed must be the same for all, and c; but if it's so, then those who see the longer diagonal path must say that time taken by light to go down and up must be longer. Therefore, despite we're talking about just one clock and one event, those two observers come to different results, so to the relativity of time and to all its implications.
Using the Pithagorean Theorem on the triangle in Fig. A2, we have:

$c^2t^2 = c^2t'^2 + V^2t^2$, from which: $\boxed{t' = t\sqrt{1-\frac{V^2}{c^2}}}$ and so, in general: $t' < t$ and t' tends to zero when V tends to c!

I remind you that t' is the time of the astronaut who is travelling with the clock, while t is the time elapsed on the Earth. If all this is true for the fastest thing (light), then it's also true for standard hands clocks and also for the biologic ones (living beings)! In the seventies, by using very sensitive atomic clocks, they proved the time dilation on the Earth, between two atomic clocks which were synchronized, at the beginning, after that one of them flew on a plane, underwenting a slight, but well felt, time dilation.

App.1-Chapter 1: Fundamental introductory concepts.

App.1-Par. 1.1: Galilean transformations.

They simply give the relations between spatial coordinates (and time), for two reference systems in relative motion, but in classic physics, where the speed of light is not an upper limit.

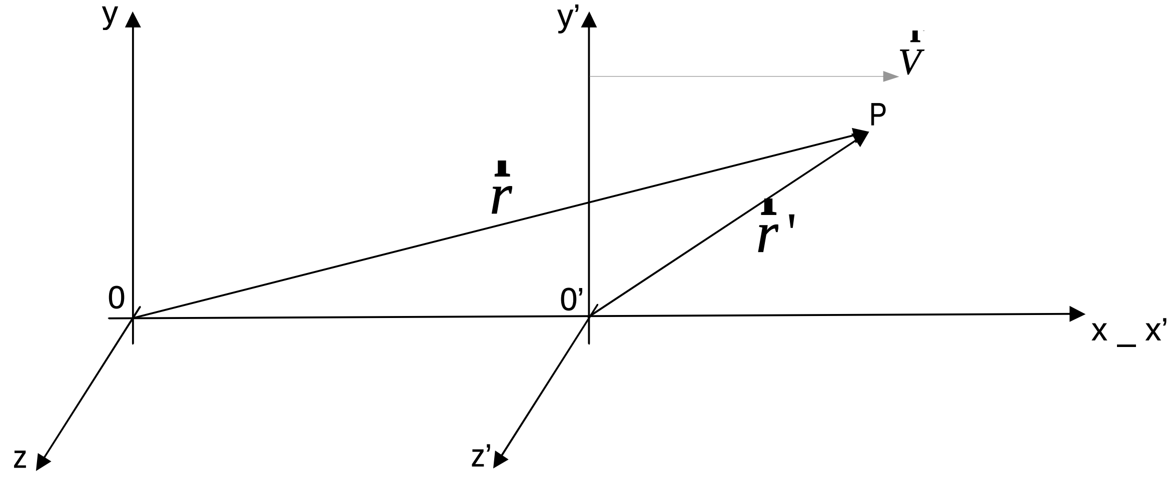


Fig. A1.1: Reference systems in relative motion.

We obviously have:

$$\vec{r} = \vec{r}' + 00' = \vec{r}' + \vec{V}t \,, \tag{A1.1}$$

from which, for the components (t=t'):

$$\begin{cases} x = x' + Vt' \\ y = y' \\ z = z' \\ t = t' \end{cases} \tag{A1.2}$$

and for the reverse ones, we obviously have:

$$\begin{cases} x' = x - Vt \\ y' = y \\ z' = z \\ t' = t \end{cases} \tag{A1.3}$$

(A1.2) and (A1.3) are the Galilean Transformations.

By deriving (A1.1), we have: $\vec{v} = \vec{v}' + \vec{V}$ which can be held as the theorem of summation of velocities in classic physics.

App.1-Par. 1.2: The (Relativistic) Lorentz Transformations.

We know that the Lorentz transformations were born before the Theory of Relativity (which is founded on them) and on an electromagnetic basis.

They correspond to the Galilean ones, but on a relativistic basis and they are in force as long as we say that the speed of light is an upper limit in the Universe and it's c for (~)every observer.

-<u>FIRST PROOF</u>:

if we suppose a relative motion along x, we correct the x components of the Galilean Transformations through a coefficient k, as follows:

$$x' = k(x - Vt) \tag{A1.4}$$

$$x = k(x' + Vt') \tag{A1.5}$$

Now, for a photon, we obviously have:

$ct' = k(c-V)t$ and $ct = k(c+V)t$, as light has the same speed c in both reference systems, from which, by mutual multiplication of the corresponding sides:

$c^2tt' = k^2tt'c^2(1-\frac{V^2}{c^2})$, from which:

$$k = \frac{1}{\sqrt{1-\frac{V^2}{c^2}}} = \frac{1}{\sqrt{1-b^2}} \qquad \text{(A1.6)}$$

Moreover, from (A1.5) we have: $t' = \frac{1}{V}(\frac{x}{k} - x')$ and using (A1.4) in it, we have:

$$t' = \frac{1}{V}[\frac{x}{k} - k(x-Vt)] = \frac{1}{kV}x - \frac{kx}{V} + kt = k[t - \frac{x}{V}(1-\frac{1}{k^2})] = k[t - \frac{V}{c^2}x] \qquad \text{(A1.7)}$$

as, for (A1.6), we have: $(1-\frac{1}{k^2}) = \frac{V^2}{c^2}$.

By the same way which led us to (A1.7), we also get the expression for t.
Finally, here are the Lorentz Transformations:

$$\begin{cases} x' = \frac{(x-Vt)}{\sqrt{1-b^2}} \\ y' = y \\ z' = z \\ t' = \frac{(t-\frac{V}{c^2}x)}{\sqrt{1-b^2}} \end{cases} \qquad \text{(A1.8)}$$

$$\begin{cases} x = \frac{(x'+Vt')}{\sqrt{1-b^2}} \\ y = y' \\ z = z' \\ t = \frac{(t'+\frac{V}{c^2}x')}{\sqrt{1-b^2}} \end{cases} \qquad \text{(A1.9)}$$

-<u>SECOND PROOF</u>:
We know that c=const in all inertial reference systems. Now, with reference to Fig. A1.1, when 0=0' and t=t', from the origin light is emitted through spherical waves and isotropically and so we can write that:

$c^2t^2 - (x^2+y^2+z^2) = 0$ and $c^2t'^2 - (x'^2+y'^2+z'^2) = 0$

as light has the same speed c in both reference systems. Therefore:

$c^2t^2 - (x^2+y^2+z^2) = c^2t'^2 - (x'^2+y'^2+z'^2)$ and for rays along x (y=y' and z=z'):

$c^2t^2 - x^2 = c^2t'^2 - x'^2$. Now we say ($i = \sqrt{-1}$): $ix = x$, $ix' = x'$, $ct = h$ and $ct' = h'$; we have:

$x^2 + h^2 = x'^2 + h'^2$, whose solution is:

$$\begin{cases} x' = x\cos q - h\sin q \\ h' = x\sin q + h\cos q \end{cases} \qquad \text{(A1.10)}$$

and in a differential form:

$$\begin{cases} dx' = dx\cos q - dh\sin q \\ dh' = dx\sin q + dh\cos q \end{cases} \qquad \text{(A1.11)}$$

Now we notice that with respect to the origin "0", $\frac{dx}{dt}=0$, as the reference system (0,x,y,z) is not moving with respect to itself. On the contrary, $\frac{dx'}{dt'}=-V$, as the system (0',x',y',z') moves with speed V with respect to "0" and, as a consequence: $\frac{dx}{dh}=0$ and $\frac{dx'}{dh'}=-i\frac{V}{c}$, but from the ratios between (A1.11) we have that:

$$\frac{dx'}{dh'}=\frac{dx\cos q-dh\sin q}{dx\sin q+dh\cos q}=\frac{\frac{dx}{dh}\cos q-\sin q}{\frac{dx}{dh}\sin q+\cos q}=\frac{0-\sin q}{0+\cos q}=-tgq$$ and so: $tgq=i\frac{V}{c}=ib$

but we know from trigonometry that: $\cos q=\frac{1}{\sqrt{1+tg^2q}}=\frac{1}{\sqrt{1-b^2}}$ and:

$\sin q=\frac{tgq}{\sqrt{1+tg^2q}}=\frac{ib}{\sqrt{1-b^2}}$ and so the (A1.10) become:

$x'=\frac{x-ihb}{\sqrt{1-b^2}}$ and $h'=\frac{ixb+h}{\sqrt{1-b^2}}$, that is:

$$\begin{cases} x'=\frac{(x-Vt)}{\sqrt{1-b^2}} \\ y'=y \\ z'=z \\ t'=\frac{(t-\frac{b}{c}x)}{\sqrt{1-b^2}} \end{cases}$$

and so (A1.8) again. By the same way, we also get (A1.9).

App.1-Par. 1.3: The contraction of length, or of Lorentz.

Moving objects with speeds close to that of ligth are shorter to not moving observers. If those observers make measurements to get the length of the running body, the best way is to use light sources (the fastest thing), by illuminating the bow and the stern of that body, in order to see the corresponding positions, moment by moment. But that light has a constant speed, and limited, too, and the result will be that of a shorted body. Reality or measuring appearance? Convince yourself immidiately that (observed) reality and the measuring appearence are the same thing, and it must be so!
Let l be the length of a segment in the O system:

$|x_B-x_A|=l$. In O', according to Lorentz Transformations:

$l'=|x'_B(t')-x'_A(t')|=\frac{|(x_B-Vt_B)-(x_A-Vt_A)|}{\sqrt{1-b^2}}=\frac{|(x_B-x_A)|}{\sqrt{1-b^2}}=\frac{l}{\sqrt{1-b^2}}=l'$, as in system O the measurement of the segment makes sense if both ends are detected "simultaneously" ($t_A=t_B$).
When v tends to c, $b=v/c$ will tend to 1 and the radical will tend to zero, as well as l!
Therefore, the length l measured in O will be that measured in O' by a value less than 1, that is, the observer at rest (O) will detect a shorter object.

App.1-Par. 1.4: Time dilation (Twin Paradox).

It sounds strange, but time, too, can be, and is, relative. Of course, every observer, in himself, sees time going by still in the same way; if you move with a speed close to that of light, you will not hear your heart beating slower. The comparison between those two observers which were in relative motion will show the difference on how those two times went by.

So, according to Lorentz Transformation, in O' (moving system): $\Delta t' = t'_B - t'_A$, whilst in O (system at rest):

$\Delta t = t_B - t_A = \dfrac{t'_B + x'_B b/c - t'_A - x'_A b/c}{\sqrt{1-b^2}}$, but, by assumption, $x'_A = x'_B$, as in the system O' (O',x',y',z') the clock is at rest, as it's travelling with the system O' itself; therefore:

$$\Delta t = \frac{\Delta t'}{\sqrt{1-b^2}}$$

When v tends to c, $b = v/c$ will tend to 1 and the radical will tend to zero, as well as $\Delta t'$!

Two twins separate as one leaves for a spatial travel one month long and at speeds close to that of ligth; once back to the Earth, he sees the other twin thirty years older! (Twin Paradox)

At a speed of 260.000 km/s you have the halving, so the radical will be ½ and those two times will go by one a half of the other.

See also the proof of the time dilation by the ligth clock (in the Introduction) and also that on a Doppler Effect basis (App.1-Par. 3.6).

App.1-Par. 1.5: The four-vector position.

Instead of writing the position vectors with the three classic components x, y and z, let's write them in a mathematically four-dimensional form, by adding time; this will be very useful. In the (justified) opinion of the writer, our Universe is three-dimensional and the adding of a fourth dimension is a purely mathematical operation; in fact, I defy you to show me the fourth dimension of a whatsoever object which is held four-dimensional. In the nowadays' STR, a real four dimension is believed!

Well, so: $\underline{x} = (x_1, x_2, x_3, x_4)$, which is the same as: (x,y,z,ct).

By this new terminology, the Lorentz Transformations become:

$$\begin{cases} x'_1 = gx_1 - bgx_4 \\ x'_2 = x_2 \\ x'_3 = x_3 \\ x'_4 = -bgx_1 + gx_4 \end{cases} \qquad \begin{cases} x_1 = gx'_1 + bgx'_4 \\ x_2 = x'_2 \\ x_3 = x'_3 \\ x_4 = bgx'_1 + gx'_4 \end{cases} \qquad \text{(A1.12)}$$

where $b = V/c$ e $g = \dfrac{1}{\sqrt{1-b^2}}$.

You can prove the space-time distance between two points is invariant for Lorentz Transformations, i.e. it is the same in all inertial reference systems:

$$(\underline{\Delta x})^2 = (\Delta x_1)^2 + (\Delta x_2)^2 + (\Delta x_3)^2 - (\Delta x_4)^2 = (\underline{\Delta x'})^2 = (\Delta x'_1)^2 + (\Delta x'_2)^2 + (\Delta x'_3)^2 - (\Delta x'_4)^2 \qquad \text{(A1.13)}$$

(This expression is a kind of Pitagorean Theorem in four dimensions).

In order to prove that, use the Lorentz transformations in (A1.13), where there are Δx_i in place of x_i, and you check the equality.

In other words, three dimension length and time are relative, while their four-dimensional composition is absolute. That's why we said the definition of physical quantities by four components would have been useful.

We can then use the Lorentz transformations on all four-vectors, in the form of (A1.12).

<u>If we use matrixes</u>, Lorentz Transformations can be written in the following way:

(remember the matrix product, with the components of the row of the first matrix which are multiplied by the corresponding components of the column of the second matrix, then summing up those products to get the component of the product matrix, indeed)

$$(x'_1, x'_2, x'_3, x'_4) = \begin{pmatrix} g & 0 & 0 & -bg \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -bg & 0 & 0 & g \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \quad \text{and} \qquad \text{(A1.14)}$$

$$(x_1, x_2, x_3, x_4) = \begin{pmatrix} g & 0 & 0 & bg \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ bg & 0 & 0 & g \end{pmatrix} \cdot \begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \\ x'_4 \end{pmatrix} \qquad \text{(A1.15)}$$

while, for the tensor form:
(use the Einstein convention, according to which, if in a term an index is repeated, the summation on that index is understood)

$x'_i = a_i^k x_k$ (i,k=1,2,3,4) on the right side, k is repeated, so you make the summation on it (A1.16)

$x_k = a'^i_k x'_i$ (i,k=1,2,3,4) (A1.17)

App.1-Par. 1.6: Relativistic Law of Transformation of Velocities.

If I'm walking with a speed of 5 km/h in a railway train car which is travelling with a speed of 100 km/h with respect to the platform, I'll have, with respect to the platform, a total speed of 105 km/h. This is classic physics. On the contrary, when those two speeds get close to that of light, the simple composition by algebraic summation cannot be used anymore, as it would show us a speed higher than that of light c, which must be impossible.
Therefore, we have, by definition:

$$\begin{cases} v_x = dx/dt \\ v_y = dy/dt \\ v_z = dz/dt \end{cases} \qquad \begin{cases} v'_x = dx'/dt' \\ v'_y = dy'/dt' \\ v'_z = dz'/dt' \end{cases}$$

Now, by differentiating the Lorentz Transformations, we have:

$$\begin{cases} dx = \dfrac{(dx' + V dt')}{\sqrt{1-b^2}} \\ dy = dy' \\ dz = dz' \\ dt = \dfrac{(dt' + \frac{V}{c^2} dx')}{\sqrt{1-b^2}} \end{cases}$$

from which:

$$\begin{cases} v_x = dx/dt = \dfrac{(dx' + V dt')}{(dt' + \frac{V}{c^2} dx')} \\ v_y = dy/dt = \dfrac{dy' \sqrt{1-b^2}}{(dt' + \frac{V}{c^2} dx')} \\ v_z = dz/dt = \dfrac{dz' \sqrt{1-b^2}}{(dt' + \frac{V}{c^2} dx')} \end{cases}$$

If now we divide numerator and denominator of last equations by dt', we have:

$$\begin{cases} v_x = \dfrac{(v'_x + V)}{(1 + \dfrac{Vv'_x}{c^2})} \\ v_y = \dfrac{v'_y \sqrt{1-b^2}}{(1 + \dfrac{Vv'_y}{c^2})} \\ v_z = \dfrac{v'_z \sqrt{1-b^2}}{(1 + \dfrac{Vv'_z}{c^2})} \end{cases} \quad \text{(A1.18)} \quad \text{and similarly:} \quad \begin{cases} v'_x = \dfrac{(v_x - V)}{(1 - \dfrac{Vv_x}{c^2})} \\ v'_y = \dfrac{v_y \sqrt{1-b^2}}{(1 - \dfrac{Vv_y}{c^2})} \\ v'_z = \dfrac{v_z \sqrt{1-b^2}}{(1 - \dfrac{Vv_z}{c^2})} \end{cases}$$

When $V/c << 1$ and $v_x/c << 1$ we are in the classic case (galileian) of algebraic sum.

If we use again the example of the railway train car, where a guy walks inside, if we say $\underline{v}' = (v'_x, 0, 0, c)$ and $\underline{v} = (v_x, 0, 0, c)$, from (A1.18) we have:

$$v_x = \frac{(v'_x + V)}{(1 + \frac{Vv'_x}{c^2})} \tag{A1.19}$$

and if $v'_x = V = c$, then $v_x = c$, and not 2c! Therefore, if the train moves with a speed c and I run inside at c, as well, with respect to the platform, I'll have a resulting speed equal to c and not 2c!
(1.19) represents the Relativistic Theorem of Addition of Velocities.
As an example: two rockets travel each with speed c/2 and meet; at which velocity v_x do they meet?

$$v_x = \frac{(c/2 + c/2)}{(1 + \frac{c^2}{4c^2})} = \frac{c}{5/4} = \frac{4}{5}c < c \ !$$

App.1-Par. 1.7: The Proper Time $d\tau$ of a particle.

If a particle is at rest in (O',x',y',z'), i.e. if it moves in O', indeed, i.e. if O' is its "proper" reference system, then: dx'= dy'= dz'=0 and:

$(d\underline{s})^2 = dx^2 + dy^2 + dz^2 - c^2dt^2 = 0 + 0 + 0 - c^2dt'^2 = -c^2dt'^2$, from which:

$$dt' = d\tau = \frac{ds}{c} = \sqrt{1 - \frac{dx^2 + dy^2 + dz^2}{c^2dt^2}}dt = \sqrt{1 - \frac{1}{c^2}(\frac{dl}{dt})^2} = \sqrt{1 - \frac{v^2}{c^2}}dt \text{ (proper time, invariant, as so is } \frac{ds}{c}\text{).}$$

Therefore, when you make calculations on a particle, you are led to use for it its proper time, indeed:

$$d\tau = \sqrt{(1 - \frac{v^2}{c^2})}dt = \frac{dt}{g} \ .$$

App.1-Chapter 2: The relativity of energies.

App.1-Par. 2.1: The momentum-energy four-vector (or linear momentum).

We know from classic physics that the linear momentum is given by the product of the mass by the velocity. Now, in the relativistic case, for what has been said so far, we will define a four-vector and then the velocity as dx/dt will have the proper time dt instead of dt, which is typical of the particle, indeed, for which we are going to define the four-vector:

$$\underline{p} = (m_0 \frac{dx_1}{dt}, m_0 \frac{dx_2}{dt}, m_0 \frac{dx_3}{dt}, m_0 \frac{dx_4}{dt}) = (\frac{m_0}{1/g} \frac{dx_1}{dt}, \frac{m_0}{1/g} \frac{dx_2}{dt}, \frac{m_0}{1/g} \frac{dx_3}{dt}, \frac{m_0}{1/g} \frac{dx_4}{dt}) =$$

$$= (m \frac{dx_1}{dt}, m \frac{dx_2}{dt}, m \frac{dx_3}{dt}, m \frac{dx_4}{dt}) = (m\vec{v}, mc) = (\vec{p}, mc) = \underline{p}$$

where $\vec{v}$ is the (three-dimension) velocity vector, $\vec{p}$ is the three-dimension linear momentum,

$mc = m \frac{dx_4}{dt}$ is the 4-dimension component, $dt = \frac{dt}{g}$ is the proper time and $m = \frac{m_0}{1/g} = gm_0$ is the dynamic mass, which is the rest mass only if v=0.

We have just begun to introduce the concept of relative mass, which is increasing with speed, and becoming infinite when v=c.

As already said before, the modules of 4-vectors are "absolute", i.e. they are invariant for Lorentz T.; in fact:

$$\left|\underline{p}\right|^2 = \left|\vec{p}\right|^2 - p_4^2 = m^2 v^2 - m^2 c^2 = -m^2 (c^2 - v^2) = -\frac{m_0^2}{(1 - \frac{v^2}{c^2})} (c^2 - v^2) = -m_0^2 c^2$$ = constant, i.e. it's not depending on v.

In Relativity there exist a Universe (mathematically 4-dimensional) described by 4-vectors, where quantities are not changing so arbitrarily with speed and where laws of nature preserve some consistency, no matter what the state of motion is.

App.1-Par. 2.2: The velocity four-vector.

We obviously define it as follows: $\underline{v} = \frac{d\underline{x}}{dt}$, where we use the proper time dt for reasons already shown. Numerator and denominator are both invariant, so also $\underline{v}$ is.

We have:

$$\underline{v} = (\frac{dx_1}{dt}, \frac{dx_2}{dt}, \frac{dx_3}{dt}, \frac{dx_4}{dt}) = (g \frac{dx_1}{dt}, g \frac{dx_2}{dt}, g \frac{dx_3}{dt}, g \frac{dx_4}{dt}) = (gv_x, gv_y, gv_z, gc) = (g\vec{v}, gc),$$

and so, as a module for such a 4-vector: $\left|\underline{v}\right|^2 = g^2 v^2 - g^2 c^2 = -c^2$ (constant!). (A2.1)

Par. 2.3: The four-force.

As in classic physics, force is the derivative of the linear momentum with respect to time, in relativity we define the 4-force as the derivative of the momentum-energy 4-vector with respect to time (proper time):

$$\underline{F} = \frac{d\underline{p}}{dt} = (F_1, F_2, F_3, F_4) = (\vec{F}, F_4) = (\frac{d}{dt}\vec{p}, \frac{d}{dt}(m_0 gc)) = (g \frac{d}{dt}(m_0 g\vec{v}), g \frac{d}{dt}(m_0 gc));$$ (A2.2)

so we have: $g \frac{d}{dt}(m_0 g\vec{v}) = g \cdot \vec{f} = \vec{F}$, (A2.3)

($\frac{d}{dt}(m_0 g\vec{v}) = \vec{f}$ is the "classic" force, and it's more so when $g = 1$)

and so: $g \frac{d}{dt}(g\vec{v}) = \frac{\vec{F}}{m_0} = \frac{d}{dt}(g\vec{v})$, and by components:

$$g \frac{d}{dt}(gv_i) = \frac{F_i}{m_0} = \frac{d}{dt}(gv_i) \qquad \text{(i=x,y,z)}$$ (A2.4)

and then, about the fourth component: $g \frac{d}{dt}(m_0 gc) = F_4$ (A2.5) and so: $g\frac{d}{dt}(gc)=\frac{F_4}{m_0}=\frac{d}{dt}(gc)$. (A2.6)

Now, we differentiate (A2.1), that is, the following equation:

$|\underline{v}|^2=g^2v^2-g^2c^2=g^2v_x^2+g^2v_y^2+g^2v_z^2-g^2c^2=-c^2$, so getting:

$$0=(\frac{d}{dt}(gv_x)^2+\frac{d}{dt}(gv_y)^2+\frac{d}{dt}(gv_z)^2-\frac{d}{dt}(gc)^2)=(2gv_x\frac{d}{dt}(gv_x)\ +2gv_y\frac{d}{dt}(gv_y)+$$

$$+2gv_z\frac{d}{dt}(gv_z)-2gc\frac{d}{dt}(gc))$$

that is: $0=(gv_x\frac{d}{dt}(gv_x)\ +gv_y\frac{d}{dt}(gv_y)++gv_z\frac{d}{dt}(gv_z)-gc\frac{d}{dt}(gc))$, and, for (A2.4) and (A2.6):

$0=(gv_x\frac{F_x}{m_0}+gv_y\frac{F_y}{m_0}++gv_z\frac{F_z}{m_0}-gc\frac{F_4}{m_0})$ and for (A2.3) $(g\cdot\vec{f}=\vec{F})$:

$0=(gv_x\frac{gf_x}{m_0}+gv_y\frac{gf_y}{m_0}++gv_z\frac{gf_z}{m_0}-gc\frac{F_4}{m_0})$, that is:

$F_4=\frac{g}{c}(\vec{f}\cdot\vec{v})$. (A2.7)

If now we go back to (A2.2), we finally have the 4-force, or Minkowski force:

$$\underline{F}=\frac{d\underline{p}}{dt}=(g\vec{f},\frac{g}{c}(\vec{f}\cdot\vec{v})) \qquad (A2.8)$$

To have the transformation equations for such a 4-force, please see Subappendix 1.4.

Par. 2.4: $E_0=m_0c^2$.

According to (A2.5), we have: $g\frac{d}{dt}(m_0gc)=F_4$, whilst for (A2.7) we have: $F_4=\frac{g}{c}(\vec{f}\cdot\vec{v})$. Therefore:

$g\frac{d}{dt}(m_0gc)=\frac{g}{c}(\vec{f}\cdot\vec{v})$, that is: $\frac{d}{dt}(m_0gc^2)=\vec{f}\cdot\vec{v}$, that is, again:

$d(m_0gc^2)=\vec{f}\cdot\vec{v}dt=dL=dE$ (A2.9)

(A2.9) is exactly the expression for the energy in classic physics, if $g=1$; the integration of (2.9) with integration constant equal to zero yields:

$$E=gm_0c^2 \qquad (A2.10)$$

In reality (A2.10) holds only for gained energies (as in particle accelerators), while for lost energies (collapsing Universe or Atomic Physics of electrons going down in energy levels) the following must be used, and I assume it as mine:

$E=\frac{1}{g}m_0c^2$ (Rubino)

(see also Appendix 2; for a convincing proof of it, please contact me: leonrubino@yahoo.it).
Therefore, a particle whose mass is m_0 has got a total energy:

$$E=gm_0c^2=\frac{m_0c^2}{\sqrt{(1-\frac{v^2}{c^2})}} \qquad (A2.11)$$

and "at rest"($v=0$ and so $g=1$) it has a "rest" energy:

$$E_0=m_0c^2 \qquad (A2.12)$$

App.1-Par. 2.5: Relativistic kinetic energy.

The difference between (A2.11) and (A2.12) obviously yields the pure kinetic energy of a particle:

$$E_K = E - E_0 = gm_0c^2 - m_0c^2 = m_0c^2(g-1) = m_0c^2\left(\frac{1}{\sqrt{(1-\frac{v^2}{c^2})}} - 1\right) \qquad \text{(A2.13)}$$

If now ve develop, according to Taylor, the expression for $g = \frac{1}{\sqrt{(1-\frac{v^2}{c^2})}} = \frac{1}{\sqrt{(1-b^2)}}$, we get, for v<<c (β<<1):

$g = \frac{1}{\sqrt{(1-b^2)}} = 1 + \frac{1}{2}b^2 + \frac{3}{8}b^4 + ...$, that is: $(g-1) \cong \frac{1}{2}b^2 = \frac{1}{2}\frac{v^2}{c^2}$ and for (A2.13):

$E_K = m_0c^2(g-1) = m_0c^2(\frac{1}{2}\frac{v^2}{c^2}) = \frac{1}{2}m_0v^2$ (v<<c) which is the well known classic expression (Newton's) for the kinetic energy!

In order to have a proof of the (A2.13) starting from a collapsing Universe characteristics, please see into Appendix 2.

App.1-Chapter 3: Relativistic phenomena.

App.1-Par. 3.1: Time and gravity: gravity slows down the time!

Also gravity slows down the time! On a mountain time elapses faster than down in the valley. Of course, on the Earth, the difference is imperceptible, but on a neutron star or in a black hole, that effect is very strong.

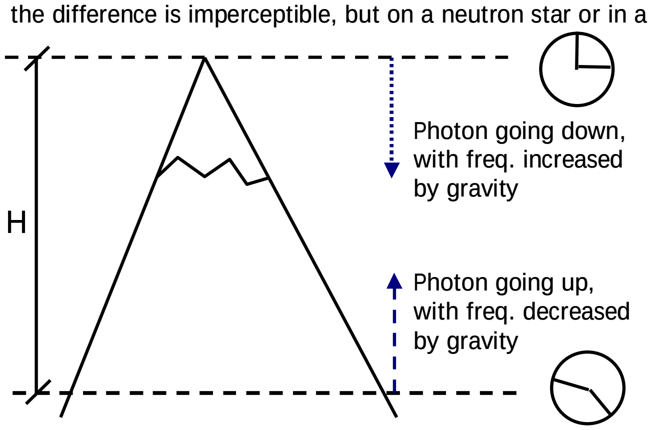


Fig. A3.1: Mountain, gravity and time.

$\Delta E = m_0gH$ (delta energy from the level difference)

$\Delta E = h\Delta n$ (delta energy due to the freq. decrease of the photon). From them: $\Delta n = m_0gH/h$.

For a photon, $E = hn$, but in relativity: $E = m_0c^2$, from which, for a photon: $m_0 = hn/c^2$ and so, for Δn : $\Delta n = ngH/c^2$ and as time is the reciprocal of the frequency, we have: $\Delta n/n = \Delta t/t$ and so:

$\Delta t = \frac{gH}{c^2}t$ Therefore, over a time t, we have a slow down Δt due to gravity!

We know that the escape velocity of a celestial body whose mass is M and radius R is: $V = \sqrt{2GM/R}$. If on that body an object is cast vertically with the escape velocity, it will quit the gravitational field of that celestial body and will go towards the infinite, without falling down anymore.

A black hole is a body so compressed (big M and small R) that the escape velocity on its surface reaches the speed of light and so not even the light can escape, from which the name of black hole; moreover, for what above said, we can say that in a black hole time is approximately stopped!

App.1-Par. 3.2: Volume of moving solids.

Moving solids appear rotated.

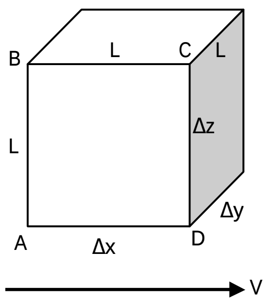


Fig. A3.2: Body seen at rest.

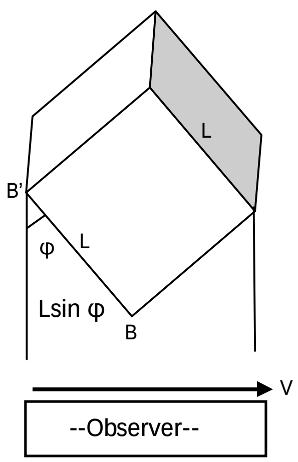


Fig. A3.3: Body seen moving.

$V = \Delta x \Delta y \Delta z$ (volume of the solid for an observer integral with it)

$V' = \Delta x' \Delta y' \Delta z' = \Delta x\sqrt{1-b^2}\Delta y \Delta z = V\sqrt{1-b^2}$ (volume of the solid for an observer who sees that solid in movement with speed V, along x).

Of course, for Lorentz Transformations, as the movement is along x, only Δx is contracted. On the other hand, the observer at rest sees point B with a delay with respect to A, and this delay is L/c, obviously. As a further consequence, B appears as moved back about a stretch which is (L/c)V=βL.

We have: Lsin φ= βL, from which: sin φ= β and φ=arcsin β.

Finally, that body appears rotated! And a sphere goes on appearing as a sphere.

App.1-Par. 3.3: The equation of waves, or of D'Alembert, holds in every inertial reference system.

Electromagnetic waves in vacuum, an so the light, too, propagates, as well known, respecting the wave equation:

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}=0=\Delta f-\frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}=\Box f$$

Now, we preliminarily notice that, according to the Lorentz Transformations, we have (by deriving them):

$$\frac{\partial x'}{\partial x}=g\ ,\ \frac{\partial x'}{\partial t}=-gV\ ,\ \frac{\partial t'}{\partial x}=-g\frac{V}{c^2}\ ,\ \frac{\partial t'}{\partial t}=g\ ,\ \frac{\partial y'}{\partial y}=\frac{\partial z'}{\partial z}=1,\ \frac{\partial x'}{\partial y}=\frac{\partial x'}{\partial z}=\frac{\partial y'}{\partial x}=......=0$$

According to mathematical analysis, we have, in 0': $f = f(\vec{r}', t')$, and so:

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial x'}\frac{\partial x'}{\partial x}+\frac{\partial f}{\partial y'}\frac{\partial y'}{\partial x}+\frac{\partial f}{\partial z'}\frac{\partial z'}{\partial x}+\frac{\partial f}{\partial t'}\frac{\partial t'}{\partial x}=g\frac{\partial f}{\partial x'}+(-V)g\frac{\partial f}{\partial t'}$$; furthermore:

$$\frac{\partial^2 f}{\partial x^2}=g\left(\frac{\partial^2 f}{\partial x'^2}+\frac{V^2}{c^4}\frac{\partial^2 f}{\partial t'^2}\right)-\frac{2V}{c^2-V^2}\frac{\partial^2 f}{\partial x'\ \partial t'}$$ and similarly, for the other coordinates:

$$\frac{\partial f}{\partial t}=-Vg\frac{\partial f}{\partial x'}+g\frac{\partial f}{\partial t'}$$ and $$\frac{\partial^2 f}{\partial t^2}=g^2\left(V^2\frac{\partial^2 f}{\partial x'^2}+\frac{\partial^2 f}{\partial t'^2}\right)-2Vg\frac{\partial^2 f}{\partial x'\ \partial t'}$$ and $$\frac{\partial^2 f}{\partial y^2}=\frac{\partial^2 f}{\partial y'^2}\ ,\ \frac{\partial^2 f}{\partial z^2}=\frac{\partial^2 f}{\partial z'^2}$$

from which, by substitution in the wave equation, we have:

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}=\frac{\partial^2 f}{\partial x'^2}+\frac{\partial^2 f}{\partial y'^2}+\frac{\partial^2 f}{\partial z'^2}-\frac{1}{c^2}\frac{\partial^2 f}{\partial t'^2}$$ that is what we wanted to prove.

App.1-Par. 3.4: The Fizeau experiment.

In 1849, a long time before the formulation of the Special Theory of Relativity by Einstein (1905), the French physicist A.H.L. Fizeau carried out studies on the speed of light in water and in moving fluids.

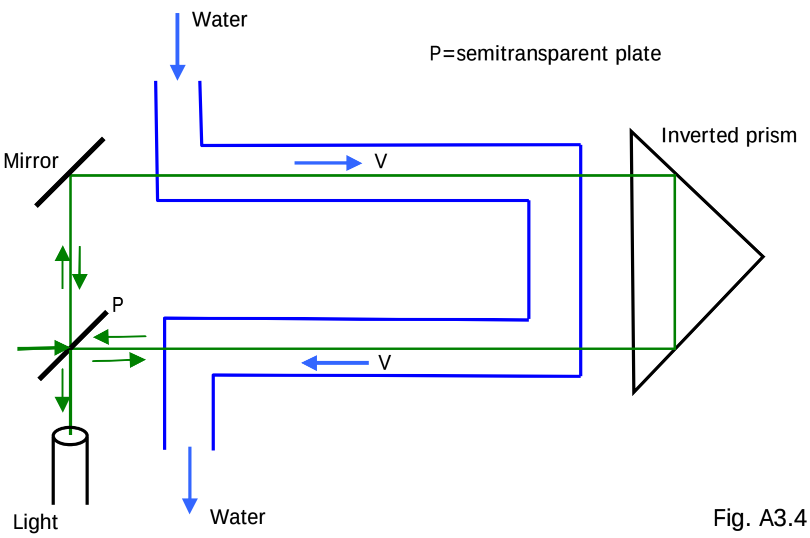


Fig. A3.4: Fizeau's experiment.

In water at rest, light has velocity $v = \frac{c}{n} = \frac{3 \cdot 10^8 m/s}{1{,}33} \cong 225.000 km/s$, where n=1,33 is the refractive index of water. If now water, or the fluid, in which we are going to measure the speed of light, flows with speed V, then, according to Fizeau's results, the total velocity of light in the flowing fluid is:

$$v = \frac{c}{n} + V(1 - \frac{1}{n^2}), \tag{A3.1}$$

in total disagreement with classic physics, according to which we should more simply have: $v = \frac{c}{n} + V$.

Years later, STR has given a theoretical explanation of (A3.1). In fact, for the Theorem of Addition of Velocities given by (A1.19), we can write that:

$v = \frac{(\frac{c}{n} + V)}{(1 + \frac{Vc}{nc^2})} = \frac{(\frac{c}{n} + V)}{(1 + \frac{V}{nc})}$; now, we multiply numerator and denominator by $(1 - \frac{V}{nc})$, and we have:

$v = \frac{(\frac{c}{n} + V)}{(1 + \frac{V}{nc})} = \frac{(\frac{c}{n} + V)(1 - \frac{V}{nc})}{1 - (\frac{V}{nc})^2} = \frac{\frac{c}{n} + V(1 - \frac{1}{n^2}) - \cancel{\frac{V^2}{nc}}}{1 - \cancel{(\frac{V}{nc})^2}}$, but quantities $\frac{V^2}{nc}$ on the numerator and $(\frac{V}{nc})^2$ on the denominator are both negligible (<<1) with respect to the other terms and can be neglected indeed, from which the assertion: $v = \frac{c}{n} + V(1 - \frac{1}{n^2})$.

App.1-Par. 3.5: Relativistic Doppler Effect (longitudinal).

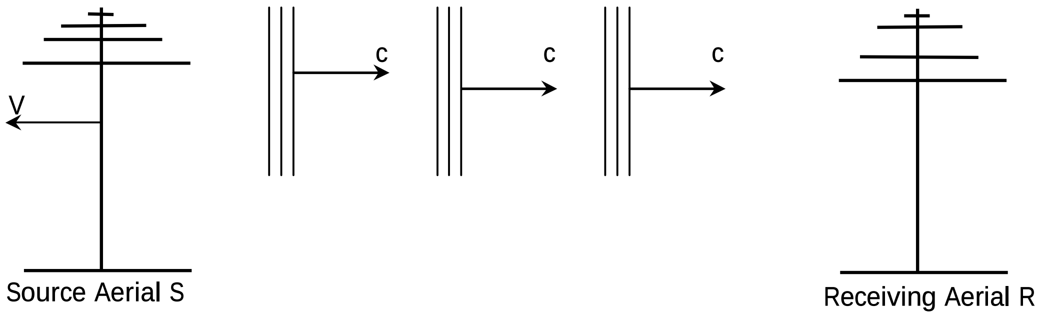


Fig. A3.5: Longitudinal Doppler Effect (example of a source S getting farther from R with speed V (β)).

The source aerial sends electromagnetic signals to the receiving one, and the periods is T_S;
Because of the time dilation, the receiving aerial will receive them with a period T'_S, so that:

$T'_S = \frac{T_S}{\sqrt{(1 - b^2)}}$; moreover, the fact that S is getting farther will cause a $\Delta T'_S$ among phase fronts, which is:

$\Delta T'_S = \frac{VT'_S}{c} = \frac{V}{c} \frac{T_S}{\sqrt{(1 - b^2)}} = b \frac{T_S}{\sqrt{(1 - b^2)}}$; so, we'll totally have T_R:

$$T_R = T'_S + \Delta T'_S = \frac{T_S}{\sqrt{(1 - b^2)}} + b \frac{T_S}{\sqrt{(1 - b^2)}} = \frac{T_S}{\sqrt{(1 - b^2)}}(1 + b) = \frac{T_S}{\sqrt{(1 - b)}\sqrt{(1 + b)}}(1 + b) =$$

$$= \frac{T_S}{\sqrt{(1 - b)}} \sqrt{(1 + b)} = T_S \frac{\sqrt{(1 + b)}}{\sqrt{(1 - b)}} = T_R$$

and we rewrite it here: $T_R = T_S \frac{\sqrt{(1+b)}}{\sqrt{(1-b)}}$; (S and R getting farther) (A3.2)

The same holds if the receiver is the one who gets farther. If, then, S and R are getting closer, through the same reasonings which led us so far, the following holds:

$$T_R = T_S \frac{\sqrt{(1-b)}}{\sqrt{(1+b)}}$$ (S and R getting closer). (A3.3)

For a more general treatment of this subject, see Subappendix 1.2.

App.1-Par. 3.6: The Twin Paradox explained by the Relativistic Doppler Effect!

Say a twin leaves for a space flight with a velocity equal to 3/5 c=180.000 km/s, getting far away from the Earth for 25 min (measured on the Earth) and then he comes back towards the Earth with the same velocity, so taking another 25 min. Out of simplicity, we neglect the acceleration and deceleration phases.
Now, in order to prove that the time dilation acts also on the cardiac (heart) rhythm of the travelling twin (but still in the opinion of the twin at rest on the Earth, and when the twins meet on the Earth, at the end of the flight), say, out of simplicity, both twins have one heartbeat per second and say the twin on the Earth transmits a radio pulse (whose speed is c) every second, i.e. every heartbeat, towards the space ship, in order to inform his travelling twin brother on his own cardiac rhythm. Now, remember that in the opinion of the "older" twin at rest on the Earth, the flight lasts, by supposition, 25+25=50 min (3000 heartbeats), while, if the time dilation is true, for the "younger" travelling twin it lasts (V=3/5 c, that is: β=3/5):

$$T_{young} = T_{old}\sqrt{(1-b^2)} = T_{old}\sqrt{(1-\frac{V^2}{c^2})} = 50\min\sqrt{(1-\frac{9c^2}{25c^2})} = 40\min$$ (=2400 heartbeats)

Moreover, under a Doppler analysis of the phenomenon, we can say that for equation (A3.2), the twin on the Earth (old) transmits his heartbeats every second, but the flying twin will receive them, during the first "to" step of the flight, every two seconds; in fact:

$T_{young}(to) = T_{old}(to)\frac{\sqrt{(1+b)}}{\sqrt{(1-b)}} = 1s\frac{\sqrt{1+(3/5)}}{\sqrt{1-(3/5)}} = 2s$, so, in the "to" step, the flying twin, whose "to" flight time is 20 min=1200s=1200 heartbeats, has received one heartbeat every 2s from his brother on the Earth, that is just 600 heartbeats.

Therefore, the flying twin, during his 20 min(=1200s) "to" flight, has counted on himself 1200 heartbeats, but has received only 600 from his brother on the Earth.

After 20 min of the flying twin, the direction of the flight is inverted and the return to the Earth starts, for another 20 min (still according to the time measured by the flying twin). During those further 20 min's return flight ("from"), on the contrary, the twin on the Earth still transmits every second, but the flying one now receives every half a second; in fact, for equation (A3.3):

$T_{young}(from) = T_{old}(from)\frac{\sqrt{(1-b)}}{\sqrt{(1+b)}} = 1s\frac{\sqrt{1-(3/5)}}{\sqrt{1+(3/5)}} = 0{,}5s$, so, during those further flyer's 20 min=1200s= =1200 heartbeats counted on himself, the flyer receives 2400 heartbeats from his brother on the Earth.

Therefore, the flying twin, during his further 20 min(=1200s) return flight, has obviously counted on himself another 1200 hearthbeats and has received 2400 (!) from his brother from the Earth.

Sum up of the counts:
Totally, during the whole space flight, of 20+20=40 min, the flying twin has obviously counted 1200+1200=2400 hearthbeats on himself and (a piece of!) 600+2400=3000 hearthbeats from his twin brother on the Earth.
He must feel younger!!!

App.1-Par. 3.7: The Michelson and Morley experiment.

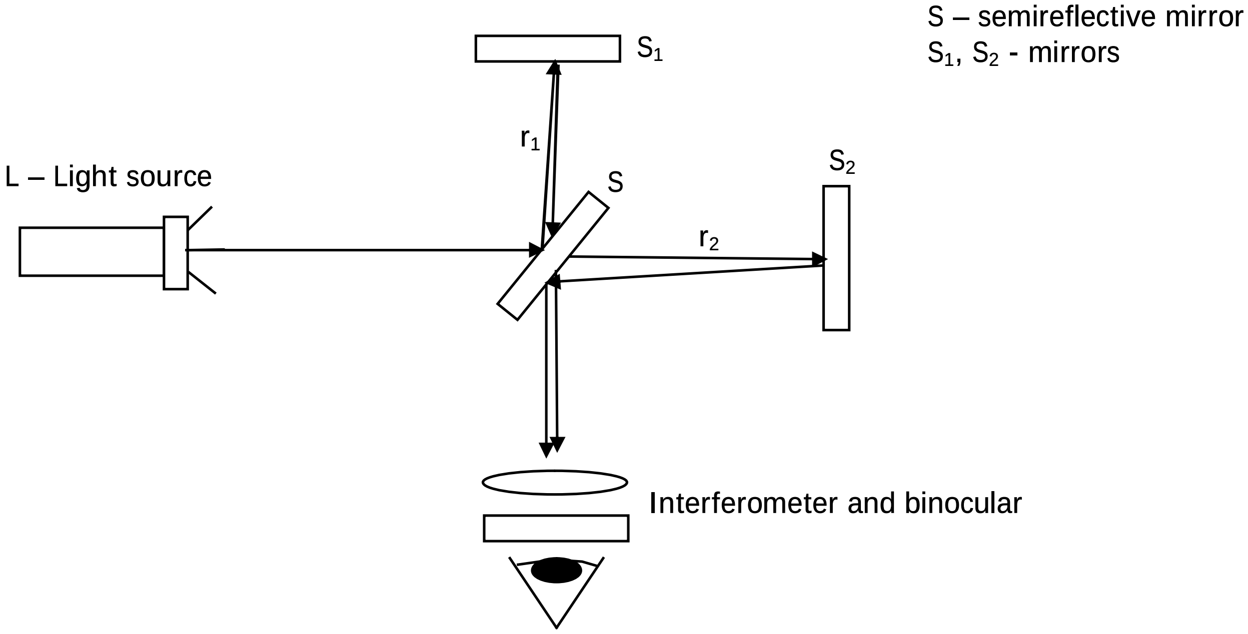


Fig. A3.6: Michelson's device (interferometer).

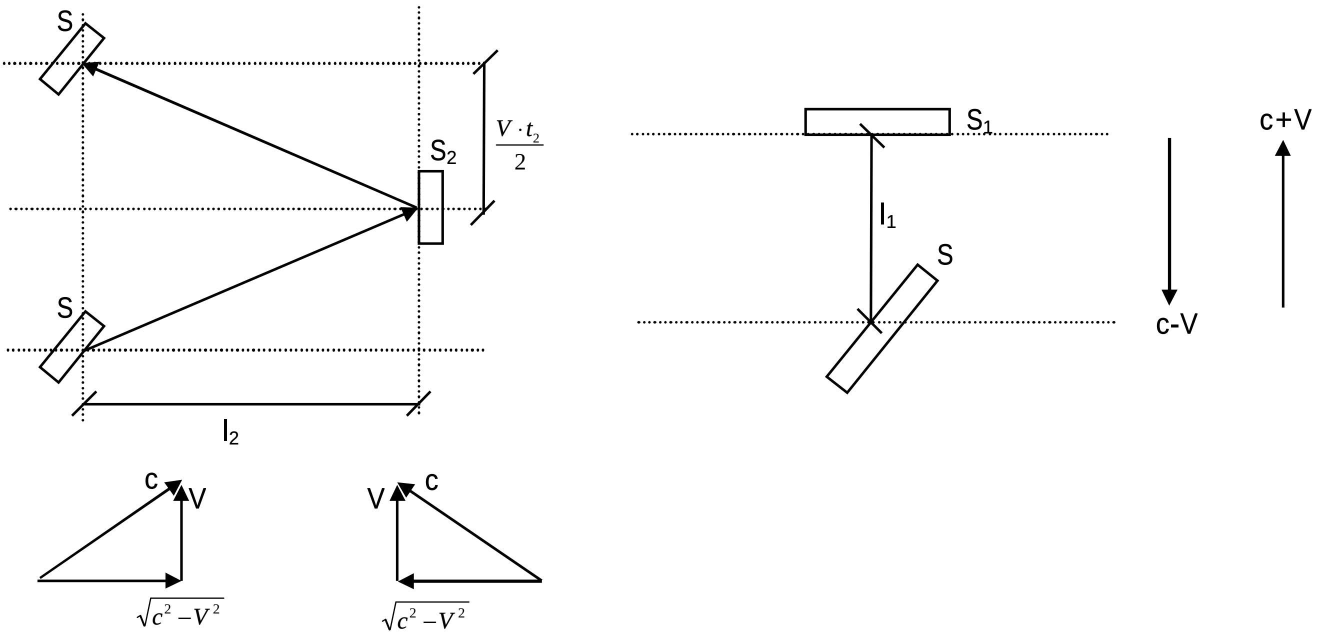


Fig. A3.7: Luminous paths and relevant velocities.

Before Einstein, they thought that electromagnetic waves, and so the light, had to propagate in a mean, as well as for sounds in the air. They supposed that the space was filled with an invisible and very light gas, the ether. The Earth rotates around the Sun with a speed V around 30 km/s, so it should move through the ether with such a speed and light emitted by a light source which is on the Earth itself should have, in general, a speed different from c (c±V along the direction of rotation of the Earth and $\sqrt{c^2 - V^2}$ transversally).

In 1886 they started to prepare the experiment which should have proved the movement of the Earth through the ether. In Cleveland they stopped the street traffic during the experiment in order not to have vibrations; the device was put on a floating stone slab in a well of mercury, to easily rotate it of 90° without vibrations.

Now, if you put l_1 along the direction of rotation of the Earth, about the - to and fro - light path, we have:

$t_1 = \frac{l_1}{c+V} + \frac{l_1}{c-V} = \frac{2l_1}{c} \frac{1}{(1 - V^2/c^2)}$ and for the transversal path along l_2:

$$t_2 = \frac{2l_2}{\sqrt{c^2 - V^2}} = \frac{2l_2}{c} \frac{1}{\sqrt{(1 - V^2/c^2)}}$$

If you make both rays enter an interferometer to make them interfere, indeed, they should arrive with a Δt:

$$\Delta t = t_2 - t_1 = \frac{2}{c}\left(\frac{l_2}{\sqrt{(1 - V^2/c^2)}} - \frac{l_1}{(1 - V^2/c^2)}\right) \cong \frac{2}{c}[l_2(1 + V^2/2c^2) - l_1(1 + V^2/c^2)]$$

as we have $V/c \cong 10^{-4}$, $V^2/c^2 \cong 10^{-8}$ and $(1+x)^k \cong 1 + kx$.

The wavelenght of the used light was $l = 5{,}5 \cdot 10^{-7} m$ and we know that λ corresponds to the full angle $2p$; therefore, we can write the following proportion, which involves the phase difference δ between the two rays and the path difference cΔt:

$\frac{l}{2p} = \frac{c\Delta t}{d}$, from which: $d = \frac{2pc\Delta t}{l}$.

By fixing the one arm length and adjust the other arm one by a micrometric screw, you can make cΔt of the same size of λ, so making the desired interference phenomenon.
Now, without bringing any change to the geometry of the device, rotate it of 90°; the roles of l_1 and l_2 are so swapped and we'll have:

$$\Delta t' = t'_2 - t'_1 = \frac{2}{c}\left(\frac{l_1}{\sqrt{(1 - V^2/c^2)}} - \frac{l_2}{(1 - V^2/c^2)}\right) \cong \frac{2}{c}[l_1(1 + V^2/2c^2) - l_2(1 + V^2/c^2)]$$

and we should also have: $\frac{\Delta d}{2p} = \frac{d - d'}{2p} = \frac{c\Delta t - c\Delta t'}{l} = \frac{l_1 + l_2}{l}\frac{V^2}{c^2} = \frac{22m}{5{,}5 \cdot 10^{-7} m} 10^{-8} \cong 0{,}4$, that is, through the rotation of the interferometer, you should see a shift of the fringes of interference of 0,4 times the distance λ between two subsequent maxima.

In reality, none of all that was observed, despite the accuracy of the devices was as good as to detect a $\frac{\Delta d}{2p} = 0{,}01$!

Michelson declared himself to be disappointed by that experiment, as he couldn't prove the movement of the Earth through the ether.
The question was solved in 1905 by an employee of the Patent Office of Berne, Albert Einstein, who suggested to cease searching for a proof of the movement of the Earth through the ether, for the simple reason that the ether is not existing!
I add that the nowadays' dark matter will soon end up like it.

App.1-Chapter 4: Relativistic Electrodynamics.

App.1-Par. 4.1: Magnetic force is simply a Coulomb's electric force(!).

Concerning this, let's examine the following situation, where we have a wire, of course made of positive nuclei and electrons, and also a cathode ray (of electrons) flowing parallel to the wire:

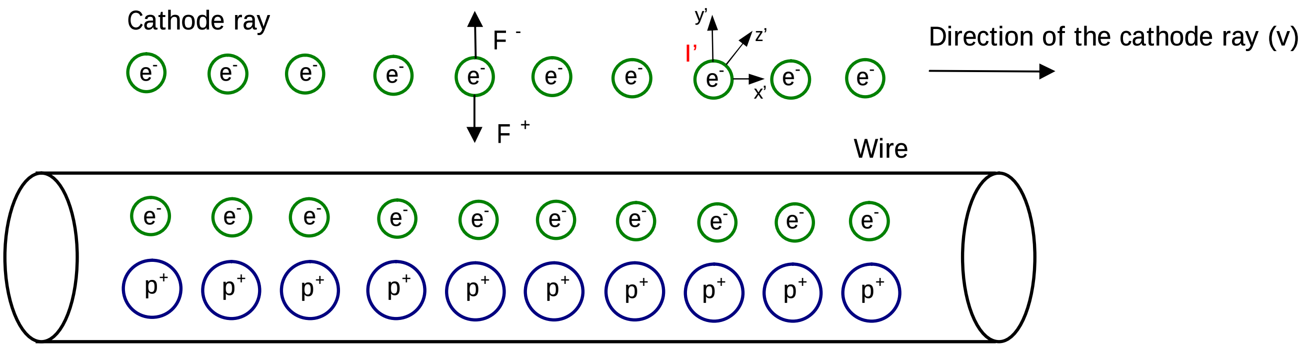


Fig. A4.1: Wire not flown by any current, seen from the cathode ray steady ref. system I' (x', y', z').

We know from magnetism that the cathode ray will not be bent towards the wire, as there isn't any current in it. This is the interpretation of the phenomenon on a magnetic basis; on an electric basis, we can say that every single electron in the ray is rejected away from the electrons in the wire, through a force F^- identical to that F^+ through which it's attracted from positive nuclei in the wire.

Now, let's examine the situation in which we have a current in the wire (e^- with speed u)

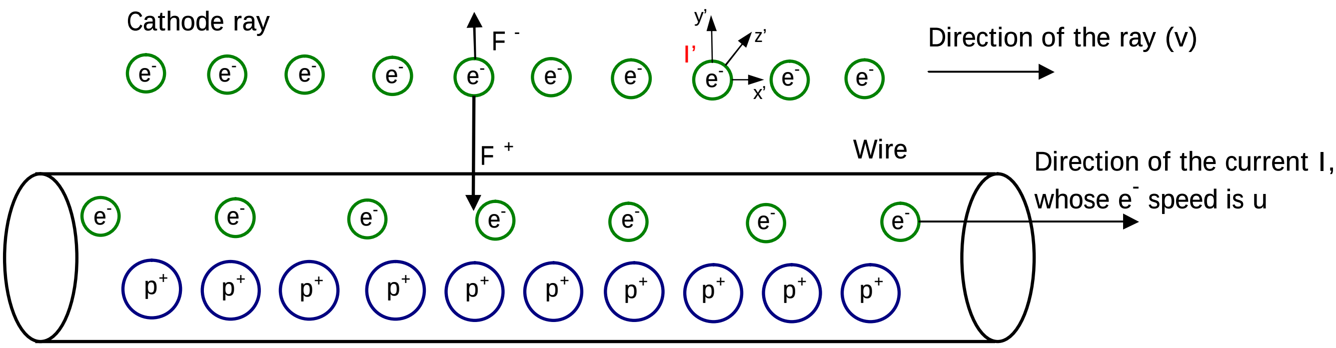


Fig. A4.2: Wire flown by a current (with e^- speed=u), seen from the cathode ray steady ref. system I' (x', y', z').

In this case we know from magnetism that the cathode ray must bend towards the wire, as we are in the well known case of parallel currents in the same direction, which must attract each other.
This is the interpretation of this phenomenon on a magnetic basis; on an electric basis, we can say that as the electrons in the wire follow those in the ray, they will have a speed lower than that of the positive nuclei, in the system I', as such nuclei are still in the wire. As a consequence of that, spaces among the electrons in the wire will undergo a lighter relativistic Lorentz contraction, if compared to that of the nuclei's, so there will be a lower negative charge density, if compared to the positive one, so electrons in the ray will be electrically attracted by the wire.
This is the interpretation of the magnetic field on an electric basis. Now, although the speed of electrons in an electric current is very low (centimeters per second), if compared to the relativistic speed of light, we must also acknowledge that the electrons are billions and billions...., so a small Lorentz contraction on so many spaces among charges, makes a substantial magnetic force to appear.
But now let's see if mathematics can prove we're quantitatively right on what asserted so far, by showing that the magnetic force is an electric one itself, but seen on a relativistic basis.
On the basis of that, let's consider a simplified situation in which an electron e^-, whose charge is q, moves with speed v and parallel to a nuclei current whose charge is Q^+ each (and speed u):

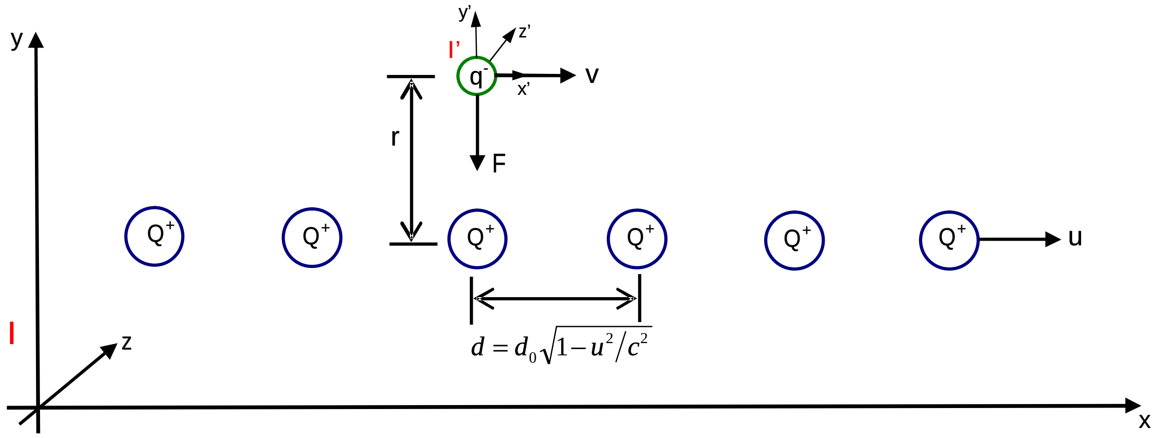


Fig. A4.3: Current of positive charge (speed u) and an electron whose speed is v, in the reader's steady system I.

a) Evaluation of F on an electro<u>magnetic</u> basis, in the system I :
First of all, we remind ourselves of the fact that if we have N charges Q in line and d spaced (as per Fig. A4.3), then the linear charge density λ will be:

$$\lambda = N \cdot Q / N \cdot d = Q/d \quad .$$

Now, still with reference to Fig. A3.3, in the system I, for the electro<u>magnetics</u> the electron will undergo the Lorentz force $F_l = q(E + v \times B)$ which is made of an originally electrical component and of a magnetic one:

$$F_{el} = E \cdot q = (\frac{1}{\varepsilon_0}\frac{\lambda}{2\pi r})q = (\frac{1}{\varepsilon_0}\frac{Q/d}{2\pi r})q$$ due to the electric attraction from a linear distribution of charges Q, and:

$$F_{magn} = \mu_0 \frac{I}{2\pi r} = \mu_0 \frac{Q/t}{2\pi r} = \mu_0 \frac{Q/(d/u)}{2\pi r} = \mu_0 \frac{uQ/d}{2\pi r}$$ (Biot and Savart).

So: $F_l = q(\frac{1}{e_0}\frac{Q/d}{2pr} - vm_0\frac{uQ/d}{2pr}) = q\frac{Q/d_0}{2pr}(\frac{1}{e_0} - m_0 uv)\frac{1}{\sqrt{1-u^2/c^2}}$, (A4.1)

where the negative sign tells us the magnetic force is repulsive, in that case, because of the real directions of those currents, and where the steady distance d_0 is contracted to d, according to Lorentz, in the system I where charges Q have got speed u ($d = d_0\sqrt{1-u^2/c^2}$).

b) Evaluation of F on an electric base, in the steady system I' of q:

in the system I' the charge q is still and so it doesn't represent any electric current, and so there will be only a Coulomb electric force towards charges Q:

$$F'_{el} = E' \cdot q = (\frac{1}{e_0}\frac{l'}{2pr})q = (\frac{1}{e_0}\frac{Q/d'}{2pr})q = q(\frac{1}{e_0}\frac{Q/d_0}{2pr})\frac{1}{\sqrt{1-u'^2/c^2}} \quad , \qquad \text{(A4.2)}$$

where u' is the speed of the charge distribution Q in the system I', which is due to u and v by means of the well known relativistic theorem of composition of speeds:

$$u' = (u-v)/(1-uv/c^2) \quad , \qquad \text{(A4.3)}$$

and d_0, this time, is contracted indeed according to u': $d' = d_0\sqrt{1-u'^2/c^2}$.

We now note that, through some algebraic calculations, the following equality holds (see (A4.3)):

$1-u'^2/c^2 = \frac{(1-u^2/c^2)(1-v^2/c^2)}{(1-uv/c^2)^2}$, which, if replacing the radicand in (A4.2), yields:

$$F'_{el} = E' \cdot q = (\frac{1}{e_0}\frac{l'}{2pr})q = (\frac{1}{e_0}\frac{Q/d'}{2pr})q = q(\frac{1}{e_0}\frac{Q/d_0}{2pr})\frac{(1-uv/c^2)}{\sqrt{1-u^2/c^2}\sqrt{1-v^2/c^2}} \qquad \text{(A4.4)}$$

We now want to compare (4.1) with (4.4), but we still cannot, as one is about I and the other is about I'; so, let's scale F'_{el} in (A4.4), to I, too, and in order to do that, we see that, by definition of the force itself, in I':

$F'_{el}(in\_I') = \frac{\Delta p_{I'}}{\Delta t_{I'}} = \frac{\Delta p_I}{\Delta t_I\sqrt{1-v^2/c^2}} = \frac{F_{el}(in\_I)}{\sqrt{1-v^2/c^2}}$, where $\Delta p_{I'} = \Delta p_I$, as Δp extends along y, and not along the direction of the relative motion, so, according to the Lorentz transformations, it doesn't change, while Δt, of course, does. So:

$$F_{el}(in\_I) = F'_{el}(in\_I')\sqrt{1-v^2/c^2} = q(\frac{1}{e_0}\frac{Q/d_0}{2pr})\frac{(1-uv/c^2)}{\sqrt{1-u^2/c^2}\sqrt{1-v^2/c^2}}\sqrt{1-v^2/c^2} =$$

$$= q(\frac{1}{e_0}\frac{Q/d_0}{2pr})\frac{(1-uv/c^2)}{\sqrt{1-u^2/c^2}} = F_{el}(in\_I) \qquad \text{(A4.5)}$$

Now we can compare (4.1) with (A4.5), as now both are related to the I system.
Let's write them one over another:

$$F_l(in\_I) = q(\frac{1}{e_0}\frac{Q/d}{2pr} - vm_0\frac{uQ/d}{2pr}) = q\frac{Q/d_0}{2pr}(\frac{1}{e_0} - m_0 uv)\frac{1}{\sqrt{1-u^2/c^2}}$$

$$F_{el}(in\_I) = q(\frac{1}{e_0}\frac{Q/d_0}{2pr})\frac{(1-uv/c^2)}{\sqrt{1-u^2/c^2}} = q\frac{Q/d_0}{2pr}(\frac{1}{e_0} - \frac{uv}{e_0 c^2})\frac{1}{\sqrt{1-u^2/c^2}}$$

Therefore we can state that these two equations are identical if the following identity holds: $c = 1/\sqrt{e_0 m_0}$, and this identity is known since 1856. As these two equations are identical, the magnetic force has been traced back to the Coulomb's electric force, so the unification of electric and magnetic fields has been accomplished!!

App.1-Par. 4.2: The Current Density four-vector.

We obviously have the following equations on charge density:

$$r_0 = \frac{dQ}{dt_0} \ , \ r = \frac{dQ}{dt} = \frac{dQ}{\sqrt{1-b^2}\,dt_0} = g r_0$$

Moreover, notice that the following equation holds; it shows the invariance of the electric charge:

$$r_0 dt_0 = r dt \ .$$

Moreover, we know from physics that the current density is:

$$\vec{j} = r\vec{v} \, .$$

So, we are led to define the current density 4-vector in the following way:

$\underline{j} = (\vec{j}, rc) = (g r_0 \vec{v}, g r_0 c)$; note the similarity with the momentum-energy 4-vector:

$\underline{p} = (\vec{p}, mc) = (g m_0 \vec{v}, g m_0 c)$. Then, we have:

$\left|\underline{j}\right|^2 = \left|\vec{j}\right|^2 - j_4^2 = -r_0^2 c^2$ and moreover, as we can apply the Lorentz Transformations to a 4-vector, as said at App.1-Par. 1.5 – eq. (A1.12), we have:

$$\begin{cases} j_1' = g j_1 - b g j_4 \\ j'_2 = j_2 \\ j'_3 = j_3 \\ j'_4 = -b g j_1 + g j_4 \end{cases} \quad \text{or also:} \quad \begin{cases} j_x' = g\,(j_x - rV) \\ j'_y = j_y \\ j'_z = j_z \\ r' = g\,(r - \frac{V}{c^2} j_x) \end{cases} \qquad \text{(A4.6)}$$

where $b = V/c$ e $g = \frac{1}{\sqrt{1-b^2}}$.

Now, in order to show by 4-vectors and in a more compact way what shown in App.1-Par. 4.2, we consider a wire flown by a stationary current I in a reference system k and let I be directed along x; if, in such a system, we put a charge q whose speed is v along x, we'll have a movement of q by the only field B, as E=0, because, on an average, we have in a conductor as many positive charges as the negative ones are. On the contrary, if we place ourselves in a k' reference system which is moving with speed V=v with respect to k, in k' q is at rest and theoretically the magnetic force should disappear; but this is unacceptable for the Principle of Relativity (see the Introduction). An electric field must so appear in k'; in fact:

$\underline{j} = \underline{j}_+ + \underline{j}_-$, $\underline{j}_+ = (\vec{j}_+, c r_+) = (0, nqc)$ [n]=[number of q involved/m^3] and

$\underline{j}_- = (\vec{j}_-, c r_-) = (-nq\vec{v}, -nqc)$, the first term is negative as the direction of v is opposite to the conventional one for I (as, here q<0) and the second one is still negative, as well, still because here, q<0.

For the Lorentz Transformations (A1.12) we so have (v=V):

$$j'_{+x} = -g n q V \qquad\qquad j'_{-x} = g\,(-nqv + nqV)$$

$$r'_+ = g n q \qquad\qquad r'_- = g\,(-nq + nq\frac{vV}{c^2})$$

and as now $r'_+ \neq -r'_-$, an electric field must appear.

App.1-Par. 4.3: The Electromagnetic Field Tensor.

Preamble on tensors:

A vector is a tensor of rank 1.
For us it's enough to say that we get a rank 2 tensor when we make the product of the components of two vectors c and b:

$$c(c_i) = (c_1, c_2, c_3, c_4) \ , \ b(b_k) = (b_1, b_2, b_3, b_4) \quad \rightarrow \quad A_{ik} = c_i b_k$$

Through (A1.16) and (A1.17) we have seen how to express the Lorentz Transformations:

$c_i = a_i^{\,l} c'_l$ (i,m=1,2,3,4) and $b_k = a_k^{\,m} b'_m$ (k,l=1,2,3,4) and so:

$$A_{ik} = c_i b_k = a_i^{\,l} a_k^{\,m} c'_l\, b'_m = \boxed{a_i^{\,l} a_k^{\,m} A'_{lm} = A_{ik}} \qquad \text{(A4.7)}$$

which is the transformation law for a rank 2 tensor.

Then, we also notice that we get a rank 2 tensor also when we derive the components of a vector b with respect to the x coordinate:

$$b_i = a_i^{\,l} b'_l \quad , \quad x'_m = a_m^{\,k} x_k \quad >>> \quad \frac{\partial x'_m}{\partial x_k} \rightarrow a_m^{\,k}$$

$$\frac{\partial b_i}{\partial x_k} = \frac{\partial b_i}{\partial x'_m}\frac{\partial x'_m}{\partial x_k} = \frac{\partial}{\partial x'_m}(a_i^{\,l} b'_l)\frac{\partial x'_m}{\partial x_k} = \boxed{a_i^{\,l} a_m^{\,k}}\frac{\partial b'_l}{\partial x'_m}$$

; therefore, such a derivative transforms as the components of a rank 2 tensor and so it's a rank 2 tensor, as well.

Preamble on electromagnetism:

We know from the electromagnetism that electric and magnetic fields (induction vector B) can be expressed as a function of electrodynamic potentials φ and A:

$$\vec{E} = -\vec{\nabla} j - \frac{\partial \vec{A}}{\partial t} \text{ and} \qquad \text{(A4.8)}$$

$$\vec{B} = \vec{\nabla} \times \vec{A} \qquad \text{(A4.9)}$$

and we also know the Lorentz Condition: $\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2}\frac{\partial j}{\partial t} = \vec{\nabla} \cdot \vec{A} + em\frac{\partial j}{\partial t} = 0$, (A4.10)

and also the Continuity Equation: $\vec{\nabla} \cdot \vec{j} + \frac{\partial r}{\partial t} = 0$. (A4.11)

Still from electromagnetism, we also know that:

$$\Delta j - \frac{1}{c^2}\frac{\partial^2 j}{\partial t^2} = -\frac{r}{e} = \Box f \quad \text{and} \qquad \text{(A4.12)}$$

$$\Delta \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -m\vec{j} = \Box \vec{A} \qquad \text{(A4.13)}$$

and we remind ourselves that we already defined the current density 4-vector j:

$$\underline{j} = (\vec{j}, cr) = (j_x, j_y, j_z, cr) \qquad (j_i = v_i r).$$

Now, we feel led to define the Potential Four-vector or Four-Potential $\underline{\Phi}$:

$\underline{\Phi} = (A_x, A_y, A_z, \frac{j}{c})$; in fact, so doing, (A4.12) and (A4.13) can be so summarized:

$\Box \Phi_k = -mj_k$ and if we also define the four-divergence $div_4 = \vec{\nabla}_4 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial x} + \frac{\partial}{\partial (ct)} = (\vec{\nabla} + \frac{\partial}{\partial (ct)})$

we easily get:

$\vec{\nabla}_4 \underline{j} = 0$ for the Continuity Equation (A4.11) and $\vec{\nabla}_4 \underline{\Phi} = 0$ for the Lorentz Condition (A4.10).

Now, from (A4.8) and (A4.9), we have:

$B_x = B_1 = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = \frac{\partial \Phi_3}{\partial x_2} - \frac{\partial \Phi_2}{\partial x_3}$, $E_x = E_1 = -\frac{\partial j}{\partial x} - \frac{\partial A_x}{\partial t} = c(\frac{\partial \Phi_4}{\partial x_1} - \frac{\partial \Phi_1}{\partial x_4})$ etc; therefore, E and B cannot be espressed through two 4-vectors, but through a rank 2 four-tensor, as we proved the derivative of a vector is a 2-tensor:

$F_{ik} = c(\frac{\partial \Phi_k}{\partial x_i} - \frac{\partial \Phi_i}{\partial x_k})$; let's write the components of F_{ik} as a matrix:

$$F_{ik} = \begin{pmatrix} 0 & cB_z & -cB_y & -E_x \\ -cB_z & 0 & cB_x & -E_y \\ cB_y & -cB_x & 0 & -E_z \\ E_x & E_y & E_z & 0 \end{pmatrix}$$

, and through (A4.7) we have proved that F_{ik} transforms in the following way: $F_{ik} = a_i^l a_k^m F'_{lm}$. (A4.14)

Notice that F_{ik} is antisymmetric, that is: $F_{ik}=-F_{ki}$.

Then, of course:

$$F'_{ik} = \begin{pmatrix} 0 & cB'_z & -cB'_y & -E'_x \\ -cB'_z & 0 & cB'_x & -E'_y \\ cB'_y & -cB'_x & 0 & -E'_z \\ E'_x & E'_y & E'_z & 0 \end{pmatrix}$$

If now we remind ourselves that on the right side of (A4.14) the summation over l and m is understood, as they are repeated there, and if we develop such an equation, we get the transformation of the electromagnetic field:

$$\begin{cases} E_x = E'_x \\ E_y = g\,(E'_y + VB'_z) \\ E_z = g\,(E'_z - VB'_y) \end{cases} \qquad \begin{cases} B_x = B'_x \\ B_y = g\,(B'_y - \frac{V}{c^2} E'_z) \\ B_z = g\,(B'_z + \frac{V}{c^2} E'_y) \end{cases}$$

and also its inverse:

$$\begin{cases} E'_x = E_x \\ E'_y = g\,(E_y - VB_z) \\ E'_z = g\,(E_z + VB_y) \end{cases} \qquad \begin{cases} B'_x = B_x \\ B'_y = g\,(B_y + \frac{V}{c^2} E_z) \\ B'_z = g\,(B_z - \frac{V}{c^2} E_y) \end{cases}$$

SUBAPPENDIXES:

Subapp. 1: Lorentz Transformations in succession.

k >>> V >>> **k'** >>> W >>> **k''**

We have three reference systems k, k' and k'' and V and W are the relevant relative velocities.

Through the following terminology: $b_1 = \frac{V}{c}$, $b_2 = \frac{W}{c}$, $g_1 = 1\big/\sqrt{1-b_1^2}$, $g_2 = 1\big/\sqrt{1-b_2^2}$, $U = \frac{V+W}{1+\frac{VW}{c^2}}$

and $b = \frac{U}{c}$, we have, for the Lorentz Transformations applied in succession:

$x = g_1(x'+Vt')$ with $t = g_1(t'+\frac{b_1}{c}x')$ and

$x' = g_2(x''+Wt'')$ with $t' = g_2(t''+\frac{b_2}{c}x'')$ and, by substitution:

$$x = g_1 g_2(x''+Wt''+Vt''+b_1 b_2 x'') = g_1 g_2[(1+b_1 b_2)x''+(V+W)t''] =$$
$$= g_1 g_2(1+b_1 b_2)(x''+\frac{V+W}{1+b_1 b_2}t'') = x \qquad \text{(A.1.1)}$$

and similarly: $t = g_1 g_2(1+b_1 b_2)(t''+\frac{1}{c^2}\frac{V+W}{(1+b_1 b_2)}x'')$; (A.1.2)

Now, we see that:

$$g_1 g_2(1+b_1 b_2) = 1\Big/\sqrt{(1-b_1^2)(1-b_2^2)/(1+b_1 b_2)^2} = 1\Big/\sqrt{[1+b_1^2 b_2^2+2b_1 b_2-(b_1^2+b_2^2+2b_1 b_2)]/(1+b_1 b_2)^2} =$$

$= 1\Big/\sqrt{1-[(b_1+b_2)/(1+b_1 b_2)]^2} = 1\Big/\sqrt{1-U^2/c^2} = g$, and so (A.1.1) and (A.1.2) can be rewritten like that:

$x = g(x''+Ut'')$ with $t = g(t''+\frac{U}{c^2}x'') = g(t''+\frac{b}{c}x'')$, so, instead of carrying out two Lorentz T. (g_1,V and g_2,W), you just make one, but using g and U.

Subapp. 2: Transversal (relativistic) Doppler Effect.

If we represent an electromagnetic wave propagating, through its electric field E:

$\vec{E} = E_0 e^{i(wt-\vec{k}\cdot\vec{r})}$; such a field propagates along r and we know that $|\vec{k}| = \frac{2p}{l}$ and $w = \frac{2p}{T}$ so:

$|\vec{k}|\, l = wT$, that is: $|\vec{k}| = w\frac{T}{l} = \boxed{\frac{w}{c} = |\vec{k}|}$; (A.2.1)

$\underline{I} = wt - \vec{k}\cdot\vec{r}$ is evidently an invariant, but it can also be expressed as the product of two 4-vectors (invariants): (position 4-vector and wave 4-vector) $\underline{I} = -\underline{r}(\vec{r},ct)\cdot\underline{k}(\vec{k},w/c)$.

We know, now, that for (A.2.1), $|\vec{k}| = \frac{w}{c}$ and let's take a light wave propagating in a system k'(V) over the plane x', y' and forming an angle θ' with x'; the components of $\vec{k}'$ will be:

$k'_1 = k'\cos q' = (w'/c)\cos q'$, $k'_2 = k'\sin q' = (w'/c)\sin q'$, $k'_3 = 0$ and $k'_4 = (w'/c) = k'$.

For the Lorentz T. , we have, on the contrary, in a system k:

$k_1 = g(k'_1+bk'_4)$, $k_2 = k'_2$, $k_3 = k'_3$ and $k_4 = g(k'_4+bk'_1)$; now, as also k_3=0, in the system k, too, the ray propagates on x,y; so, we have:

$\underline{k} = (\frac{w}{c}\cos q, \frac{w}{c}\sin q, 0, \frac{w}{c})$; now, we calculate ω and θ: on this purpose, from the transformation of k'_4 , we have:

$\frac{w}{c} = g(\frac{w'}{c} + b\frac{w'}{c}\cos q')$, or:

$$w = w'\frac{(1+b\cos q')}{\sqrt{(1-b^2)}} = w'g(1+b\cos q') \qquad \text{(A.2.2)}$$

while from the transformation of k'_1 , we have: $\frac{w}{c}\cos q = g(\frac{w'}{c}\cos q'+b\frac{w'}{c})$ and if we consider (A.2.2), we have:

$$\cos q = \frac{w'}{w}g(\cos q'+b) = \frac{(\cos q'+b)}{(1+b\cos q')} \qquad \text{(A.2.3)}$$

Then, we notice that the transformation of k'$_2$ and (A.2.2) yield:

$$\sin q = \frac{w'}{w}\sin q' = \frac{\sqrt{(1-b^2)}}{(1+b\cos q')}\sin q' = \frac{\sin q'}{g(1+b\cos q')} \qquad \text{(A.2.4)}$$

and from (A.2.3) and (A.2.4) we also have: $\sin q' = \dfrac{\sin q}{g(1-b\cos q)}$, which is, then, (A.2.4) with θ' and θ swapped and with (-β) in place of β; all this, for the relativity of the movement.

Now, suppose a source ω' is at rest in a system k'(θ'); then, from (A.2.3) we have:

$\cos q' = \dfrac{(\cos q - b)}{(1-b\cos q)}$ (useful to get θ immediately, from θ') (it's the (A.2.3) with θ' and θ swapped and with (-β) in place of β), from which:

$(1+b\cos q') = \dfrac{(1-b^2)}{1-b\cos q}$ and (A.2.4) becomes:

$$w = w'\frac{\sqrt{(1-b^2)}}{1-b\cos q} \qquad \text{(A.2.5)}$$

Here ω' is the ω of the moving source and $w' \neq w$. Therefore, if in the system k you see the radiation under an angle $q = p$, this means that the radiation comes from right, from the system k' which is getting farther along the x axis, and so we can talk about a Longitudinal Doppler Effect (Par. 3.5) and, in this case, $\cos q = \cos p = -1$ and from (A.2.5) with the source getting farther ($q = p$), we have: $w = w'\sqrt{\dfrac{(1-b)}{(1+b)}}$ (A.2.6)

\>>> $T = T'\sqrt{\dfrac{(1+b)}{(1-b)}}$, whilst, when the system k' getting closer (θ=0 >>> cos θ=1), we have:

$$w = w'\sqrt{\frac{(1+b)}{(1-b)}} \qquad \text{(A.2.7)}$$

\>>> $T = T'\sqrt{\dfrac{(1-b)}{(1+b)}}$, just like in App.1-Par. 3.5.

Curio: by a series development on β (<<1), (A.2.6) and (A.2.7) give: $w \cong w'(1-b)$ and $w \cong w'(1+b)$ so:

$\dfrac{w-w'}{w'} = \dfrac{\Delta w}{w'} = m b$; this formula is very used in astrophysics and cosmology for the red shift (remember that $w = 2pn$ and $l n = c$).

In order to go from the case of the moving emitter to that of the moving observer and vice versa, you just swap β with – β (V with –V) and ω' with ω. In any case, when there is an approaching (or a getting farther) situation, formulae for the Longitudinal Relativistic Doppler Effect are the same, no matter who is approaching.

If, on the contraty, system k sees the radiation coming under $q = p/2$, from the top, then we can talk about a TRANSVERSAL Relativistic Doppler Effect; in this case you don't have either a getting farther or an approaching situation, but the only Doppler kind effect is just due to the time dilation; in fact, also from (A.2.5), with $q = p/2$, we have: $w = w'\sqrt{(1-b^2)}$. By developing, with β<<1, we have: $w \cong w'(1-\dfrac{b^2}{2})$ (second degree in β, so, a lighter effect, with respect to the longitudinal one). Such an effect was first observed by Ives in 1938 and this plainly proved the theory. Moreover, the diversity between θ' and θ also confirms the phenomenon of the light ABERRATION, according to which, if you are moving, you see light coming to you under a different angle, something like when you are driving a car in a rainy day and you see the rain falling askew on the windscreen. And from (A.2.3) and (A.2.4), we have:

$$tg q = \frac{\sin q'\sqrt{(1-b^2)}}{(\cos q' + b)} \ .$$

Subapp. 3: The Transformations of the four-velocity.

We have defined the velocity 4-vector in App.1-Par. 2.2:

$$\underline{v}=(\frac{dx_1}{dt},\frac{dx_2}{dt},\frac{dx_3}{dt},\frac{dx_4}{dt})=(g\,\frac{dx_1}{dt},g\,\frac{dx_2}{dt},g\,\frac{dx_3}{dt},g\,\frac{dx_4}{dt})=(gv_x,gv_y,gv_z,gc)=(g\vec{v},gc)=(u_1,u_2,u_3,u_4)$$

By applying the Lorentz T. , we have: $u'_1=\Gamma(u_1-bu_4)$, $u'_2=u_2$, $u'_3=u_3$, $u'_4=\Gamma(u_4-bu_1)$;
now, you can see that Γ is defined by the V of the moving system 0' (and β=V/c), while g is referred to the velocity v had by a particle in 0, and g' is to v' had by a particle in 0'. Now, by replacing the u by the relevant values: $g'v'_x=\Gamma(gv_x-gV)$, $g'v'_y=gv_y$, $g'v'_z=gv_z$, $g'c=\Gamma(gc-gbv_x)$;

from the last one, we have: $\frac{g}{g'}=\frac{1}{\Gamma(1-\frac{V}{c^2}v_x)}$, (A.3.1)

and if you put it in the first three ones:

$v'_x=\frac{g}{g'}\Gamma(v_x-V)$, $v'_y=\frac{g}{g'}v_y$, $v'_z=\frac{g}{g'}v_z$, it yields the transformations of the velocities.

For the case of the inverse transformations, in place of the (A.3.1), we'll have: $\frac{g}{g'}=\Gamma(1+\frac{V}{c^2}v_x)$.

It follows from (A.3.1) that if the particle is at rest in 0 (v=0 >>$g=1$), then $g'=\Gamma$, from which v'=-V , that is, in 0' the particle has a velocity –V (of course).

Subapp. 4: The transformations of the four-force.

At App.1-Par. 2.3 we have introduced the Minkowski Force, or 4-force: $\underline{F}=\frac{d\underline{p}}{dt}=(g\vec{f},\frac{g}{c}(\vec{f}\cdot\vec{v}))=(F_1,F_2,F_3,F_4)$.

According to the Lorentz T.: $F'_1=\Gamma(F_1-bF_4)$, $F'_2=F_2$, $F'_3=F_3$, $F'_4=\Gamma(F_4-bF_1)$.

If now we introduce in such equations the components of the Minkowski Force, we have:

$g'\ f'_x=\Gamma(gf_x-\frac{b}{c}g(\vec{f}\cdot\vec{v}))$, $g'f'_y=gf_y$, $g'f'_z=gf_z$, $g'(\vec{f}'\cdot\vec{v}')=\Gamma(g(\vec{f}\cdot\vec{v})-Vgf_x)$, from which:

$f'_x=\frac{g}{g'}\Gamma(f_x-\frac{b}{c}(\vec{f}\cdot\vec{v}))$, $f'_y=\frac{g}{g'}f_y$, $f'_z=\frac{g}{g'}f_z$, $(\vec{f}'\cdot\vec{v}')=\frac{g}{g'}\Gamma((\vec{f}\cdot\vec{v})-Vf_x)$, which, for (A.3.1), becomes:

$$f'_x=\frac{(f_x-\frac{b}{c}(\vec{f}\cdot\vec{v}))}{1-\frac{V}{c^2}v_x}\ ,\quad f'_y=\frac{f_y\sqrt{1-b^2}}{1-\frac{V}{c^2}v_x}\ ,\quad f'_z=\frac{f_z\sqrt{1-b^2}}{1-\frac{V}{c^2}v_x}\ ,\quad (\vec{f}'\cdot\vec{v}')=\frac{(\vec{f}\cdot\vec{v})-Vf_x}{1-\frac{V}{c^2}v_x}$$

which are the transformation equations of the 4-force.

Subapp. 5: The acceleration four-vector and the transformations of the acceleration.

Of course, the 4-vector acceleration can be defined as follows:

$$\underline{a}=\frac{d^2\underline{x}}{dt^2}=(\frac{d^2x_1}{dt^2},\frac{d^2x_2}{dt^2},\frac{d^2x_3}{dt^2},\frac{d^2x_4}{dt^2})=\frac{d\underline{v}}{dt}=(a_1,a_2,a_3,a_4)$$

For the first three (spatial) components, we have: ($a=1,2,3$)

$$a_a=\frac{d}{dt}(gv_a)\frac{dt}{dt}=gv_a\frac{dg}{dt}+g^2\frac{dv_a}{dt}=\frac{\dot{v}_a}{1-b^2}+\frac{v_a(\vec{v}\cdot\dot{\vec{v}})}{c^2(1-b^2)^2}$$, as:

$\dot{g} = \frac{dg}{dt} = \frac{d}{dt}(\frac{1}{\sqrt{(1-b^2)}}) = g^3 b\dot{b} = \frac{g^3 v\dot{v}}{c^2}$. About a_4 :

$a_4 = \frac{d}{dt}(gc)\frac{dt}{dt} = cg\frac{dg}{dt} = \frac{c}{2}\frac{dg^2}{dt} = cg^4 b\dot{b} = \frac{(v\dot{v})}{c(1-b^2)^2}$ and so:

$\underline{a} = (g\frac{d}{dt}(gv), \frac{c}{2}\frac{dg^2}{dt}) = (g^2\dot{v} + g^4 b\dot{b}v, \frac{(v\dot{v})}{(1-b^2)^2})$; notice that in the system where the particle is at rest, (v=0, β=0) we have: $a_1^{(0)} = \dot{v}_x$, $a_2^{(0)} = \dot{v}_y$, $a_3^{(0)} = \dot{v}_z$, $a_4^{(0)} = 0$, that is, the spatial part of $\underline{a}$ is equal to the common three-dimensional one. Moreover: $|\underline{a}|^2 = \dot{v}^2 > 0$ and, as $|\underline{a}|^2$ is an invariant, the inequality is always true and so $\underline{a}$ is space-type.

In order to get the transformation equations for the acceleration, know that:

$\dot{v} = \frac{dv}{dt}$ and $\dot{v}' = \frac{dv'}{dt}$; moreover, $v_x = v_x(t)$ and $v'_x = v'_x(t')$ (consistently). Then, let's name:

$b_x = \frac{v_x}{c}$ and $b'_x = \frac{v'_x}{c}$; so, we get from (A1.18) that:

$$\frac{dv_x}{dv'_x} = \frac{1}{g^2(1+bb'_x)^2} \ , \ \frac{dv_y}{dv'_y} = \frac{1-b(b'_y\frac{dv'_x}{dv'_y} - b'_x)}{g(1+bb'_x)^2} \text{ and } \frac{dv_z}{dv'_z} = \frac{1-b(b'_z\frac{dv'_x}{dv'_z} - b'_x)}{g(1+bb'_x)^2} \ .$$

Then, from them, we have:

$$dv_x = \frac{dv'_x}{g^2(1+bb'_x)^2} \ , \ dv_y = \frac{dv'_y - b(b'_y dv'_x - b'_x dv'_y)}{g(1+bb'_x)^2} \text{ and } dv_z = \frac{dv'_z - b(b'_z dv'_x - b'_x dv'_z)}{g(1+bb'_x)^2} \ ;$$

if now we divide these equations by the following well known equation (of the Lorentz T.) $dt = g(dt' + \frac{b}{c}dx')$, we have:

$$a_x = \frac{1}{g^3(1+bb'_x)^3}a'_x \ , \ a_y = \frac{a'_y + b(b'_x a'_y - b'_y a'_x)}{g^2(1+bb'_x)^3} \text{ and } a_z = \frac{a'_z + b(b'_x a'_z - b'_z a'_x)}{g^2(1+bb'_x)^3}$$

which are the equations for the transformation of the accelerations, indeed.

At last, we notice those equations have got the velocity inside; therefore, if the three-dimension acceleration is constant, in an inertial reference system, it will change with time in all others!

App. 2: As I see the Universe (Gravity from Electromagnetism).

Contents of App. 2:

App. 2-Chapter 1: A new Universe, 100 times bigger, more massive and older.

App. 2-Par. 1.1: No dark matter!

ON DISCREPANCIES BETWEEN CALCULATED AND OBSERVED DENSITIES **ρ**Univ :
The search for 99% of matter in the Universe, after that it has been held invisible sounds somewhat strange. And it's a lot of matter, as dark matter should be much more than the visible one (from 10 to 100 times more).

Astrophysicists measure a ρ value of visible Universe which is around: $r \cong 2 \cdot 10^{-30} kg/m^3$.
Prevailing cosmology nowadays gives the following value of ρ: (see also (A1.6)):

$$r_{Wrong} = H_{local}^2 / (\frac{4}{3} pG) \cong 2 \cdot 10^{-26} kg/m^3 \text{ (too high!)} . \quad \text{(A1.1)}$$

Let's use the following plausible value for Hlocal (local Hubble's constant – see (A1.7) below):

$$H_{local} \cong 75 km/(s \cdot Mpc) \cong 2{,}338 \cdot 10^{-18} [(\frac{m}{s}) \Big/ m] \quad \text{(A1.2)}$$

confirmed by many measurements on Coma cluster, for instance, (see (A1.7) below) and this also confirms that the farthest objects ever observed are travelling away with a speed close to that of light:

$$H_{local} \approx c / R_{Universe-Old} \text{ , from which: } R_{Univ-Old} \approx c / H_{local} \approx 4000 Mpc \approx 13{,}5 \cdot 10^9 light\_year \quad \text{(A1.3)}$$

Moreover, one can easily calculate the speed of a "gravitating" mass m at the edge of the visible Universe, by the following equality between centrifugal and gravitational forces:

$$m \cdot a = m \cdot \frac{c^2}{R_{Univ-Old}} = G \cdot m \cdot M_{Univ-Old} / R^2_{Univ-Old} \quad , \qquad \text{(A1.4)}$$

from which, also considering (A1.3), we have:

$$M_{Univ-Old} = c^3 / (G \cdot H_{local}) \cong 1{,}67 \cdot 10^{53}\, kg \qquad \text{(A1.5)}$$

and so:

$$r_{Wrong} = M_{Univ-Old} / (\frac{4}{3} p R^3_{Univ-Old}) = (c^3 / GH_{local}) \Big/ [\frac{4}{3} p\, (\frac{c}{H_{local}})^3] = H^2_{local} / (\frac{4}{3} pG) \cong 2 \cdot 10^{-26}\, kg / m^3 \qquad \text{(A1.6)}$$

i.e. (A1.1) indeed (too high value!)
Good..., sorry, bad; this value is ten thousand times higher than the observed density value, which has been measured by astrophysicists. Moreover, galaxies are too “light” to spin so fast (see further on). As a consequence, they decided to take up searching for dark matter, and a lot of, as it should be much more than the visible one (from 10 to 100 times more).

On the contrary, astrophysicists detect a value for ρ around: $r \cong 2 \cdot 10^{-30}\, kg / m^3$.

Let’s try to understand which arbitrary choices, through decades, led to this discrepancy. From Hubble’s observations on, we understood far galaxies and clusters got farther with speeds determined by measurements of the red shift. Not only; the farthest ones have got higher speeds and it quite rightly seems there’s a law between the distance from us of such objects and the speeds by which they get farther from us.
Fig. A1.1 below is a picture of the Coma cluster, about which hundreds of measurements are available; well, we know the following data about it:

distance Δx=100 Mpc = 3,26 10^8 l.y. = 3,09 10^{24} m

speed Δv=6870 km/s=6,87 10^6 m/s.

Fig. A1.1: Coma cluster.

If we use data on Coma cluster to figure out the Hubble’s constant Hlocal, we get:

$$H_{local} = \Delta v \Big/ \Delta x \cong 2{,}22 \cdot 10^{-18} [(\frac{m}{s}) \Big/ m] , \qquad \text{(A1.7)}$$

That is a good value for “local” Hubble’s constant.

App. 2-Par. 1.2: The cosmic acceleration a_{Univ}.

As a confirmation of all we just said, we also got the same Hlocal value from (A1.3) when we used data on the visible Universe of 13,5 10^9 l.y. radius and ~c speed, instead of data on Coma cluster. By the same reasonings which led us so far to get the Hlocal constant definition, we can also state that if galaxies increase their own speeds with going farther, then they are accelerating with an acceleration we call a_{Univ} , and, from physics, we know that:

$\Delta x = \frac{1}{2} a \cdot \Delta t^2 = \frac{1}{2}(a \cdot \Delta t) \cdot \Delta t = \frac{1}{2} \Delta v \cdot \Delta t$, from which: $\Delta t = \frac{2 \cdot \Delta x}{\Delta v}$, which, if used in the definition of acceleration a_{Univ} , yields:

$$a_{Univ} = \frac{\Delta v}{\Delta t} = \frac{\Delta v}{\frac{2 \cdot \Delta x}{\Delta v}} = \frac{(\Delta v)^2}{2 \cdot \Delta x} = a_{Univ} \cong 7{,}62 \cdot 10^{-12} m/s^2 , \quad \text{cosmic acceleration} \qquad \text{(A1.8)}$$

after that we used data on Coma cluster.
This is the acceleration by which all our visible Universe is accelerating towards the center of mass of the whole Universe.
Now, we say the Universe is 100 times bigger and heavier:

$$R_{Univ-New} \cong 100 R_{Univ} \cong 1{,}17908 \cdot 10^{28} m \qquad \text{(A1.9)}$$

$$M_{Univ-New} \cong 100 M_{Univ} \cong 1{,}59486 \cdot 10^{55} kg \qquad \text{(A1.10)}$$

This value of radius is 100 times the one previously calculated in (A1.3) and it should represent the radius between the center of mass of the Universe and the place where we are now, place in which the speed of light is c.
((as we are not exactly on the edge of such a Universe, we can demonstrate the whole radius is larger by a factor $\sqrt{2}$, that is R_{Univ}=1,667 10^{28}m.))

Anyway, we are dealing with linear dimensions 100 times those supported in the prevailing cosmology nowadays. We can say that there is invisible matter, but it is beyond the range of our largest telescopes and not inside galaxies or among them; the dark matter should upset laws of gravitations, but they hold very well.

By these new bigger values, we also realize that:

$$c^2 = \frac{GM_{Univ}}{R_{Univ}} \quad ! \qquad \text{(A1.11)}$$

By the assumptions in the (A1.9) and (A1.10), we get:

$$r = M_{Univ-New} / (\frac{4}{3} p \cdot R^3_{Univ-New}) = 2.32273 \cdot 10^{-30} kg/m^3 \quad ! \qquad \text{(A1.12)}$$

which is the right measured density!

And we also see that:

$$a_{Univ} = \frac{c^2}{R_{Univ-New}} = 7{,}62 \cdot 10^{-12}\ m/s^2$$, (as we know, from physics, that $a = \frac{v^2}{r}$)

as well as:

$$a_{Univ} = G \cdot M_{Univ-New} / R^2_{Univ-New} = 7{,}62 \cdot 10^{-12}\ m/s^2$$ (from the Newton's Universal Law of Gravitation)

The new density in the (A1.12) is very very close to that observed and measured by astrophysicists and already reported at page 69.
Nature fortunately sends encouraging and convincing signs on the pursuit of a way, when confirmations on what one has understood are coming from branches of physics very far from that in which one is investigating.
On the basis of that, let's remind ourselves of the classic radius of an electron ("stable" and base particle in our Universe!), which is defined by the equality of its energy $E=m_e c^2$ ant its electrostatic one, imagined on its surface (in a classic sense):

$$m_e \cdot c^2 = \frac{1}{4pe_0} \frac{e^2}{r_e}$$, so:

$$r_e = \frac{1}{4pe_0} \frac{e^2}{m_e \cdot c^2} \cong 2{,}8179 \cdot 10^{-15} m \qquad \text{(A1.13)}$$

Now, still in a classic sense, if we imagine, for instance, to figure out the gravitational acceleration on an electron, as if it were a small planet, we must easily conclude that: $m_x \cdot g_e = G\frac{m_x \cdot m_e}{r_e^2}$, so:

$$g_e = G\frac{m_e}{r_e^2} = 8p^2 e_0^2 \frac{Gm_e^3 c^4}{e^4} = a_{Univ} = 7{,}62 \cdot 10^{-12}\ m/s^2 \text{ !!!} \qquad \text{(A1.14)}$$

that is the very value obtained in (A1.8) through different reasonings, macroscopic, and not microscopic, as it was for (A1.14). All in all, why should gravitational behaviours of the Universe and of electrons (making it) be different?

App. 2-Par. 1.4: Further considerations on the meaning of a_{Univ}.

Well, we have to admit that if matter shows mutual attraction as gravitation, then we are in a harmonic and oscillating Universe in contraction towards a common point, that is the center of mass of all the Universe. As a matter of fact, the acceleration towards the center of mass of the Universe and the gravitational attractive properties are two faces of the same medal. Moreover, all the matter around us shows it want to collapse: if I have a pen in my hand and I leave it, it drops, so showing me it wants to collapse; then, the Moon wants to collapse into the Earth, the Earth wants to collapse into the Sun, the Sun into the centre of the Milky Way, the Milky Way into the centre of the cluster and so on; therefore, all the Universe is collapsing. Isn't it?

So why do we see far matter around us getting farther and not closer? Easy. If three parachutists jump in succession from a certain altitude, all of them are falling towards the center of the Earth, where they would ideally meet, but if parachutist n. 2, that is the middle one, looks ahead, he sees n. 1 getting farther, as he jumped earlier and so he has a higher speed, and if he looks back at n. 3, he still sees him getting farther as n. 2, who is making observations, jumped before n. 3 and so he has a higher speed. Therefore, although all the three are accelerating towards a common point, they see each other getting farther. Hubble was somehow like parachutist n. 2 who is making observations here, but he didn't realize of the background acceleration g (a_{Univ}).

At last, I remind you of the fact that recent measurements on Ia type supernovae in far galaxies, used as standard candles, have shown an accelerating Universe; this fact is against the theory of our supposed current post Big Bang expansion, as, after that an explosion has ceased its effect, chips spread out in expansion, ok, but they must obviously do that without accelerating.

Moreover, on abundances of U^{235} and U^{238} we see now (trans-CNO elements created during the explosion of the primary supernova, we see that (maybe) the Earth and the solar system are just (approximately) five or six billion years old, but all this is not against all what just said on the real age of the Universe, as there could have been sub-cycles from which galaxies and solar systems originated, whose duration is likely less than the age of the whole Universe.

About T_{Univ} of the Universe, we know from physics that: v=ωR and $w = 2p/T$, and, for the whole Universe: $c=\omega R_{Univ}$ and $w = 2p/T_{Univ}$, from which:

$$T_{Univ} = \frac{2pR_{Univ}}{c} = 2{,}47118 \cdot 10^{20} s \qquad \text{(7.840 billion years)} \qquad \text{(A1.15)}$$

About the angular frequency: $w_{Univ} \cong c/R_{Universo-New} = 2{,}54 \cdot 10^{-20}\ rad/s$, and it is a right parameter for a reinterpretation of the global Hubble's constant H_{global} , whose value is H_{local} only in the portion of Universe visible by us ($w_{Univ} = H_{Global}$).

App. 2-Par. 1.5: Further confirmations and encouragements from other branches of physics.

1) Stephan-Boltzmann's law:

$e = sT^4$ [W/m²], where $s = 5{,}67 \cdot 10^{-8}\ W/(m^2K^4)$

It's very interesting to notice that if we imagine an electron ("<u>stable</u>" and base particle in our Universe!) irradiating all energy it's made of in time T_{Univ} , we get a power which is exactly ½ of Planck's constants, expressed in watt!

In fact:

$$L_e = \frac{m_e c^2}{T_{Univ}} = \frac{1}{2}h_W = 3{,}316 \cdot 10^{-34} W$$

(One must not be surprised by the coefficient ½; in fact, at fundamental energy levels, it's always present, such as, for instance, on the first orbit of the hydrogen atom, where the circumference of the orbit of the electron (2πr) really is $\frac{1}{2}l_{DeBroglie}$ of the electron. The photon, too, can be represented as if it were contained in a small cube whose side is $\frac{1}{2}l_{photon}$).

2) Moreover, we notice that an electron and the Universe have got the same luminosity-mass ratio:

in fact, $L_{Univ} = \frac{M_{Univ}c^2}{T_{Univ}} = 5{,}80 \cdot 10^{51} W$ (by definition) and it's so true that:

$$\frac{L_{Univ}}{M_{Univ}} = \frac{\frac{M_{Univ}c^2}{T_{Univ}}}{M_{Univ}} = \frac{c^2}{T_{Univ}} = \frac{L_e}{m_e} = \frac{\frac{m_e c^2}{T_{Univ}}}{m_e} = \frac{c^2}{T_{Univ}} = \frac{\frac{1}{2}h_W}{m_e}$$

and, according to Stephan-Boltzmann's law, we can consider that both an "electron" and the Universe have got the same temperature, the cosmic microwave background one:

$$\frac{L}{4pR^2} = sT^4 \text{, so: } T = (\frac{L}{4pR^2 s})^{1/4} = (\frac{L_{Univ}}{4pR_{Univ}{}^2 s})^{1/4} = (\frac{L_e}{4pr_e{}^2 s})^{1/4} = (\frac{\frac{1}{2}h}{4pr_e{}^2 s})^{1/4} = 2{,}73K \quad !!!$$

And all this is no more true if we use data from the prevailing cosmology!

3) The Heisenberg Uncertainty Principle as a consequence of the essence of the macroscopic and a_{Univ} accelerating Universe:

according to this principle, the product Δx Δp must keep above $\mathbf{h}/2$, and with the equal sign, when Δx is at a maximum, Δp must be at a minimum, and vice versa:

$\Delta p \cdot \Delta x \geq \mathbf{h}/2$ and $\Delta p_{max} \cdot \Delta x_{min} = \mathbf{h}/2$ ($\mathbf{h} = h/2p$)

Now, as Δp_{max} we take, for the electron ("stable" and base particle in our Universe!), $\Delta p_{max} = (m_e \cdot c)$ and as Δx_{min} for the electron, as it is a harmonic of the Universe in which it is (just like a sound can be considered as made of its harmonics), we have: $\Delta x_{min} = a_{Univ}/(2p)^2$, as a direct consequence of the characteristics of the Universe in which it is; in fact, from (A1.15), $R_{Univ} = a_{Univ}/w_{Univ}^2$, as we know from physics that $a = w^2 R$, and then $w_{Univ} = 2p/T_{Univ} = 2pn_{Univ}$, and as w_e of the electron (which is a harmonic of the Universe) we therefore take the "n_{Univ}–th" part of w_{Univ}, that is:

$|w_e| = |w_{Univ}/n_{Univ}|$ like if the electron of the electron-positron pairs can make oscillations similar to those of the Universe, but through a speed-amplitude ratio which is not the (global) Hubble Constant, but through H_{Global} divided by n_{Univ}, and so, if for the whole Universe: $R_{Univ} = a_{Univ}/w_{Univ}^2$, then, for the electron:

$$\Delta x_{min} = \frac{a_{Univ}}{(w_e)^2} = \frac{a_{Univ}}{(|w_{Univ}/n_{Univ}|)^2} = \frac{a_{Univ}}{(|H_{Global}/n_{Univ}|)^2} = \frac{a_{Univ}}{(2p)^2}$$

, from which:

$\Delta p_{max} \cdot \Delta x_{min} = m_e c \frac{a_{Univ}}{(2p)^2} = 0{,}527 \cdot 10^{-34}$ [Js] and such a number ($0{,}527 \cdot 10^{-34}$ Js), as chance would have it, is really $\mathbf{h}/2$!!

4) As we previously did, let's remind ourselves of the classic radius of an electron ("stable" and base particle in our Universe!), which is defined by the equality of its energy $E=m_e c^2$ ant its electrostatic one, imagined on its surface (in a classic sense):

$$m_e \cdot c^2 = \frac{1}{4pe_0}\frac{e^2}{r_e} \text{, so:}$$

$$r_e = \frac{1}{4pe_0}\frac{e^2}{m_e \cdot c^2} \cong 2{,}8179 \cdot 10^{-15} m$$

Now, still in a classic sense, if we imagine, for instance, to figure out the gravitational acceleration on an electron, as if it were a small planet, we must easily conclude that: $m_x \cdot g_e = G\frac{m_x \cdot m_e}{r_e^2}$, so:

$$g_e = G\frac{m_e}{r_e^2} = 8p^2 e_0^2 \frac{Gm_e^3 c^4}{e^4} = a_{Univ} = 7{,}62 \cdot 10^{-12}\ m/s^2 \ !!!$$

5) We know that $a = \frac{1}{137}$ is the value of the Fine structure Constant and the following formula $\frac{Gm_e^2}{r_e} \Big/ hn$ yields the same value only if n is the one of the Universe we just described, that is: $a = \frac{1}{137} = \frac{Gm_e^2}{r_e} \Big/ hn_{Univ}$, where, clearly: $n_{Univ} = \frac{1}{T_{Univ}}$ (see (A1.15)) !!

6) If I suppose, out of simplicity, that the Universe is made of just harmonics, as electrons e^- (and/or positrons e^+), their number will be: $N = \frac{M_{Univ}}{m_e} \cong 1{,}75 \cdot 10^{85}$ (~Eddington); the square root of such a number is: $\sqrt{N} \cong 4{,}13 \cdot 10^{42}$ (~Weyl).

Now, we are surprised to notice that $\sqrt{N} r_e \cong 1{,}18 \cdot 10^{28} m$ (!), that is, the very R_{Univ} value we had in (A1.9) ($R_{Univ} = \sqrt{N} r_e \cong 1{,}18 \cdot 10^{28} m$) !!!

App. 2-Par. 1.6: On discrepancies between calculated and observed rotation speeds of galaxies.

Fig. A1.2: Andromeda galaxy (M31).

<u>Andromeda galaxy (M31)</u>:

Distance: 740 kpc; R_{Gal}=30 kpc;
Visible Mass M_{Gal} = 3 $10^{11} M_{Sun}$;
Suspect Mass (+Dark) M_{+Dark} = 1,23 $10^{12} M_{Sun}$;
M_{Sun}=2 10^{30} kg; 1 pc= 3,086 10^{16} m;

By balancing centrifugal and gravitational forces for a star at the edge of a galaxy:

$m_{star}\frac{v^2}{R_{Gal}} = G\frac{m_{star}M_{Gal}}{R_{Gal}^2}$, from which: $v = \sqrt{\frac{GM_{Gal}}{R_{Gal}}}$

On the contrary, if we also consider the tidal contribution due to a_{Univ} , i.e. the one due to all the Universe around, we get:

$v = \sqrt{\frac{GM_{Gal}}{R_{Gal}} + a_{Univ}R_{Gal}}$; let's figure out, for instance, in M31, how many R_{Gal} (how many k times) far away from the center of the galaxy the contribution from a_{Univ} can save us from supposing the existence of dark matter:

$\sqrt{\frac{GM_{+Dark}}{kR_{Gal}}} = \sqrt{\frac{GM_{Gal}}{kR_{Gal}} + a_{Univ}kR_{Gal}}$, so: $k = \sqrt{\frac{G(M_{+Dark} - M_{Gal})}{a_{Univ}R_{Gal}^2}} \cong 4$, therefore, at 4R_{Gal} far away, the existence of a_{Univ} makes us obtain the same high speeds observed, without any dark matter. Moreover, at 4R_{Gal} far away, the contribution due to a_{Univ} is dominant.

At last, we notice that a_{Univ} has no significant effect on objects as small as the solar system; in fact:

$G\frac{M_{Sun}}{R_{Earth-Sun}} \cong 8{,}92 \cdot 10^8 >> a_{Univ}R_{Earth-Sun} \cong 1{,}14$.

All these considerations on the link between a_{Univ} and the rotation speed of galaxies are widely open to further speculations and the equation through which one can take into account the tidal effects of a_{Univ} in the galaxies can have a somewhat different and more difficult look, with respect to the above one, but the fact that practically all galaxies have dimensions in a somewhat narrow range (3 – 4 $R_{Milky\ Way}$ or not so much more) doesn't seem to be like that just by chance, and, in any case, none of them have radii as big as tents or hundreds of $R_{Milky\ Way}$, but rather by just some times. In fact, the part due to the cosmic acceleration, by zeroing the centripetal acceleration in some phases of the revolution of galaxies, would fringe the galaxies themselves, and, for instance, in M31, it equals the gravitational part at a radius equal to:

$\frac{GM_{M31}}{R_{Gal-Max}} = a_{Univ} R_{Gal-Max}$, from which: $R_{Gal-Max} = \sqrt{\frac{GM_{M31}}{a_{Univ}}} \cong 2{,}5 R_{M31}$; in fact, maximum radii ever observed in galaxies are roughly this size.

App. 2-Chapter 2: The unification of electromagnetic and gravitational forces (Rubino).

App. 2-Par. 2.1: The effects of M_{Univ} on particles.

We remind you that from the definition of r_e in (A1.13): $\frac{1}{4pe_0} \cdot \frac{e^2}{r_e} = m_e c^2$ and from the (A1.11): $c^2 = \frac{GM_{Univ}}{R_{Univ}}$ (~Eddington), we get:

$$\boxed{\frac{1}{4pe_0} \cdot \frac{e^2}{r_e} = \frac{GM_{Univ} m_e}{R_{Univ}}} \quad !! \qquad \text{(A2.1)}$$

As an alternative, we know that the Fine structure Constant is 1 divided by 137 and it's given by the following equation:

$a = \frac{1}{137} = \frac{\frac{1}{4pe_0} e^2}{\frac{h}{2p} c}$ (Alonso-Finn), but we also see that $\frac{1}{137}$ is given by the following equation, which can be considered suitable, as well, as the Fine structure Constant:

$a = \frac{1}{137} = \frac{\frac{Gm_e^2}{r_e}}{hn_{Univ}} = \frac{E_{Box\_Min}}{E_{Emanable}}$, where $n_{Univ} = \frac{1}{T_{Univ}}$. E_{Box_Min} is the smallest box of energy in the Universe (the electron), while $E_{Emanable}$ is the smallest emanable energy, as n_{Univ} is the smallest frequency.

Besides, a is also given by the speed of an electron in a hydrogen atom and the speed of light ratio: $a = v_{e\_in\_H} / c = e^2 / 2e_0 hc$, or also as the ratio between Compton wavelenght of the electron (which is the minimum λ of e⁻ when it's free and has the speed of light c) and the wavelength of e⁻ indeed, on the first orbit of H: $a = l_{Compton} / l_{1-H} = (h/m_e c)/(h/m_e v_{e\_in\_H})$. Moreover, $a = \sqrt{r_e/a_0}$, where $a_0 = 0{,}529$ Å is the Bohr's radius.

So, we could set the following equation and deduce the relevant consequences (Rubino):

$(a = \frac{1}{137}) = \frac{\frac{1}{4pe_0} e^2}{\frac{h}{2p} c} = \frac{\frac{Gm_e^2}{r_e}}{hn_{Univ}}$, from which: $\frac{1}{4pe_0} e^2 = \frac{c}{2pn_{Univ}} \frac{Gm_e^2}{r_e} = \frac{c}{H_{global}} \frac{Gm_e^2}{r_e} = R_{Univ} \frac{Gm_e^2}{r_e}$

after that (A1.15) has been used.

Therefore, we can write: $\frac{1}{4pe_0} \frac{e^2}{R_{Univ}} = \frac{Gm_e^2}{r_e}$ (and this intermediate equation, too, shows a deep relationship between electromagnetism and gravitation, but let's go on...)

Now, if we temporarily imagine, out of simplicity, that the mass of the Universe is made of N electrons e^- and positrons e^+, we could write:

$M_{Univ} = N \cdot m_e$, from which: $\frac{1}{4pe_0} \frac{e^2}{R_{Univ}} = \frac{GM_{Univ}m_e}{\sqrt{N}\sqrt{N}r_e}$,

or also: $\frac{1}{4pe_0} \cdot \frac{e^2}{(R_{Univ}/\sqrt{N})} = \frac{GM_{Univ}m_e}{\sqrt{N}r_e}$. (A2.2)

If now we suppose that $R_{Univ} = \sqrt{N}r_e$ (see also (A4.2)), or, by the same token, $r_e = R_{Univ}/\sqrt{N}$, then (A2.2) becomes:

$$\boxed{\frac{1}{4pe_0} \cdot \frac{e^2}{r_e} = \frac{GM_{Univ}m_e}{R_{Univ}}}$$!! (Rubino) that is (A2.1) again.

Now, first of all we see that the supposition $R_{Univ} = \sqrt{N}r_e$ is very right, as from the definition of N above given (A1.10), we have:

$N = \frac{M_{Univ}}{m_e} \cong 1{,}75 \cdot 10^{85}$ (~Eddington), from which: $\sqrt{N} \cong 4{,}13 \cdot 10^{42}$ (~Weyl) and

$R_{Univ} = \sqrt{N}r_e \cong 1{,}18 \cdot 10^{28} m$, that is the very R_{Univ} value obtained in (A1.9).

App. 2-Par. 2.2: The discovery of the common essence of gravity and electromagnetism.

Now, (A2.1) is of a paramount importance and has got a very clear meaning (Rubino) as it tells us that the electrostatic energy of an electron in an electron-positron pair (e^+e^- adjacent) is exactly the gravitational energy given to this pair by the whole Universe M_{Univ} at an R_{Univ} distance! (and vice versa)

Therefore, an electron gravitationally cast by an enormous mass M_{Univ} for a very long time T_{Univ} and through a long travel R_{Univ}, gains a gravitationally originated kinetic energy so that, if later it has to release it all together, in a short time, through a collision, for instance, and so through an oscillation of the e^+e^- pair - spring, it must transfer a so huge gravitational energy indeed, stored in billion of years that if this energy were to be due just to the gravitational potential energy of the so small mass of the electron itself, it should fall short by many orders of size. Therefore, the effect due to the immediate release of a big stored energy, by e^-, which is known to be $\frac{GM_{Univ}m_e}{R_{Univ}}$, makes the electron "appear", in the very moment, and in a narrow range (r_e), to be able to release energies coming from forces stronger than the gravitational one, or like if it were able to exert a special gravitational force, through a special Gravitational Universal Constant G', much bigger than G:

$(\frac{1}{4pe_0} \cdot \frac{e}{m_e} \cdot \frac{e}{m_e}) \cdot \frac{m_e m_e}{r_e} = G' \cdot \frac{m_e m_e}{r_e}$; it's only that during the sudden release of energy by the electron, there is a run taking effect due to its eternal free (gravitational) falling in the Universe. And, at the same time, gravitation is an effect coming from the composition of many small electric forces.

I also remark here, that the energy represented by (A2.1), as chance would have it, is really $m_e c^2$!!!, that is a sort of run taking kinetic energy, had by the free falling electron-positron pair , and that Einstein assigned to the rest matter, unfortunately without telling us that such a matter is never at rest with respect to the center of mass of the Universe, as we all are inexorably free falling, even though we see one another at rest; from which is its essence of gravitationally originated kinetic energy $m_e c^2$:

$$m_e c^2 = \frac{1}{4pe_0} \cdot \frac{e^2}{r_e} = \frac{GM_{Univ}m_e}{R_{Univ}} .$$

App. 2-Par. 2.3: The oscillatory essence of the whole Universe and of its particles.

We're talking about oscillations as this is the way the energy is transferred, and also in collisions, such as those among billiards balls, where there do are oscillations in the contact point, and how, even though we cannot directly see them (those of peripheral electrons, of molecules, of atoms etc, in the contact point).
So, we're properly talking about oscillations also because, for instance, a Sun/planet system or a single hydrogen atom, or a e^+e^- pair, which are ruled by laws of electromagnetism, behave as real springs: in fact, in polar coordinates, for an electron orbiting around a proton, there is a balancing between the electrostatic attraction and the centrifugal force:

$$F_r = -\frac{1}{4pe_0}\frac{e^2}{r^2} + m_e\left(\frac{dj}{dt}\right)^2 r = -\frac{1}{4pe_0}\frac{e^2}{r^2} + \frac{p^2}{m_e r^3} \text{ , where } \frac{dj}{dt} = w \text{ and } p = m_e v \cdot r = m_e wrr = m_e wr^2$$

Let's figure out the corresponding energy by integrating such a force over the space:

$$U = -\int F_r dr = -\frac{1}{4pe_0}\frac{e^2}{r} + \frac{p^2}{2m_e r^2} . \quad \text{(A2.3)}$$

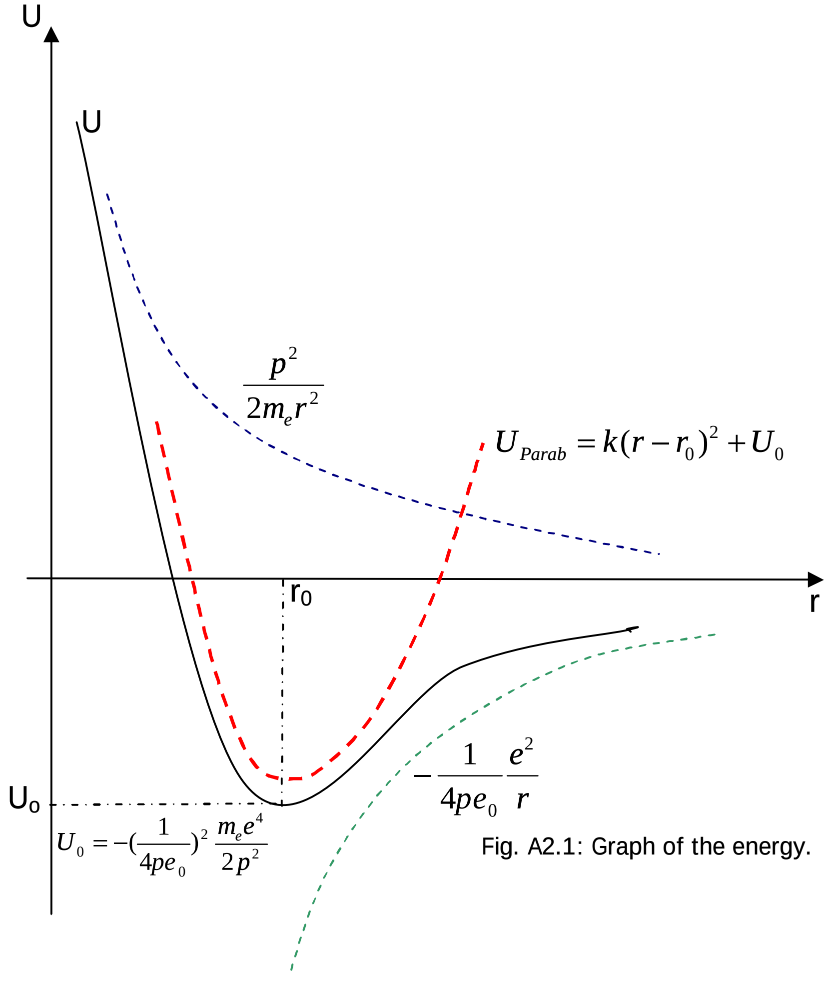
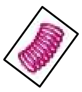


Fig. A2.1: Graph of the energy.

The point of minimum in (r_0,U_0) is a balance and stability point (F_r=0) and can be calculated by zeroing the first derivative of (A2.3) (i.e. setting F_r=0 indeed).
Moreover, around r_0, the curve for U is visibly replaceable by a parabola U_{Parab}, so, in that neighbourhood, we can write:
$U_{Parab} = k(r-r_0)^2 + U_0$, and the relevant force is: $F_r = -\partial U_{Parab}/\partial r = -2k(r-r_0)$

Which is, as chance would have it, an elastic force ($F = -kx$ - Hooke's Law). 

Moreover, the gravitational law which is followed by the Universe is a force which changes with the square value of the distance, just like the electric one, so the gravitational force, too, leads to the Hooke's law for the Universe.

By means of (A2.1) and of its interpretation, we have turned the essence of the electric force into that of the gravitational one; now we do the same between the electric and magnetic force, so accomplishing the unification of electromagnetic and gravitational fields. At last, all these fields are traced back to a_{Univ} , as gravitation does.

App. 2-Chapter 3: The unification of magnetic and electric forces.

App. 6-Par. 3.1: Magnetic force is simply a Coulomb's electric force(!).

Concerning this, let's examine the following situation, where we have a wire, of course made of positive nuclei and electrons, and also a cathode ray (of electrons) flowing parallel to the wire:

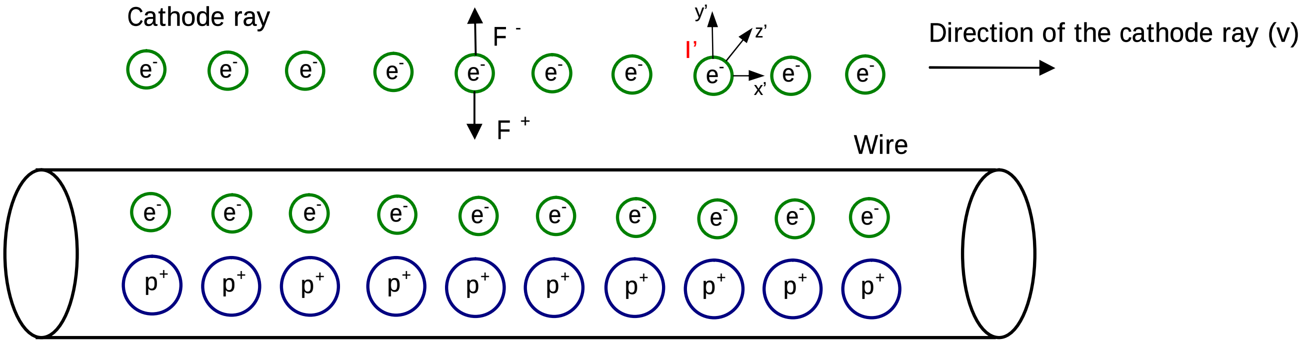


Fig. A3.1: Wire not flown by any current, seen from the cathode ray steady ref. system I' (x', y', z').

We know from magnetism that the cathode ray will not be bent towards the wire, as there isn't any current in it. This is the interpretation of the phenomenon on a magnetic basis; on an electric basis, we can say that every single electron in the ray is rejected away from the electrons in the wire, through a force F^- identical to that F^+ through which it's attracted from positive nuclei in the wire.

Now, let's examine the situation in which we have a current in the wire (e^- with speed u)

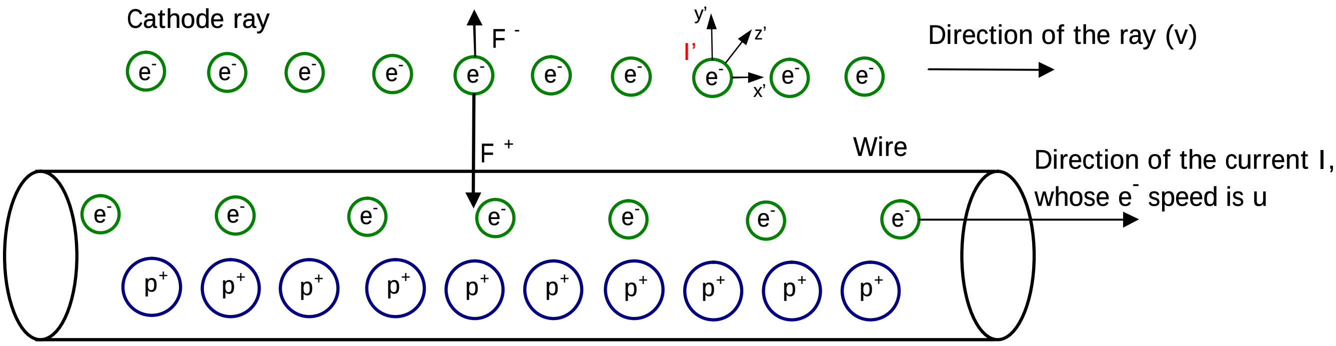


Fig. A3.2: Wire flown by a current (with e^- speed=u), seen from the cathode ray steady ref. system I' (x', y', z').

In this case we know from magnetism that the cathode ray must bend towards the wire, as we are in the well known case of parallel currents in the same direction, which must attract each other.

This is the interpretation of this phenomenon on a magnetic basis; on an electric basis, we can say that as the electrons in the wire follow those in the ray, they will have a speed lower than that of the positive nuclei, in the system I', as such nuclei are still in the wire. As a consequence of that, spaces among the electrons in the wire will undergo a lighter relativistic Lorentz contraction, if compared to that of the nuclei's, so there will be a lower negative charge density, if compared to the positive one, so electrons in the ray will be electrically attracted by the wire.

This is the interpretation of the magnetic field on an electric basis. Now, although the speed of electrons in an electric current is very low (centimeters per second), if compared to the relativistic speed of light, we must also acknowledge that the electrons are billions and billions...., so a small Lorentz contraction on so many spaces among charges, makes a substantial magnetic force to appear.

But now let's see if mathematics can prove we're quantitatively right on what asserted so far, by showing that the magnetic force is an electric one itself, but seen on a relativistic basis.

On the basis of that, let's consider a simplified situation in which an electron e^- , whose charge is q, moves with speed v and parallel to a nuclei current whose charge is Q^+ each (and speed u):

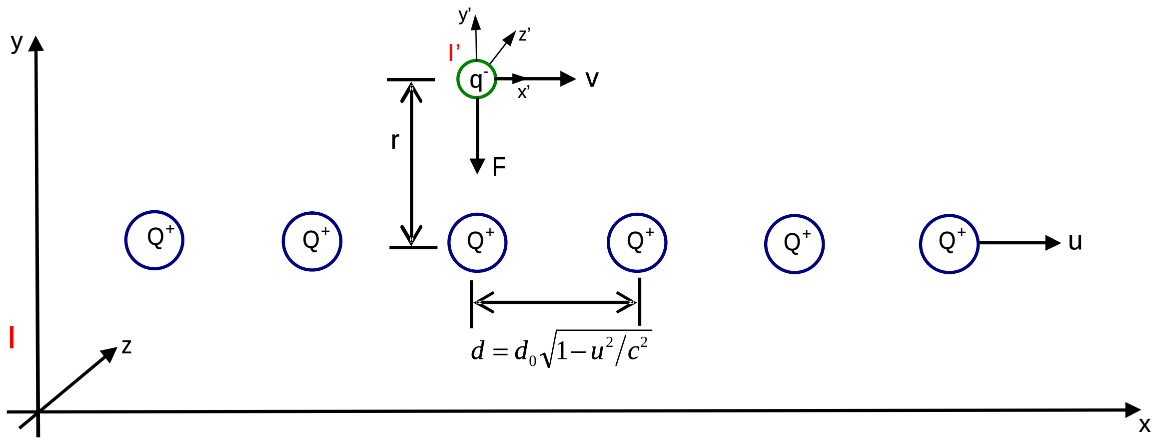


Fig. A3.3: Current of positive charge (speed u) and an electron whose speed is v, in the reader's steady system I.

a) Evaluation of F on an electromagnetic basis, in the system I :
First of all, we remind ourselves of the fact that if we have N charges Q in line and d spaced (as per Fig. A3.3), then the linear charge density λ will be:

$$l = N\cdot Q/N\cdot d = Q/d \quad .$$

Now, still with reference to Fig. A3.3, in the system I, for the electromagnetics the electron will undergo the Lorentz force $F_l = q(E + v\times B)$ which is made of an originally electrical component and of a magnetic one:

$$F_{el} = E\cdot q = (\frac{1}{e_0}\frac{l}{2pr})q = (\frac{1}{e_0}\frac{Q/d}{2pr})q$$ due to the electric attraction from a linear distribution of charges Q, and:

$$F_{magn} = m_0\frac{I}{2pr} = m_0\frac{Q/t}{2pr} = m_0\frac{Q/(d/u)}{2pr} = m_0\frac{uQ/d}{2pr}$$ (Biot and Savart).

So: $$F_l = q(\frac{1}{e_0}\frac{Q/d}{2pr} - vm_0\frac{uQ/d}{2pr}) = q\frac{Q/d_0}{2pr}(\frac{1}{e_0} - m_0uv)\frac{1}{\sqrt{1-u^2/c^2}} \quad , \qquad \text{(A3.1)}$$

where the negative sign tells us the magnetic force is repulsive, in that case, because of the real directions of those currents, and where the steady distance d_0 is contracted to d, according to Lorentz, in the system I where charges Q have got speed u ($d = d_0\sqrt{1-u^2/c^2}$).

b) Evaluation of F on an electric base, in the steady system I' of q:

in the system I' the charge q is still and so it doesn't represent any electric current, and so there will be only a Coulomb electric force towards charges Q:

$$F'_{el} = E'q = (\frac{1}{e_0}\frac{l'}{2pr})q = (\frac{1}{e_0}\frac{Q/d'}{2pr})q = q(\frac{1}{e_0}\frac{Q/d_0}{2pr})\frac{1}{\sqrt{1-u'^2/c^2}} \quad , \qquad \text{(A3.2)}$$

where u' is the speed of the charge distribution Q in the system I', which is due to u and v by means of the well known relativistic theorem of composition of speeds:

$$u' = (u-v)/(1-uv/c^2) \quad , \qquad \text{(A3.3)}$$

and d_0, this time, is contracted indeed, according to u': $d' = d_0\sqrt{1-u'^2/c^2}$.
We now note that, through some algebraic calculations, the following equality holds (see (A3.3)):

$$1-u'^2/c^2 = \frac{(1-u^2/c^2)(1-v^2/c^2)}{(1-uv/c^2)^2}$$, which, if replacing the radicand in (A3.2), yields:

$$F'_{el} = E'q = (\frac{1}{e_0}\frac{l'}{2pr})q = (\frac{1}{e_0}\frac{Q/d'}{2pr})q = q(\frac{1}{e_0}\frac{Q/d_0}{2pr})\frac{(1-uv/c^2)}{\sqrt{1-u^2/c^2}\sqrt{1-v^2/c^2}} \qquad \text{(A3.4)}$$

We now want to compare (A3.1) with (A3.4), but we still cannot, as one is about I and the other is about I'; so, let's scale F'_{el} in (A3.4), to I, too, and in order to do that, we see that, by definition of the force itself, in I':

$F'_{el}(in\_I') = \frac{\Delta p_{I'}}{\Delta t_{I'}} = \frac{\Delta p_I}{\Delta t_I \sqrt{1-v^2/c^2}} = \frac{F_{el}(in\_I)}{\sqrt{1-v^2/c^2}}$, where $\Delta p_{I'} = \Delta p_I$, as Δp extends along y, and not along the direction of the relative motion, so, according to the Lorentz transformations, it doesn't change, while Δt, of course, does. So:

$$F_{el}(in\_I) = F'_{el}(in\_I')\sqrt{1-v^2/c^2} = q\left(\frac{1}{e_0}\frac{Q/d_0}{2pr}\right)\frac{(1-uv/c^2)}{\sqrt{1-u^2/c^2}\sqrt{1-v^2/c^2}}\sqrt{1-v^2/c^2} =$$

$$= q\left(\frac{1}{e_0}\frac{Q/d_0}{2pr}\right)\frac{(1-uv/c^2)}{\sqrt{1-u^2/c^2}} = F_{el}(in\_I) \qquad \text{(A3.5)}$$

Now we can compare (A3.1) with (A3.5), as now both are related to the I system.
Let's write them one over another:

$$F_l(in\_I) = q\left(\frac{1}{e_0}\frac{Q/d}{2pr} - v m_0 \frac{uQ/d}{2pr}\right) = q\frac{Q/d_0}{2pr}\left(\frac{1}{e_0} - m_0 uv\right)\frac{1}{\sqrt{1-u^2/c^2}}$$

$$F_{el}(in\_I) = q\left(\frac{1}{e_0}\frac{Q/d_0}{2pr}\right)\frac{(1-uv/c^2)}{\sqrt{1-u^2/c^2}} = q\frac{Q/d_0}{2pr}\left(\frac{1}{e_0} - \frac{uv}{e_0 c^2}\right)\frac{1}{\sqrt{1-u^2/c^2}}$$

Therefore we can state that these two equations are identical if the following identity holds: $c = 1/\sqrt{e_0 m_0}$, and this identity is known since 1856. As these two equations are identical, the magnetic force has been traced back to the Coulomb's electric force, so the unification of electric and magnetic fields has been accomplished!!

App. 2-Chapter 4: Justification of the equation $R_{Univ} = \sqrt{N} r_e$ previously used for the unification of electric and gravitational forces (Rubino).

App. 2-Par. 4.1: The equation $R_{Univ} = \sqrt{N} r_e$ (!).

First of all, we have already checked the validity of the equation $R_{Univ} = \sqrt{N} r_e$, used in (A2.2), as it has proved to be numerically correct. And it's also justified on an oscillatory basis and now we see how; such an equation tells us the radius of the Universe is equal to the classic radius of the electron multiplied by the square root of the number of electrons (and positrons) N in which the Universe can be thought as made of. (We know that in reality almost all the matter in the Universe is not made of e^+e^- pairs, but rather of p^+e^- pairs of hydrogen atoms H, but we are now interested in considering the Universe as made of basic bricks, or in fundamental harmonics, if you like, and we know that electrons and positrons are basic bricks, as they are stable, while the proton doesn't seem so, and then it's neither a fundamental harmonic, and so nor a basic brick). Suppose that every pair e^+e^- (or, for the moment, also p^+e^- (H), if you like) is a small spring (this fact has been already supported by reasonings made around (A2.3)), and that the Universe is a big oscillating spring (now contracting towards its center of mass) with an oscillation amplitude obviously equal to R_{Univ} which is made of all microoscillations of e^+e^- pairs. And, at last, we confirm that those micro springs are all randomly spread out in the Universe, as it must be; therefore, one is oscillating to the right, another to the left, another one upwards and another downwards, and so on. Moreover e^+ and e^- components of each pair are not fixed, so we will not consider N/2 pairs oscillating with an amplitude $2r_e$, but N electrons/positrons oscillating with an amplitude r_e.

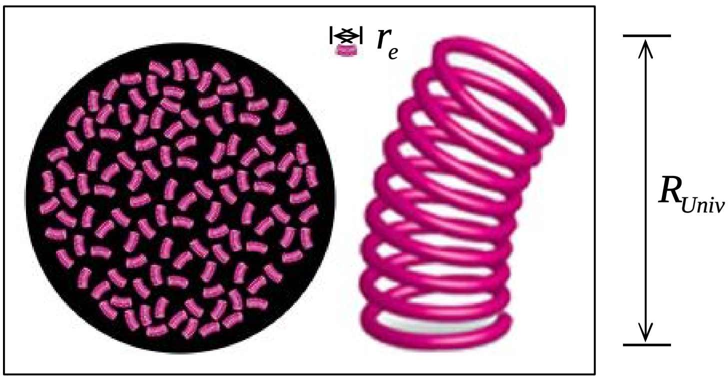


Fig. A4.1: The Universe represented as a set of many (N) small springs, oscillating on random directions, or as a single big oscillating spring.

Now, as those micro oscillations are randomly oriented, their random composition can be shown as in Fig. A4.2.

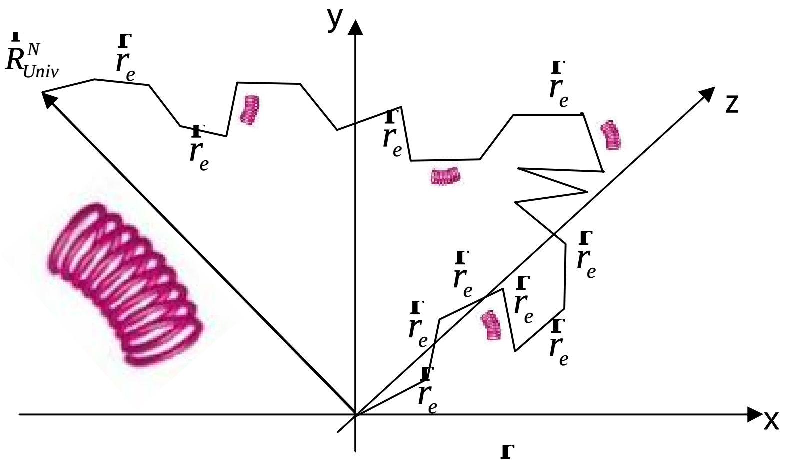


Fig. A4.2: Composition of N micro oscillations $\vec{r}_e$ randomly spread out, so forming the global oscillation R_{Univ}.

We can obviously write that: $\vec{R}^N_{Univ} = \vec{R}^{N-1}_{Univ} + \vec{r}_e$ and the scalar product $\vec{R}^N_{Univ}$ with itself yields: $\vec{R}^N_{Univ} \cdot \vec{R}^N_{Univ} = (R^N_{Univ})^2 = (R^{N-1}_{Univ})^2 + 2\vec{R}^{N-1}_{Univ} \cdot \vec{r}_e + r_e^2$; we now take the mean value:

$$\left\langle (R^N_{Univ})^2 \right\rangle = \left\langle (R^{N-1}_{Univ})^2 \right\rangle + \left\langle 2\vec{R}^{N-1}_{Univ} \cdot \vec{r}_e \right\rangle + \left\langle r_e^2 \right\rangle = \left\langle (R^{N-1}_{Univ})^2 \right\rangle + \left\langle r_e^2 \right\rangle, \qquad \text{(A4.1)}$$

as $\left\langle 2\vec{R}^{N-1}_{Univ} \cdot \vec{r}_e \right\rangle = 0$, because $\vec{r}_e$ can be oriented randomly over 360° (or over $4p$ sr, if you like), so a vector averaging with it, as in the previous equation, yields zero.

We so rewrite (A4.1): $\left\langle (R^N_{Univ})^2 \right\rangle = \left\langle (R^{N-1}_{Univ})^2 \right\rangle + \left\langle r_e^2 \right\rangle$ and proceeding, on it, by induction:
(by replacing N with N-1 and so on):

$\left\langle (R^{N-1}_{Univ})^2 \right\rangle = \left\langle (R^{N-2}_{Univ})^2 \right\rangle + \left\langle r_e^2 \right\rangle$, and then: $\left\langle (R^{N-2}_{Univ})^2 \right\rangle = \left\langle (R^{N-3}_{Univ})^2 \right\rangle + \left\langle r_e^2 \right\rangle$ etc, we get:

$\left\langle (R^N_{Univ})^2 \right\rangle = \left\langle (R^{N-1}_{Univ})^2 \right\rangle + \left\langle r_e^2 \right\rangle = \left\langle (R^{N-2}_{Univ})^2 \right\rangle + 2\left\langle r_e^2 \right\rangle = ...= 0 + N\left\langle r_e^2 \right\rangle = N\left\langle r_e^2 \right\rangle$, that is:

$\left\langle (R^N_{Univ})^2 \right\rangle = N\left\langle r_e^2 \right\rangle$, from which, by taking the square roots of both sides:

$\sqrt{\left\langle (R^N_{Univ})^2 \right\rangle} = R_{Univ} = \sqrt{N}\sqrt{\left\langle r_e^2 \right\rangle} = \sqrt{N} \cdot r_e$, that is:

$$R_{Univ} = \sqrt{N} \cdot r_e \quad \text{!!!} \quad \text{(Rubino)} \qquad \text{(A4.2)}$$

Anyway, it's well known that, in physics, for instance, the walk R made over N successive steps r, and taken in random directions, is really the square root of N by r (see, for instance, studies on Brownian movement).

App. 2-Chapter 5: "a$_{Univ}$"as absolute responsible of all forces.

App. 2-Par. 5.1: Everything from "a$_{Univ}$".

Still in agreement with what has been said so far, the cosmic acceleration itself a_{Univ} is responsible for gravity all, and so for the terrestrial one, too. In fact, just because the Earth is dense enough, it's got a gravitational acceleration on its surface g=9,81 m/s^2, while if today we could consider it as composed of electrons randomly spread, just like in Fig. A4.1 for the Universe, then it would have a radius $\sqrt{\frac{M_{Earth}}{m_e}} \cdot r_e = \sqrt{N_{Earth}} \cdot r_e$, and the gravitational acceleration on its surface would be:

$$g_{New} = G\frac{M_{Earth}}{(\sqrt{N_{Earth}} \cdot r_e)^2} = a_{Univ} = 7{,}62 \cdot 10^{-12} m/s^2 \quad \text{!!!}$$

Therefore, once again we can say that the gravitational force is due to the collapsing of the Universe by a_{Univ}, and all gravitational accelerations we meet, time after time, for every celestial object, are different from a_{Univ} according to how much such objects are compressed.

App. 2-Par. 5.2: Summarizing table of forces.

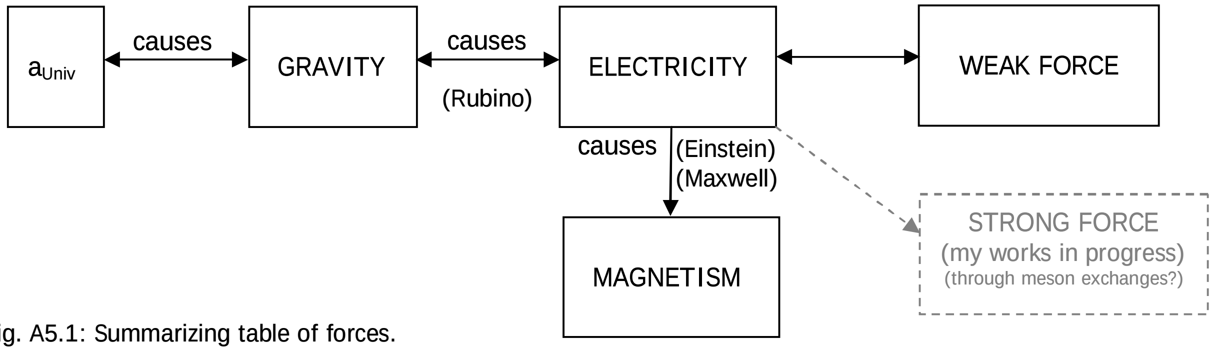


Fig. A5.1: Summarizing table of forces.

App. 2-Par. 5.3: Further considerations on composition of the Universe in pairs + / -.
The full releasing of every single small spring which stands for the electron-positron pair, is nothing but the annihilation, with turning into photons of those two particles. In such a way, that pair wouldn't be represented anymore by a pointed wave, pointed in certain place and time, (for instance $\sin(x-vt)/(x-vt)$, or the similar $d(x-vt)$ of Dirac), where the pointed part would stand for the charge of the spring, but it will be represented by a function like $\sin(x-ct)$, omogeneous along all its trajectory, and this is what a photon is. This will happen when the collapsing of the Universe in its center of mass will be accomplished.
Moreover, the essence of the pairs e^+e^-, or, in this era, of e^-p^+, is necessary in order not to violate Principle of Conservation of Energy. In fact, the Universe seems to vanish towards a singularity, after its collapsing, or taking place from nothing, during its inverse Big Bang-like process, and so doing, it would be a violation of such a conservation principle, if not supported by the Indetermination Principle, according to which an energy ΔE is legitimated to appear anyhow, unless it lasts less than Δt, in such a way that $\Delta E \cdot \Delta t \le \mathbf{h}/2$; in other words, it can appear provided that the observer doesn't have enough time, in comparison to his means of measure, to figure it out, so coming to the ascertainment of a violation. And, by the same token, the whole Universe, which is made of pairs +/-, has this property. And the appearing of a ΔE made of a pair of particles, shows the particles to reject each other first, so showing the same charge, while the successive annihilation after Δt shows a successive attraction, showing now opposite charges. So, the appearing and the annihilation correspond to the expansion and collapsing of the Universe. Therefore, if we were in an expanding Universe, we wouldn't have any gravitational force, or it were opposite to how it is now, and it's not true that just the electric force can be repulsive, but the gravitational force, too, can be so (in an expanding Universe); now it's not so, but it was!
The most immediate philosophical consideration which could be made, in such a scenario, is that, how to say, anything can be born (can appear), provided that it dies, and quick enough; so the violation is avoided, or better, it's not proved/provable, and the Principle of Conservation of Energy is so preserved, and the contradiction due to the appearing of energy from nothing is gone around, or better, it is contradicted it itself.
The proton, then, plays the role of a positron, with respect to the electron and it's heavier than it because of the possibility to exist that the fate couldn't deny to it, around the Anthropical Cosmological Principle, as such a proton brings to atoms and cells for life which investigates over it.
When the collapse of the Universe will happen, the proton will irradiate all its mass and become a positron, ready to annihilate with the electron. And through all this, we also answer the question on the unexplained prevailing of matter over the antimatter: in fact, that's not true; if we consider the proton, that is a future and ex positron, as the antimatter of the electron, and vice versa, the balance is perfect.

App. 2-Par. 5.4: The Theory of Relativity is just an interpretation of the oscillating Universe just described, contracting with speed c and acceleration a_{univ}.
On composition of speeds:

1) Case of a body whose mass is m. If in our reference system I, where we (the observers) are at rest, there is a body whose mass is m and it's at rest, we can say: $v_1 = 0$ and $E_1 = \frac{1}{2}mv_1^2 = 0$. If now I give kinetic energy to it, it will

jump to speed v_2, so that, obviously: $E_2 = \frac{1}{2} m v_2^2$ and its delta energy of GAINED energy $\Delta_{\uparrow} E$ (delta up) is:

$$\Delta_{\uparrow} E = E_2 - E_1 = \frac{1}{2} m v_2^2 - 0 = \frac{1}{2} m (v_2 - 0)^2 = \frac{1}{2} m (\Delta v)^2$$, with $\Delta v = v_2 - v_1$.

Now, we've obtained a Δv which is simply $v_2 - v_1$, but this is a PARTICULAR situation and it's true only when it starts from rest, that is, when $v_1 = 0$.

On the contrary: $\Delta_{\uparrow} E = E_2 - E_1 = \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2 = \frac{1}{2} m (v_2^2 - v_1^2) = \frac{1}{2} m (\Delta_V v)^2$, where Δ_V is a vectorial delta: $\Delta_V v = \sqrt{(v_2^2 - v_1^2)}$; therefore, we can say that, apart from the particular case when we start from rest ($v_1 = 0$), if we are still moving, we won't have a simple delta, but a vectorial one; this is simple base physics.

2) Case of the Earth. In our reference system I, in which we (the observers) are at rest, the Earth (E-Earth) rotates around the Sun with a total energy:

$E_{Tot} = \frac{1}{2} m_E v_E^2 - G \frac{M_{Sun} m_E}{R_{E-S}}$, and with a kinetic energy $E_K = \frac{1}{2} m_E v_E^2$. If now we give the Earth a delta up $\Delta_{\uparrow} E$ of kinetic energy in order to make it jump from its orbit to that of Mars (M-Mars), then, just like in the previous point 1, we have:

$\Delta_{\uparrow} E = \frac{1}{2} m_E v_E^2 - \frac{1}{2} m_E v_M^2 = \frac{1}{2} m_E (v_E^2 - v_M^2) = \frac{1}{2} m_E (\Delta_V v)^2$, with $\Delta_V v = \sqrt{(v_E^2 - v_M^2)}$, and so also here the speed deltas are vectorial-like (Δ_V).

3) Case of the Universe. In our reference system I, where we (the observers) are at rest, if we want to make a body, whose mass is m_0 and originally at rest, get speed V, we have to give it a delta v indeed, but for all what has been said so far, as we are already moving in the Universe, (and with speed c), as for above points 1 and 2, such a delta v must withstand the following (vectorial) equality:

$$V = \Delta_V v = \sqrt{(c^2 - v_{New-Abs-Univ-Speed}^2)} , \qquad \text{(A5.1)}$$

where $v_{New-Abs-Univ-Speed}$ is the new absolute speed the body (m_0) looks to have, not with respect to us, but with respect to the Universe and its center of mass.

As a matter of fact, a body is inexorably linked to the Universe where it is, in which, as chance would have it, it already moves with speed c and therefore has got an intrinsic energy $m_0 c^2$.

In more details, as we want to give the body (m_0) a kinetic energy E_k , in order to make it gain speed V (with respect to us), and considering that, for instance, in a spring which has a mass on one of its ends, for the harmonic motion law, the speed follows a harmonic law like:

$v = (w X_{Max}) \sin a = V_{Max} \sin a$ ($v_{New-Abs-Univ-Speed} = c \sin a$, in our case),

and for the harmonic energy we have a harmonic law like:

$E = E_{Max} \sin a$ ($m_0 c^2 = (m_0 c^2 + E_K) \sin a$, in our case),

we get $\sin a$ from the two previous equations and equal them, so getting:

$$v_{New-Abs-Univ-Speed} = c \frac{m_0 c^2}{m_0 c^2 + E_K} ,$$

now we put this expression for $v_{New-Abs-Univ-Speed}$ in (A5.1) and get:

$V = \Delta_V v = \sqrt{(c^2 - v_{New-Abs-Univ-Speed}^2)} = \sqrt{[c^2 - (c \frac{m_0 c^2}{m_0 c^2 + E_K})^2]} = V$, and we report it below:

$$V = \sqrt{[c^2 - (c \frac{m_0 c^2}{m_0 c^2 + E_K})^2]} \qquad \text{(A5.2)}$$

If now we get E_K from (A5.2), we have:

$$E_K = m_0 c^2 \left(\frac{1}{\sqrt{1-\frac{V^2}{c^2}}} - 1 \right)$$!!! which is exactly the Einstein's relativistic kinetic energy!

If now we add to E_K such an intrinsic kinetic energy of m_0 (which also stands "at rest" – rest with respect to us, not with respect to the center of mass of the Universe), we get the total energy:

$$E = E_K + m_0 c^2 = m_0 c^2 + m_0 c^2 \left(\frac{1}{\sqrt{1-\frac{V^2}{c^2}}} - 1 \right) = \frac{1}{\sqrt{1-\frac{V^2}{c^2}}} m_0 c^2 = g \cdot m_0 c^2$$, that is the well known

$$E = g \cdot m_0 c^2$$ (of the Special Theory of Relativity). (A5.3)

All this after that we supposed to bring kinetic energy to a body at rest (with respect to us). Equation (A5.3) works wery well on particle accelerators, where particles gain energy, but there are cases (collapsing Universe and Atomic Physics) where masses lose energy and radiate, instead of gaining it, and in such cases (A5.3) is completely inapplicable, as it's in charge for added energies, not for lost ones.

App. 2-Par. 5.5: On "Relativity" of lost energies.

In case of lost energies (further phase of the harmonic motion), the following one must be used:

$$E = \frac{1}{g} \cdot m_0 c^2$$ (Rubino) (A5.4)

which is intuitive just for the simple reason that, with the increase of the speed, the coefficient $1/g$ lowers m_0 in favour of the radiation, that is of the lost of energy; unfortunately, this is not provided for by the Theory of Relativity, like in (A5.4).

For a convincing proof of (A5.4) and of some of its implications, I have further files about.

By using (A5.4) in Atomic Physics in order to figure out the ionization energies $\Delta_{\downarrow} E_Z$ of atoms with just one electron, but with a generic Z, we come to the following equation, for instance, which matches very well the experimental data:

$$\Delta_{\downarrow} E_Z = m_e c^2 [1 - \sqrt{1 - (\frac{Ze^2}{2e_0 hc})^2}]$$ (A5.5)

and for atoms with a generic quantum number n and generic orbits:

$$\Delta_{\downarrow} E_{Z-n} = m_e c^2 [1 - \sqrt{1 - (\frac{Ze^2}{4ne_0 hc})^2}]$$ (Wåhlin) (A5.6)

Orbit (n)	Energy (J)	Orbit (n)	Energy (J)
1	$2{,}1787\ 10^{-18}$	5	$8{,}7147\ 10^{-20}$
2	$5{,}4467\ 10^{-19}$	6	$6{,}0518\ 10^{-20}$
3	$2{,}4207\ 10^{-19}$	7	$4{,}4462\ 10^{-20}$
4	$1{,}3616\ 10^{-19}$	8	$3{,}4041\ 10^{-20}$

Tab. A5.1: Energy levels in the hydrogen atom H (Z=1), as per (A5.6).

On the contrary, the use of the here unsuitable (A5.3) doesn't match the experimental data, but brings to complex corrections and correction equations (Fock-Dirac etc), which tries to "correct", indeed, an unsuitable use.

Again, in order to have clear proofs of (A5.5) and (A5.6), I have further files about.

App. 2-SUBAPPENDIXES.

App. 2-Subppendix 1: Physical constants.

Boltzmann's Constant k: $1{,}38 \cdot 10^{-23} J / K$

Cosmic Acceleration a_{Univ}: $7{,}62 \cdot 10^{-12} m / s^2$

Distance Earth-Sun AU: $1{,}496 \cdot 10^{11} m$

Mass of the Earth M_{Earth}: $5{,}96 \cdot 10^{24} kg$

Radius of the Earth R_{Earth}: $6{,}371 \cdot 10^{6} m$

Charge of the electron e: $-1{,}6 \cdot 10^{-19} C$

Number of electrons equivalent of the Universe N: $1{,}75 \cdot 10^{85}$

Classic radius of the electron r_e: $2{,}818 \cdot 10^{-15} m$

Mass of the electron m_e: $9{,}1 \cdot 10^{-31} kg$

Fine structure Constant $a\,(\cong 1/137)$: $7{,}30 \cdot 10^{-3}$

Frequency of the Universe n_{Univ} : $4{,}05 \cdot 10^{-21} Hz$

Pulsation of the Universe $w_{Univ} (= H_{global})$: $2{,}54 \cdot 10^{-20}\, rad/s$

Universal Gravitational Constant G: $6{,}67 \cdot 10^{-11} Nm^2 / kg^2$

Period of the Universe T_{Univ} : $2{,}47 \cdot 10^{20} s$

Light Year l.y.: $9{,}46 \cdot 10^{15} m$

Parsec pc: $3{,}26\_a.l. = 3{,}08 \cdot 10^{16} m$

Density of the Universe ρ_{Univ}: $2{,}32 \cdot 10^{-30} kg / m^3$

Microwave Cosmic Radiation Background Temp. T: $2{,}73K$

Magnetic Permeability of vacuum μ_0: $1{,}26 \cdot 10^{-6} H / m$

Electric Permittivity of vacuum ε_0: $8{,}85 \cdot 10^{-12} F / m$

Planck's Constant h: $6{,}625 \cdot 10^{-34} J \cdot s$

Mass of the proton m_p: $1{,}67 \cdot 10^{-27} kg$

Mass of the Sun M_{Sun}: $1{,}989 \cdot 10^{30} kg$

Radius of the Sun R_{Sun}: $6{,}96 \cdot 10^{8} m$

Speed of light in vacuum c: $2{,}99792458 \cdot 10^{8} m / s$

Stephan-Boltzmann's Constant σ: $5{,}67 \cdot 10^{-8} W / m^2 K^4$

Radius of the Universe (from the centre to us) R_{Univ}: $1{,}18 \cdot 10^{28} m$

Mass of the Universe (within R_{Univ}) M_{Univ}: $1{,}59 \cdot 10^{55} kg$

Thank you for your attention.
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