Classical Mechanics
(General Definitions)

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Definitions for a single particle A

$$m = m_{\mathrm{A}}$$
$$\mathbf{r} = \mathbf{r}_{\mathrm{A}}$$
$$\mathbf{v} = \mathbf{v}_{\mathrm{A}}$$
$$\mathbf{a} = \mathbf{a}_{\mathrm{A}}$$

Work

$$W = \int_{\mathbf{r}_1}^{\mathbf{r}_2} m\,\mathbf{a} \cdot d\mathbf{r} \;\; \longrightarrow \;\; W = \Delta \tfrac{1}{2}\, m\,\mathbf{v}^2$$

Impulse

$$\mathbf{I} = \int_{t_1}^{t_2} m\,\mathbf{a}\,dt \;\; \longrightarrow \;\; \mathbf{I} = \Delta m\,\mathbf{v}$$

Conservation of Energy

$$\Delta E = \Delta \tfrac{1}{2}\, m\,\mathbf{v}^2 - \int_{\mathbf{r}_1}^{\mathbf{r}_2} m\,\mathbf{a} \cdot d\mathbf{r} \;\; \longrightarrow \;\; \Delta E = 0 \;\; \longrightarrow \;\; E = const$$

Conservation of Momentum

$$\Delta \mathbf{M} = \Delta m\,\mathbf{v} - \int_{t_1}^{t_2} m\,\mathbf{a}\,dt \;\; \longrightarrow \;\; \Delta \mathbf{M} = 0 \;\; \longrightarrow \;\; \mathbf{M} = const$$

Principle of Least Action

$$\delta \int_{t_1}^{t_2} \tfrac{1}{2}\, m\,\mathbf{v}^2 dt + \int_{t_1}^{t_2} m\,\mathbf{a} \cdot \delta\mathbf{r}\, dt = 0$$

Definitions for a single biparticle AB

$$m = m_\mathrm{A}\, m_\mathrm{B}$$
$$\mathbf{r} = \mathbf{r}_\mathrm{A} - \mathbf{r}_\mathrm{B}$$
$$\mathbf{v} = \mathbf{v}_\mathrm{A} - \mathbf{v}_\mathrm{B}$$
$$\mathbf{a} = \mathbf{a}_\mathrm{A} - \mathbf{a}_\mathrm{B}$$

Work

$$W = \int_{\mathbf{r}_1}^{\mathbf{r}_2} m\,\mathbf{a}\cdot d\mathbf{r} \;\; \longrightarrow \;\; W = \Delta\, \tfrac{1}{2}\, m\,\mathbf{v}^2$$

Impulse

$$\mathbf{I} = \int_{t_1}^{t_2} m\,\mathbf{a}\, dt \;\; \longrightarrow \;\; \mathbf{I} = \Delta\, m\,\mathbf{v}$$

Conservation of Energy

$$\Delta E = \Delta\, \tfrac{1}{2}\, m\,\mathbf{v}^2 - \int_{\mathbf{r}_1}^{\mathbf{r}_2} m\,\mathbf{a}\cdot d\mathbf{r} \;\; \longrightarrow \;\; \Delta E = 0 \;\; \longrightarrow \;\; E = const$$

Conservation of Momentum

$$\Delta\mathbf{M} = \Delta\, m\,\mathbf{v} - \int_{t_1}^{t_2} m\,\mathbf{a}\, dt \;\; \longrightarrow \;\; \Delta\mathbf{M} = 0 \;\; \longrightarrow \;\; \mathbf{M} = const$$

Principle of Least Action

$$\delta \int_{t_1}^{t_2} \tfrac{1}{2}\, m\,\mathbf{v}^2 dt + \int_{t_1}^{t_2} m\,\mathbf{a}\cdot \delta\mathbf{r}\, dt = 0$$

Definitions for a single particle A (vector $\mathbf{u}$)

$$m = m_{\mathrm{A}}$$

$$\mathbf{u} = \cdots \text{ or } (\mathbf{r}_{\mathrm{A}}) \text{ or } (\mathbf{v}_{\mathrm{A}}) \text{ or } (\mathbf{a}_{\mathrm{A}}) \text{ or } \cdots$$

$$\dot{\mathbf{u}} = d\mathbf{u}/dt$$

$$\ddot{\mathbf{u}} = d^2\mathbf{u}/dt^2$$

Work

$$W = \int_{\mathbf{u}_1}^{\mathbf{u}_2} m\ddot{\mathbf{u}} \cdot d\mathbf{u} \;\rightarrow\; W = \Delta \tfrac{1}{2} m\dot{\mathbf{u}}^2$$

Impulse

$$\mathbf{I} = \int_{t_1}^{t_2} m\ddot{\mathbf{u}}\, dt \;\rightarrow\; \mathbf{I} = \Delta m\dot{\mathbf{u}}$$

Conservation of Energy

$$\Delta E = \Delta \tfrac{1}{2} m\dot{\mathbf{u}}^2 - \int_{\mathbf{u}_1}^{\mathbf{u}_2} m\ddot{\mathbf{u}} \cdot d\mathbf{u} \;\rightarrow\; \Delta E = 0 \;\rightarrow\; E = const$$

Conservation of Momentum

$$\Delta \mathbf{M} = \Delta m\dot{\mathbf{u}} - \int_{t_1}^{t_2} m\ddot{\mathbf{u}}\, dt \;\rightarrow\; \Delta \mathbf{M} = 0 \;\rightarrow\; \mathbf{M} = const$$

Principle of Least Action

$$\delta \int_{t_1}^{t_2} \tfrac{1}{2} m\dot{\mathbf{u}}^2 dt + \int_{t_1}^{t_2} m\ddot{\mathbf{u}} \cdot \delta\mathbf{u}\, dt = 0$$

Definitions for a single biparticle AB (vector $\mathbf{u}$)

$$m = m_\text{A}\, m_\text{B}$$

$$\mathbf{u} = \cdots \;\text{ or }\; (\mathbf{r}_\text{A} - \mathbf{r}_\text{B}) \;\text{ or }\; (\mathbf{v}_\text{A} - \mathbf{v}_\text{B}) \;\text{ or }\; (\mathbf{a}_\text{A} - \mathbf{a}_\text{B}) \;\text{ or }\; \cdots$$

$$\dot{\mathbf{u}} = d\mathbf{u}/dt$$

$$\ddot{\mathbf{u}} = d^2\mathbf{u}/dt^2$$

Work

$$W = \int_{\mathbf{u}_1}^{\mathbf{u}_2} m\,\ddot{\mathbf{u}} \cdot d\mathbf{u} \;\rightarrow\; W = \Delta \tfrac{1}{2}\, m\,\dot{\mathbf{u}}^2$$

Impulse

$$\mathbf{I} = \int_{t_1}^{t_2} m\,\ddot{\mathbf{u}}\, dt \;\rightarrow\; \mathbf{I} = \Delta m\,\dot{\mathbf{u}}$$

Conservation of Energy

$$\Delta E = \Delta \tfrac{1}{2}\, m\,\dot{\mathbf{u}}^2 - \int_{\mathbf{u}_1}^{\mathbf{u}_2} m\,\ddot{\mathbf{u}} \cdot d\mathbf{u} \;\rightarrow\; \Delta E = 0 \;\rightarrow\; E = const$$

Conservation of Momentum

$$\Delta \mathbf{M} = \Delta m\,\dot{\mathbf{u}} - \int_{t_1}^{t_2} m\,\ddot{\mathbf{u}}\, dt \;\rightarrow\; \Delta \mathbf{M} = 0 \;\rightarrow\; \mathbf{M} = const$$

Principle of Least Action

$$\delta \int_{t_1}^{t_2} \tfrac{1}{2}\, m\,\dot{\mathbf{u}}^2 dt + \int_{t_1}^{t_2} m\,\ddot{\mathbf{u}} \cdot \delta\mathbf{u}\, dt = 0$$

Appendix

If we consider a single particle of mass m then

$$\mathbf{a} - \mathbf{a} = 0 \tag{1}$$

$$m\mathbf{a} - m\mathbf{a} = 0 \tag{2}$$

$$(m\mathbf{a} - m\mathbf{a}) \cdot \delta\mathbf{r} = 0 \tag{3}$$

$$\delta \int_{t_1}^{t_2} \tfrac{1}{2} m\mathbf{v}^2 dt \; + \int_{t_1}^{t_2} m\mathbf{a} \cdot \delta\mathbf{r}\, dt = 0 \tag{4}$$

$$\frac{d}{dt}\left(\frac{\partial \frac{1}{2} m\mathbf{v}^2}{\partial \dot{q}_k}\right) - \frac{\partial \frac{1}{2} m\mathbf{v}^2}{\partial q_k} = m\mathbf{a} \cdot \frac{\partial \mathbf{r}}{\partial q_k} \tag{5}$$

Equation (3) is the D'Alembert's Principle.

Equation (4) is the Hamilton's Principle.

Equations (5) are the Euler-Lagrange Equations.

D'Alembert's Principle

In equation (3) if $\mathbf{a} = \mathbf{F}/m$ then

$(\mathbf{F} - m\mathbf{a}) \cdot \delta\mathbf{r} = 0$

Hamilton's Principle

In equation (4) if $\mathbf{a} = \mathbf{F}/m$ then

$$\delta \int_{t_1}^{t_2} \tfrac{1}{2} m \mathbf{v}^2 dt \ + \int_{t_1}^{t_2} \mathbf{F} \cdot \delta \mathbf{r} \, dt = 0$$

If $-\delta V = \mathbf{F} \cdot \delta \mathbf{r}$ and since $T = \frac{1}{2} m \mathbf{v}^2$ then

$$\delta \int_{t_1}^{t_2} (T - V) \, dt = 0$$

Since $L = T - V$ then

$$\delta \int_{t_1}^{t_2} L \, dt = 0$$

Euler-Lagrange Equations

In equations (5) if $\mathbf{a} = \mathbf{F}/m$ and $Q_k = \mathbf{F} \cdot \partial \mathbf{r} / \partial q_k$ then

$$\frac{d}{dt} \left(\frac{\partial \frac{1}{2} m \mathbf{v}^2}{\partial \dot{q}_k} \right) - \frac{\partial \frac{1}{2} m \mathbf{v}^2}{\partial q_k} = Q_k$$

If $-\partial V / \partial q_k = Q_k$ and $\partial V / \partial \dot{q}_k = 0$ and since $T = \frac{1}{2} m \mathbf{v}^2$ then

$$\frac{d}{dt} \left(\frac{\partial (T - V)}{\partial \dot{q}_k} \right) - \frac{\partial (T - V)}{\partial q_k} = 0$$

Since $L = T - V$ then

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = 0$$

Appendix II

$$k = \cdots \text{ or } (m_{\mathrm{A}}) \text{ or } \cdots$$

$$k = \cdots \text{ or } (m_{\mathrm{A}}\, m_{\mathrm{B}}) \text{ or } \cdots$$

$$\mathbf{u} = \cdots \text{ or } (\mathbf{r}_{\mathrm{A}}) \text{ or } (\mathbf{v}_{\mathrm{A}}) \text{ or } (\mathbf{a}_{\mathrm{A}}) \text{ or } \cdots$$

$$\mathbf{u} = \cdots \text{ or } (\mathbf{r}_{\mathrm{A}} - \mathbf{r}_{\mathrm{B}}) \text{ or } (\mathbf{v}_{\mathrm{A}} - \mathbf{v}_{\mathrm{B}}) \text{ or } (\mathbf{a}_{\mathrm{A}} - \mathbf{a}_{\mathrm{B}}) \text{ or } \cdots$$

$$\dot{\mathbf{u}} = d\mathbf{u}/dt$$

$$\ddot{\mathbf{u}} = d^2\mathbf{u}/dt^2$$

Conservation of Scalar $\dot{\mathrm{U}}$

$$\Delta\dot{\mathrm{U}} = \left(\Delta\, \tfrac{1}{2}\, k\, \dot{\mathbf{u}}^2 - \int_{\mathbf{u}_1}^{\mathbf{u}_2} k\, \ddot{\mathbf{u}} \cdot d\mathbf{u}\right) / \, (k)$$

$$\Delta\dot{\mathrm{U}} = 0$$

$$\dot{\mathrm{U}} = const$$

Conservation of Vector $\dot{\mathbf{U}}$

$$\Delta\dot{\mathbf{U}} = \left(\Delta\, k\, \dot{\mathbf{u}} - \int_{t_1}^{t_2} k\, \ddot{\mathbf{u}}\, dt\right) / \, (k)$$

$$\Delta\dot{\mathbf{U}} = 0$$

$$\dot{\mathbf{U}} = const$$