How infinite primes relate to other views about mathematical infinity?

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August 24, 2011

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Abstract

Infinite primes is a purely TGD inspired notion. The notion of infinity is number theoretical and infinite primes have well defined divisibility properties. One can partially order them by the real norm. p-Adic norms of infinite primes are well defined and finite. The construction of infinite primes is a hierarchical procedure structurally equivalent to a repeated second quantization of a supersymmetric arithmetic quantum field theory. At the lowest level bosons and fermions are labelled by ordinary primes. At the next level one obtains free Fock states plus states having interpretation as bound many particle states. The many particle states of a given level become the single particle states of the next level and one can repeat the construction ad infinitum. The analogy with quantum theory is intriguing and I have proposed that the quantum states in TGD Universe correspond to octonionic generalizations of infinite primes.

It is interesting to compare infinite primes (and integers) to the Cantorian view about infinite ordinals and cardinals. The basic problems of Cantor's approach which relate to the axiom of choice, continuum hypothesis, and Russell's antinomy: all these problems relate to the definition of ordinals as sets. In TGD framework infinite primes, integers, and rationals are defined purely algebraically so that these problems are avoided. It is not surprising that these approaches are

not equivalent. For instance, sum and product for Cantorian ordinals are not commutative unlike for infinite integers defined in terms of infinite primes.

Set theory defines the foundations of modern mathematics. Set theory relies strongly on classical physics, and the obvious question is whether one should reconsider the foundations of mathematics in light of quantum physics. Is set theory really the correct approach to axiomatization?

1. Quantum view about consciousness and cognition leads to a proposal that p-adic physics serves as a correlate for cognition. Together with the notion of infinite primes this suggests that number theory should play a key role in the axiomatics.
2. Algebraic geometry allows algebraization of the set theory and this kind of approach suggests itself strongly in physics inspired approach to the foundations of mathematics. This means powerful limitations on the notion of set.
3. Finite measurement resolution and finite resolution of cognition could have implications also for the foundations of mathematics and relate directly to the fact that all numerical approaches reduce to an approximation using rationals with a cutoff on the number of binary digits.
4. The TGD inspired vision about consciousness implies evolution by quantum jumps meaning that also evolution of mathematics so that no fixed system of axioms can ever catch all the mathematical truths for the simple reason that mathematicians themselves evolve with mathematics.

I will discuss possible impact of these observations on the foundations of physical mathematics assuming that one accepts the TGD inspired view about infinity, about the notion of number, and the restrictions on the notion of set suggested by classical TGD.

1 Cantorian view about infinity

The question which I have but repeatedly under the rug during the last fifteen years concerns the relationship of infinite primes to the notion of infinity as Cantor and his followers have understood it. I must be honest: I have been too lazy to even explain to myself what Cantor really said. Therefore the reading of the New Scientist article "The Ultimate logic: to infinity and beyond" [3] was a pleasant surprise since it gave a bird's eye of view about how the ideas about infinity have evolved after Cantor as a response to severe difficulties in the set theoretic formulation for the foundations of Mathematics.

1.1 Cantor's paradize

I try to summarize Cantor's view about infinity first. Cantor was the pioneer of set theory, in particular the theory of infinite sets. Cantor started his work around 1870. His goal was to formulate all notions of mathematics in terms of sets, in particular natural numbers. Cardinals and ordinals define two kind of infinite numbers in Cantor's approach.

1. Cantor realized that real numbers are "more numerous" than natural numbers and understood the importance of one-to-one correspondence (bijection) in set theory. One can say that two sets related by bijection have same cardinality. This led to the notion of cardinal number. Cardinals are represented as sets and two cardinals are same if a bijection exists between the corresponding sets. For instance, the infinite cardinals assignable to natural numbers and reals are different since no bijection between them exists.

2. The definition of ordinal relies on successor axiom of natural numbers generalized to allow infinitely large ordinals. Given ordinal can be identified as the union of all ordinals strictly smaller than it. Well ordering is a closely related notion and states that every subset of ordinals has smallest element. One can classify ordinals to three types: 0, elements with predecessor, and elements without predecessor such as ω, which corresponds to the ordinal defined as the union of all natural numbers.

   The number of ordinals much larger than the number of cardinals. This is clear since the notion of ordinal involves additional structure coming from their ordering. A given cardinal corresponds to infinitely many ordinals and one can identify the cardinal as the smallest ordinal of this kind. For instance, ω and $\omega+n$ correspond to same cardinal α_0 (countable infinity) for all finite values of n.

3. Cantor introduced the notion of power set as the set of all subsets of the set and proved that the cardinality of the power set is larger than that of set. Cantor introduced also the continuum hypothesis stating that there are no cardinals between the cardinal $\aleph_0$ *resp.* $\aleph_1$ assignable to natural numbers *resp.* reals. Hilbert represented continuum hypothesis as one of his 23 problems in his talk at the 1900 International Congress of Mathematicians in Paris. Hilbert was also a defender of Cantor and introduced the term Cantor's paradize.

4. Cantor developed the arithmetics of ordinals based on sum, product, and power: each of these operations is expressible in terms of set theoretic concepts. For infinite ordinals multiplication and sum are not commutative anymore. This looks highly counter intuitive and requires detailed definition of the sum and product. Sum means just writing the ordered sequences representing ordinals in succession. To see the non-commutativity of sum it is enough to notice that the number of elements having predecessor is not the same for $\omega + n$ and $n + \omega$.

   To see the non-commutativity of product it is enough to notice that the product is define as cartesian product $S \times T$ of the ordered sets representing the ordinals. This means that every element of T is replaced with S. It is easy to see that $n \times \omega$ and $\omega \times n$ are different.

   One can define also the powers (exponentials) in the arithemetics of ordinals: exponent must reduce to the notion of power set X^Y, which can be realized as the set of maps $Y \to X$ and has formally $\#X^{\#Y}$ elements.

It is pity that the we physicists have so pragmatic attitude to mathematics that we do not have time to realize the beauty of the idea about reduction of all mathematics to set theory. This is even more regrettable since it might well be that the manner to make progress in physics might require replacing the mathematics with a mathematics which does not rely on set theory alone.

1.2 Snakes in Cantor's paradize

Cantor's paradize is extremely beautiful place but there are snakes there. Continuum hypothesis looked to Cantor intuitively obvious but the attempts to prove it failed. Betrand Russel showed in 1901 that the logical basis of Cantor's set theory was flawed. This manifested itself via a simple paradox. Assume that it makes sense to speak about the set of all ordinals. This is by definition ordinal itself since ordinal is a set consisting of all ordinals strictly smaller than it. But this would mean that the set of all ordinals is its own member! The famous barber's paradox is a more concrete manner to express Russel's antinomy. One cannot speak of the set of ordinals and must introduce the notion of class. Russell introduced also the notion of types and type theory.

At 1920 Ernst Zermelo and Abraham Fraenkel devised a series of rules for manipulating sets but these rules did not allow to resolve the status of the continuum hypothesis. The stumbling block was the rule known as "axiom of choice" stating that if you have a collection of sets you can form a new set by picking one element from each of them. At first this sounds rather obvious but in the case when there is no obvious rule telling how to do it, situation becomes non-trivial. Then Polish mathematicians Stefan Banach and Alfred Tarski managed to show how the axiom would allow the division of a spherical ball to six subsets which can then be arranged to two balls with the same size as the original ball using only rotations and translations. These six sets are non-measurable in terms of Lebesque measure. The non-intuitive outcome must relate to the definition of the volume of the ball that is integration or measure theory: the axioms of measure theory should bring in constraints preventing construction of the six sets.

Around 1931 Kurt Gödel proved the incompleteness theorem that it is not possible to axiomatize arithmetics using any axiom system. There always remain unprovable propositions, which are true and cannot be proved to be true. This kind of statement is analogous to "I am a statement which cannot be proved to be true". If this statement could be proved to be true it would not be true.

1.3 Constructing logical universes

The attempts to expel the snakes from Cantor's paradize led to the idea that by posing some constraints it might be possible to construct logically consistent set theory obeying Zermelo-Fraenkel axioms such that continuum hypothesis and the axiom of choice would hold true and which would be free of paradoxes such as Banach-Tarski paradox.

Around 1938 Gödel introduced what he called "constructible universe" or L world satisfying these constraints. The structure of L world is hierarchical and one can say that the successor idea manifests itself directly in the construction. The levels are labeled by ordinals and one can always add a new level. The introduction of a new level to the hierarchy means that new axioms are introduced to the system bringing in meta level to the mathematical structure. The axiom system can be extended indefinitely. Gödel's theorem holds true at given level of hierarchy but by adding new levels non-probable truths can be made provable.

1963 Paul Cohen however demonstrated that there is infinite number of this kind of L worlds. In some of them continuum hypothesis holds true, in some of them the number of cardinals between $\aleph_0$ andα_1 can be arbitrary large - even infinite. This initiated a boom of constructions brings in mind the inflation of GUTs in particle physics and tge endless variety of brane constructions and the landscape misery of M-theory. From the point of view of physicist the non-uniqueness in foundations of mathematics does not seem to matter much since the everyday mathematics would remain the familiar one. One can of course ask what about quantum theory: should quantum physics replace classical physics in the formulation of fundamental fo mathematics.

For instance, von Neuman proposed one particular L world. In von Neumann unverse one starts from natural numbers and constructs its power set and at each step in the construction one consideres power set assigned to the sete obtained at the previous level. It is clear that one imagine several options. One could consider all subsets, only finite subsets, or only subsets which have cardinality smaller than the set itself. Power sets identified as the set of all finite subsets would give minimal option. Power set identified as the set of all subsets would give the maximal option.

The work of Hugh Woodin represented in 2010 International Congress of Mathematicians in Hyderabad, India represents the last twist in the story. Woodin argues that one must step outside the system that is conventional mathematical world to solve the problem. Woodin has introduced so called Woodin cardinals whose existence implies that all "projective" subsets of reals have a measurable size: it is not an accident that the word "measure" appears here when one recalls what Banach-Tarski paradox states. Woodin was motivated by the problems of set theory. He expresses this by saying "Set theory is riddled with unsolvability. Almost any problem of set theory is unsolvable".

Woodin proposed his own constructive universe which he calls ultimate L. It has all the desired properties: in particular, continuum hypothesis holds true. Physicists reader need not get frustrated if he fails to intuit why this is the case: for a decade ago Wooding himself did not believe in this. Also this L world is infinite tower to which one can add new levels.

2 The notion of infinity in TGD Universe

The construction of infinite primes, integers, and rationals brings strongly in mind the L worlds of Gödel and followers and this inspires the idea about concrete comparison of these approaches to see the differences.

2.1 Rule of thumb

It is good to start with a rule of thumb allowing to make strong conclusions about the cardinalities of infinite primes. If one considers the set formed by all finite subsets of a countable set you get a countable set because these subsets can be expressed as bit sequences with finite number if non-vanishing binary digits telling whether given element of set belongs to the subset or not: this bit sequence corresponds to a unique integer. If *all* subsets (also infinite) are allowed the set is not countably finite. If continuum hypothesis holds true it has at least as many elements as real line.

2-adic integers are good example. Consider first all 2-adic numbers with a *finite* number of non-vanishing bits (finite as real numbers). You get a countably infinite set since you can map these bit sequences to natural numbers in an obvious manner.

Consider next all possible bit sequences: most of them have infinite number bits. These numbers form naturally 2-adic continuum with 2-adic topology and differentiability. 2-adics can be mapped to real continuum in simple manner: canonical identification allows to do this continuously. The cardinality of these bit sequences is same as for reals as the rule of thumb would predict.

The hierarchy of infinite integers is based on number theoretical view about infinity and it would seem that these infinities are between the countable infinity and infinity defining the number of points of real axis. This reflects the fact that number theoretic infinity is much more refined notion than the infinities associated with cardinals and even ordinals. For instance, one can divide these infinities whereas Cantorian arithmetics contains only sum, product and power.

2.2 How Cantor's ordinals relate to the construction of infinite primes?

The fascinating question is whether the comparison of the construction of infinite primes, integers and rationals could relate to the work of Cantor and Gödel and his followers could provide new insights about infinite primes themselves.

1. What is intriguing that L-worlds are defined as infinite hierarchies just as the hierarchy of infinite primes and associated hierarchies. The great idea is that these constructions are essentially set theoretic in accordance with the vision that mathematics should reduce to set theory. As already noticed, naive set theory however leads to paradoxes which motivates the work of Gödel and followers. The basic physical philosophy is the identification of physical state as a set: this is essentially a notion belonging to classical physics.

2. TGD approach is algebraic rather than set theoretic. The construction is based on explicit formulas assuming the existence of weird quantities defined as product of all primes at previous level. These quantities can be taken as purely algebraic notions without any attempt to find a set theoretic definition.

   The possibility to interpret the construction as a repeated second quantization of a supersymmetric arithmetic quantum field theory with bosons and fermions labeled by ordinary primes at the loweset level of hierarchy replaces the set theoretic picture. These weird products of all primes represent Dirac sea at a given level of hierarchy and the many particles states of previous level become elementary particles at the new level of hierarchy. This construction is proposed to have a direct physical realization in terms of many-sheeted space-time and generalized to the level of octonionic primes is suggested to allow number theoretic interpretation of standard model quantum numbers.

   Perhaps it is not mere arrogance of quantum physics to argue that the classical set theoretic view about physical state is replaced with quantum view about it. Algebra replaces set theory and real and p-adic topologies are essential: for instance, infinite primes are infinite only in real topology.

One can raise many interesting questions. Although the underlying philosophies are very different, one can ask whether it might be possible to reduce TGD inspired construction to set theory playing key role in the construction of ordinals?

1. Can one assign to a given infinite integer a set in a natural manner? At the lowest level of hierarchy infinite prime can be mapped to a rational. Could one assign to this rational a set in cartesian product $N \times N$? Does this argument generalize to higher levels? Could the construction discussed in [3] allow to realize the set theoretic representation?

2. The notion of divisibility and explicit formulas for infinite integers obviously imply that the number of infinite numbers is much larger than cardinals of Cantor. This is true also for the ordinals of Cantor. How infinite integers relate to the ordinals of Cantor for which successor axiom is true? Also now it makes sense to form successors and in general they correspond to products of infinite primes which can be mapped to polynomials of several variables. For infinite integers however also the predecessor always exists. For instance $X \pm 1$ are infinite primes, where X represents the product of primes at previous level. Only zero fails to have predecessor for infinite natural numbers.

3. In TGD framework one loses the very essential notion of well-orderedness stating that every ordinal corresponds to a set with smallest element: that is element without predecessor. For instance, the infinite numbers known as limits and by definition are infinite and have no predecessor, the simplest example about limit is ω, which corresponds to the union of all natural numbers. The study of predecessors allowed to conclude that the sum and product are non-commutative for ordinals. Since the notion of well-ordered set does not make sense for infinite integers, one cannot identify infinite integers as ordinals.

One must however remember that just the well-orderedness hypothesis together with successor axiom allows to express ordinal as a union of strictly smaller ordinals. This in turn leads to the conclusion that ordinals cannot form a set and to Russel's antinomy and are responsible for the many problems of set theory forcing Wooding to sigh "Set theory is riddled with unsolvability. Almost any problem of set theory is unsolvable". Maybe the well-orderedness axiom is simply too strong for infinite ordinals.

4. Sum, product, and power are the basic operations in the arithmetics of ordinals. All they reduce to set theoretic constructions. One can however define these operations purely algebraically. The algebraic definition of sum and product makes sense since one can map the infinite integers to polynomials of several variables. The possibly existing set theoretic definition of infinite integers using infinite sets cannot be consistent with the commutativity of sum and product defined algebraically. Either algebra or set theory but not both!

5. Also the notion of power makes sense for ordinals and relies on the notion of power set. Could the algebraic definition of exponential make sense? If the exponent N of M^N is finite integer, then the exponent makes sense for infinite M. If N is infinite integer it does not. Hence it seems that the analogs of numbers like ω^ω do not exist in TGD inspired L universe.

6. The failure of set theoretic reductionism brings in mind the motivic approach to integration as purely algebraic approach applied to the symbol defining the integral instead of a number approach based on set theoretic notions. The motivation of the motivic approach in p-adic context is that p-adic numbers are not well-ordered so that one loses the notion of boundary and orientation as topological concepts although they can make sense algebraically.

For the hierarchy infinite integers the notion of infinity relies on real norm, which is essentially length rather than on the cardinality of a set. This infinity is essentially non-Cantorian and it is perhaps useless to try to relate it to that of ordinal or cardinal. There is just an infinite hierarchy of infinities which replaces the hiererachy of ordinals and for which the real norm of ratio makes possible partial ordering. Clearly the notion of infinity is extremely slippery and one must carefully specify what one means with infinite.

2.3 Cardinals in TGD Universe

What about cardinals in TGD framework? There seems to be no reason for giving them up and the first guess is that TGD replaces cardinals and ordinals of Cantor with cardinals and the hierarchy of infinite primes, integers, and rationals.

1. The first question is what is the cardinal assignable to infinite primes at the first level of hierarchy. For the set of finite primes the cardinal is $\aleph_0$. For the first level of infinite primes the situation is not so simple. The simple infinite primes correspond to free Fock states constructed from fermions and bosons labelled by primes. They are in one-one correspondence with rationals. There is however infinite number of many particle bound states representable as products of irreducible polynomials of one variable with integer coefficients and having finite number of roots which are algebraic numbers. The set of algebraic numbers is countable. This suggests that the cardinality of set of infinite primes at the first level of hierarchy corresponds to $\aleph_0$. This if course assuming that infinite integers and rationals for a set although they themselves cannot be described as sets.

   If one allows Fock states containing infinite number of particles and having thus infinite energy one obtains formally polynomials of infinite degree identifiable as Taylor expansions. In this case the roots can be transcendental numbers and one expects that a cardinal larger than $\aleph_0$, say $\aleph_1$ emerges. In von Neumann's Universe one indeed allows all subsets and $\aleph_1$ appears already at the first level. The higher cardinals appear at higher levels.

   One cannot exclude the Fock states containing infinite number of quanta if one accepts the idea that infinite prime representing quantum state characterizing entire Universe make sense. Does this mean that $\aleph_1$ has meaning only for entire universe and for states carrying infinite energy (in ZEO the positive energy part of zero energy state would carry the infinite energy)?

2. What happens at the next levels of the hierarchy? One possibility is that infinite primes at each level define a countable set. The point is that in polynomials representation one considers only finite degree polynomials depending on finite number of variables, having rational coefficients. Therefore everything at the level of definitions is countable and finite and the product X of primes of previous level is just an algebraic symbol identifiable as a variable of polynomial.

3. In an alternative construction of infinite integers suggested in [?]ne considers the first level of the hierarchy the set of finite subsets of algebraic numbers and the set of finite subsets of this set at the next level and so on. All these sets are countable which suggests that the number of infinite primes at each level of the hierarchy is countable and that only the completion of algebraic number to reals or p-adic can give rise to $\aleph_1$. This would conform with the fact that quantum physics is basically based on counting of quanta and that finite measurement resolution is an essential restriction on what we can know.

2.4 What about the axiom of choice?

Axiom of choice has several variants. One variant is axiom of countable choice. Second variant is generalized continuum hypothesis states that the cardinality of an infinite set is between that of infinite set S and its power set: in other words there is no cardinal satisfying $\aleph_\alpha < \lambda < 2^{\aleph_\alpha}$ or equivalently: $\aleph_{\alpha+1} = 2^{\aleph_\alpha}$. For a finite collection of sets it can be proved but already when on has a countable collection of nonempty set and in the case that one cannot uniquely specify some preferred element of each set, axiom of choice must be postulated. For instance, each subset of natural numbers has smallest element so that there is no need to postulate axiom of choice separately. Also closed intervals of real axis have smallest element.

What happens to the axiom of choice in TGD Universe. TGD is a physical theory and this means that the laws of classical physics strong considerations on the allowed sets. Classical physics is in TGD framework the dictated by the Kähler action and by a principle selecting its preferred extremals. Although several almost formulations for this principle exist, it is far from being well-understood and it is not clear whether one can give explicit formula for preferred extremals. One formulation is as quaternionic sub-manifolds of 8-D imbedding space allowing octonionic structure in its tangent space and defined by octonionic representation of the gamma matrices defining the Clifford algebra.

1. The world of classical worlds can be regarded as the space of preferred extremals of Kähler action identifiable as certain 4-surfaces in $M^4 \times CP_2$. The mere extremal property implies also smoothness of the partonic 2-surfaces so that very powerful constraints are involved: therefore situation is very far from the extreme generality of set theory where one does assumes neither continuity nor smoothness. Zero energy ontology means that this space effectively reduces to a collection of spaces assignable to causal diamonds. Strong form of holography reduces this space effectively to the space consisting of collectings of partonic 2-surfaces at the light-like boundaries of CD plus 4-D tangent space data at them which very probably cannot be chosen freely.

2. In this kind of situation it might well happen that all collections of sets, say are finite or in the case that that they are countable they allow a unique choice of preferred point. Axiom of choice would not be needed. The specification of a preferred point of every 4-surface in the collection does not look a problem for a pragmatic physicist, since one can restrict the consideration to the boundaries of causal diamonds and consider for instance minimum of light-like radial coordinate. In fact, finite measurement resolution leads to the effective replacement of partonic 2-surfaces with the collection of ends of braid strands and the ends of braid strands define the preferred points. One might say, that physics with finite measurement resolution performs the choice automatically. A stronger form of this choice is that the points in question are rational points for a natural choice of the imbedding space coordinates.

2.5 Generalization of real numbers inspired by infinite integers

Surreal numbers define a generalization of reals obtained by introducing a hierarchy of infinite reals and infinitesimals as their inverses. Infinite integers and rationals in TGD sense could give rise to a similar generalization so that one would have an infinite hierarchy of 8-D imbedding space such that at given level previous level would represent infinitesimals.

TGD suggests another generalization of reals. One can construct from infinite integers rationals with unit norm. A possible interpretation would be as zero energy states with denominator and numerator represention positive and negative energy parts of the zero energy state and vanishing of total quantum numbers represented by real unit property. These numbers would have arbitrarily complex number theoretical anatomy however.

This structure has enormous representative power and one could dream that the world of classical worlds and spinor fields in this space could allow representation in terms of these real units. Brahman Atman Identity would be realized: the structure of single space-time point invisible to ordinary physics would represent the world of classical worlds! Single space-time point would be the Platonia!

Could one say that the space of all infinite rationals which are real units is countable? If previous arguments are correct this would seem to be true. If this is true, then TGD inspired notion of infinity would be extremely conservative as compared to the view proposed by Cantor and his followers using the Cantorian criteria. Just $\aleph_n$, $n = 0, 1$ and hierarchy of infinite integers which are countable sets. One can of course, ask how many surfaces WCW contains, what $\aleph$ is in question. This depends on the properties of preferred extremals. If partonic 2-surfaces can be choosen freely at the boundaries of CDs the restrictions come only from smoothness of the imbedding of the partonic 2-surfaces and tangent space data. The space of all functions from reals to reals has cardinality $2^{\aleph_1}$ which suggests that the cardinality is not larger than this, perhaps smaller since continuity and smoothness poses strong conditions. The natural guess is that the tangent space of WCW can be modelled as and infinite-dimensional separable Hilbert space which has cardinality $\aleph_1$.

TGD leads also a second generalization of the number concept motivated by number theoretical universality inspiring the attempt to glue different number fields (reals and various p-adics) together among common numbers -rationals in particular- to form a larger structure [4].

To sum up, the distinctions between Cantorian and TGD inspired approaches are clear. Cantorian approach relies on set theory and TGD on number theory. What is common is the hierarchy of infinities.

3 What could be the foundations of physical mathematics?

Theoretical physicists do not spend normally their time for questioning the foundations of mathematics. They calculate. There are exceptions: Von Neuman was both a theoretical physicist developing mathematical foundations of quantum theory and mathematician buildingthe mathematics of quantum theory and also proposing his own L world for foundations of mathematics.

A physicist posing the question "What should be done for the foundations of mathematics?" sounds blasphemous and the physicist should add the attribute "physical" to "mathematics" to avoid irritation. In any case, the fact is that the problems plaguing set theory and therefore the foundations of mathematics had been discovered roughly century ago and no commonly accepted solution to these problems have been found. The foundations of mathematics rely on classical physics and quantum view about existence suggests that the foundations of mathematics might need a revision.

Again the work of von Neuman comes readily into mind. The goal of von Neuman was to build a non-commutative measure theory: the outcome was the three algebras bearing his name and defining the mathematical backbone of three kinds of quantum theories. Factors of type I are natural for wave mechanism with finite number of degreees of freedom. In QFT hyperfinite factors of type III appear. In TGD framework hyperfinite factors of type II (and possibly of type III) are natural.

Connes who has studied von Neumann algebras highly relevant to quantum physics proposed the notion of non-commutative geometry in terms of a spectral triplet defined by C^* algebra A, Hilbert space H, and Dirac operator D with some additional properties. As a special case one re-discovers Riemannian manifolds using commutative function algebra, the Hilbert space of continuous functions, and certain kind of Dirac operator.

Physicists are usually mathematical opportunists and do not want to use time to ponder the foundations of mathematics My belief is that physicists should get rid of this attitude and make fool of themselves by posing the childish questions of physicist in the hope that some real mathematician might get interested. In order to not irritate mathematicians too much I will talk about physical mathematics instead of mathematics in the sequel.

3.1 Does it make sense to speak about physical set theory?

For the physicist set theory looks quite too general. In the recent day physical theories sets are typically manifolds, submanifolds, or orbifolds. Feynman diagrams represent example of 1-D singular manifolds and in TGD generalized Feynman diagrams of TGD fail to be 3-manifolds only at the vertices represented as 2-D partonic surfaces. In string theories and in twistor approach to gauge theories algebraic geometry is important. Branes are typically algebraic surfaces. The spaces are endowed with various structures: besides metric induced topology one differential structure, differential forms, metric, spinor structure, complex and Kähler structure, etc...

1. In algebraic geometry sets are replaced with varieties and basic set theoretic operations such as intersection and union are algebraized. Physicists should not fail to realize how profound this algebraization of the set theory is. The price that must be paid is that varieties are manifolds only locally. What limitations does this mean for set theory? Is it enough to formulate set theory algebraically? In TGD framework this could be possible in the intersection of real and p-adic worlds (WCWs) since set theoretic operations would have algebraic representation. For instance, $A \subset B$ would be formulated by adding additional functions for which the intersection of zero locus with B defines A.

   The algebraic notion of set as a variety is extremely restrictive: maybe the problems of set theory are partially due to the neglect of the fact that allowed sets must have a physical realization. Every physicists of course has her own pet theory, which he regards as the real physics, and one naturalcondition on any accceptable physics is that it can emulate sufficiently general spaces - to act as a kind of mathematical Turing machine. At least real and complex manifolds with arbitrary dimension should have some kind of physical representation. One can imagine this kind of representation in terms of unions of partonic 2-surfaces since union can be regarded also as a Cartesian product as long as the surfaces do not intersect.

2. The introduction of topology is the first step in bringing structure to the set theoretic primordial chaos. Metric topology is standard in physics at space-time level. More refined topologies can be certainly found in highly technical mathematical physics articles. In algebraic geomery Zariski topology is important but has its problems realized by Groethendienck in his attempts to build a universal cohomology theory working in all number fields. The closed sets of Zariski topology are varieties. Their complements would be open sets open also in norm based topology. Zariski topology is obviously much rougher than the metric opology. Zariski topology makes sense also for p-adic number fields. This kind of topology might make sense in TGD framework if one restricts the consideration to the intersection of real and p-adic worlds identified at the level of WCW as the space of algebraic surfaces defined using polynomials with rational coefficients and having finite degree.

3. In TGD framework preferred extremals of Kähler action define space-time surfaces and strong form of holography makes the situation effectively 2-dimensional. The conjecture is that preferred extremals correspond to quaternionic surfaces of octonionic 8-space. Octonionic structure is associated with the octonionic representation of the imbedding space gamma matrices (not actually matrices any more!) defining the Clifford algebra. Associativity would be the basic dynamical principle. Does this mean that number theory- in particular classical number fields- should appear in the formulation of the foundations ofphysical mathematics? This idea is attractive even when one does not assume that TGD Universe is the Universe.

What is beautiful that algebraic geometry brings in also number theory. One might hope that the foundations of physical mathematics could be based on the fusion of set theory, geometry, algebra, and number theory.

3.2 Quantum Boolean algebra instead of Boolean algebra?

Mathematical logic relies on the notion of Boolean algebra, which has a well-known representation as the algebra of sets which in turn has in algebraic geometry a representation in terms of algebraic varieties. This is not however attractive at space-time level since the dimension of the algebraic variety is different for the intersection *resp.* union representing AND *resp.* OR so that only only finite number of ANDs can appear in the Boolean function. TGD inspired interpretation of the fermionic sector of the theory in terms of Boolean algebra inspires more concrete ideas about the the realization of Boolean algebra at both quantum level and classical space-time level and also suggests a geometric realization of the basic logical functions respecting the dimension of the representative objects.

1. In TGD framework WCW spinors correspond to fermionic Fock states and an attractive interpretation for the basis of fermionic Fock states is as Boolean algebra. In zero energy ontology one consider pairs of positive and negative energy states and zero energy states could be seen as physical correlates for statements $A \to B$ or $A \leftrightarrow B$ with individual state pairs in the quantum superposition representing various instances of the rule $A \to B$ or $A \leftrightarrow B$. The breaking of time reversal invariance means that either the positive or negative energy part of the state (but not both) can correspond to a state with precisely definine number of particles with precisely defining quantum numbers such as four-momentum. At the second end one has scattered state which is a superposition of this kind of many-particle states. This would suggest that $A \to B$ is the correct interpretation.

2. In quantum group theory [2] the notion of co-algebra [1] is very natural and the binary algebraic operations of co-algebra are in a well-defined sense time reversals of those of algebra. Hence there is a great temptation to generalize Boolean algebra to include its co-algebra [5] so that one might speak about quantum Boolean algebra. The vertices of generalized Feynman diagrams represent two topological binary operations for partonic two surfaces and there is a strong temptation to interpret them as representations for the operations of Boolean algebra and its co-algebra.

   (a) The first vertex corresponds to the analog of a stringy trouser diagram in which partonic 2-surface decays to two and the reversal of this representing fusion of partonic 2-surfaces. In TGD framework this diagram does not represent classically particle decay or fusion but the propagation of particle along two paths after the decay or the reversal of this process. The Boolean analog would be logical OR ($A \vee B$) or set theoretical union $A \cup B$ *resp.* its co-operation. The partonic two surfaces would represent the arguments (*resp.* co-arguments) A and B.

   (b) Second one corresponds to the analog of 3-vertex for Feynman diagram: the three 3-D "lines" of generalized Feynman diagram meet at the partonic 2-surface. This vertex (co-vertex) is the analog of Boolean AND ($A \wedge B$) or intersection $A \cap B$ of two sets *resp.* its co-operation.

   (c) I have already earlier ended up with the proposal that only three-vertices appear as fundamental vertices in quantum TGD [1]. The interpretation of generalized Feynman diagrams as a representation of quantum Boolean algebra would give a deeper meaning for this proposal.

   These vertices could therefore have interpretation as a space-time representation for operations of Boolean algebra and its co-algebra so that the space-time surfaces could serve as classical correlates for the generalized Boolean functions defined by generalized Feynman diagrams and expressible in terms of basic operations of the quantum Boolean algebra. For this representation the dimension of the variety representing the value of Boolean function at classical level is the same as as the dimension of arguments: that is two. Hence this representation is not equivalent with the representation provided by algebraic geometry for which the dimension of the geometric variety representing $A \wedge B$ and $A \vee B$ in general differs from that for A and B. If one however restricts the algebra to that assignable to braid strands, statements would correspond to points at partonic level, so that one would have discrete sets and the set theoretic representation of quantum Boolean algebra could make sense. Discrete sets are indeed the only possibility since otherwise the dimension of intersection and union are different if algebraic varieties are in question.

3. The breaking of time reversal invariance is accompanied by a generation of entropy and loss of information. The interpretation at the level of quantum Boolean algebra would be following. The Boolean function and and OR assign to two statements a single statement: this means a gain of information and at the level of physics this is indeed the case since entropy is reduced in the process reducing the number of particles. The occurrence of co-operations of AND and

OR corresponds to particle decays and uncertainty about the path along which particle travels (dispersion of wave packet) and therefore loss of information.

(a) The "most logical" interpretation for the situation is in conflict with the identification of the arrow of logic implication with the arrow of time: the direction of Boolean implication arrow and the arrow of geometric time would be opposite so that final state could be said to imply the initial state. The arrow of time would weaken logical equivalence to implication arrow.

(b) If one naively identifies the arrows of logical implication and geometric time so that initial state can be said to imply the final state, second law implies that logic becomes fuzzy. Second law would weak logical equivalence to statistical implication arrow.

(c) The natural question is whether just the presence of both algebra and co-algebra operations causing a loss of information in generalized Feynman diagrams could lead to what might be called fuzzy Boolean functions expressing the presence of entropic element appears at the level of Boolean cognition.

4. This picture requires a duality between Boolean algebra and its co-algebra and this duality would naturally correspond to time reversal. Skeptic can argue that there is no guarantee about the existence of the extended algebra analogous to Drinfeld double [4] that would unify Boolean algebra and its dual. Only the physical intuition suggests its existence.

These observations suggest that generalized Feynman diagrams and their space-time counterparts could have a precise interpretation in quantum Boolean algebra and that one should perhaps consider the extension of the mathematical logic to quantum logic. Alternatively, one could argue that quantum Boolean algebra is more like a model for what mathematical cognition could be in the real world.

3.3 The restrictions of mathematical cognition as a guideline?

With the birth of quantum theory physicists ceased to be outsiders since it was impossible to consider quantum measurement as something not affecting the measured system in any way. With the advent of consciousness theory physicists have been forced to give up the idea about uni-directional action with with reality and have become a part of quantum Universe - self. This also requires dramatic modification of the basic ontology forcing to give up the physicalistic dogmas. Consciousness involves free will manifested in ability to select and create something completely new in each quantum jump. Physical Universe is not given but is re-created again and again and evolves.

In standard mathematics mathematician is still a complete outsider, and the possible limitations of mathematical cognition are not considered seriously in the attempts to formulate the foundations of mathematics. Mathematicians still choose effortlessly one element from each set of infinite collection of sets. We know that in numerics one is always bound to introduce cutoff on the number of bits and use finite subset of rational numbers but also this has not been taken into account in the formulation of foundations as far as I know. If one takes consciousness theory seriously one is led to wonder what are the physical restrictions on mathematical cognition and therefore on physical mathematics. What looks obvious that the idea about mathematics based on fixed axiomatics must be given up. The evolution of the physical universe and of consciousness means also the evolution of (at least physical) mathematics. The paradox of self reference plaguing conventional view about consciousness and leading to infinite regress disappears when this regress is replaced with evolution.

Suppose that life resides and cognitive representations are realized in the intersection of real and p-adic worlds reducing to intersections of real and p-adic variants of partonic 2-surfaces at space-time level. At the level of WCW the intersection of real and p-adic worlds could correspond to the space of partonic 2-surfaces defined by rational functions constructed using polynomials of finite degree with rational coefficients.

What kind of restrictions of this picture poses set theory, topology, and logic? The reader can of course imagine restrictions on some other fields of mathematics involved. The question in the case of the set theory and topology has been already touched. In the case of logic the key question seems to concern the operational meaning of $\forall$ and $\exists$, when the finite resolution of measurements and cognitive representation are taken into account. What these universal quantors really mean: what is their domain of definition?

Consider first the domain of definition at space-time level.

1. Should all theorems be formulated using $\forall$ and $\exists$ restricted to the dense subset rationals of 8-D imbedding space. Since continuous function is fixed from its values in a dense subset, this assumption is not so strong unless there are other restrictions.

2. At space-time surface and partonic 2-surfaces the situation is different. The assumption that only the common rational points of real and p-adic surfaces define cognitive representations poses a strong limitation since typically the number of rational points of 2-surface is expected to be finite. Algebraic extensions of p-adic numbers extend the number of common points and one can imagine an evolutionary hierarchy of mathematics realized in terms of geometry of partonic 2-surfaces reflecting itself as the geometry of space-time surfaces by strong form of holography.

3. The orbits of the rational points selected at the ends of partonic 2-surfaces are braids along light-like 3-surfaces. At space-time level one has world sheets or strings which form in general case 2-braids. This picture leads to a what I have used to call almost topological QFT.

What about the domain of definition of existence quantors at the level of WCW? The natural conjecture is that the surfaces in the intersection of real and p-adic worlds form a dense set of full WCW so that everything holding true in the intersection would hold true generally and one coul dhope that systems which are living in the proposed sense are able to discover interesting mathematics.

Suppose that the partonic 2-surfaces decompose into patches such that in each patch the surface is a zero locus of polynomials with rational coefficients. Since polynomials can be seen as Taylor series with cutoff one can hope that they form a dense subset. Since rationals are dense subset of reals, one can hope that also the restriction to rational coefficients preserves the dense subset property and living subsystems are able to represent all that is needed and completion takes care of the rest as it does for rationals. The notion of completion leading from rationals to various algebraic numbers fields and also to reals and complex numbers would become the fundamental principle leading from number theory to metric topology.

Physicist reader has certainly noticed that "rational point" does not represent a general coordinate invariant notion.

1. The coordinates of point are rational in preferred coordinates and the symmetries of the 8-D imbedding space suggest families of preferred coordinates. The moduli space for CDs would be characterized by the choice of these preferred coordinates dictating also the choice of quantization axes so that quantum measurement theory would be realized as a decomposition of WCW to a union corresponding to different choices. State function reduction would involve also a localization determining quantization axes.

2. There are many possible choices of quantization axes/preferred coordinates and this means a restriction of general coordinate invariance from group of all coordinate transformations to a discrete subgroup of isometries which is not unique. Cognition would break the general coordinate invariance. The world in which the mathematician thinks using spherical coordinates differs in some subtle manner from the world in which she thinks using Cartesian coordinates. Mathematician does not remain outside Platonia anymore just as quantum physicists is not outside the physical Universe!

Axiom of choice relates to selection, which can be regarded as a cognitive act. The question whether axiom of choice is needed at all has been already discussed but a couple of clarifying comments are in order.

1. At quantum level selection would be naturally assigned with state function reduction, also the state function reduction selecting quantization axes. The cascade of state function reductions - starting from the scale of CD and proceeding fractally downwards sub-CD by sub-CD and stopping when only negentropic entanglement stabilized by NMP remains - could be how Nature performs the choice. State function reduction would involve also the choice of quantization axes dictating possible subsequence choices. Note that non-deterministic element would be involved with the quantum choice.

2. If life and cognitive representations are at the intersection of real and p-adic worlds, it would seem that rational points are chosen at space-time level and algebraic 2-surfaces at WCW level.

As explained, it is easy to imagine the collection of sets from which one selects points is always finite or that there is a natural explicit criterion allowing to select preferred point from each set. Finite measurement resolution implying braids and string world sheets could provide this criterion. If so, the axiom of choice would be un-necessary in physical mathematics. Finite measurement resolution suggests that for partonic 2-surfaces the ends of braid strands define preferred points.

Platonia is a strange place about which many mathematicians claim to visit regularly. I already proposed that the generalization of space-time point by bringing in the infinite number theoretical anatomy of real (and octonionic) units might allow to realize number theoretical Brahman=Atman identity by representing WCW in terms of the number theoretic anatomy of space-time points. This kind of representation would certainly be the most audacious idea that physical mathematician could dare to think of.

3.4 Is quantal Boolean reverse engineering possible?

The quantal version of Boolean algebra means that the basic logical functions have quantum inverses. The inverse of $C = A \wedge B$ represents the quantum superposition of all pairs A and B for which $A \wedge B = C$ hols true. Same is true for $\vee$. How could these additional quantum logical functions with no classical counterparts extend the capacities of logician?

What comes in mind is logical reverse engineering. Consider the standard problem solving situation repeatedly encountered by my hero Hercule Poirot. Someone has been murdered. Who could have done it? Who did it? Actually scientists who want to explain instead of just applying the method to get additional items to the CVC, meet this kind of problem repeatedly. One has something which looks like an experimental anomaly and one has to explain it. Is this anomaly genuine or is it due to a systematic error in the information processing? Could the interpretation of data be somehow wrong? Is the model behind experiments based on existing theory really correct or has something very delicate been neglected? If a genuine anomaly is in question (someone has been really murdered- this is always obvious in the tales about the deeds of Hercule Poirot since the mere presence of Hercule guarantees the murder unless it has been already done), one encounters what might be called Poirot problem in honor of my hero. As a matter fact, from the point of view of Boolean algebra, one has the same reverse Boolean engineering problem irrespective of whether it was a genuine anomaly or not.

This brings in my mind the enormously simplified problem. The logical statement C is found to be true. Which pairs A, B could have implied C as $C = A \wedge B$ (or $A \vee B$). Of course, much more complex situations can be considered where C corresponds to some logical function $C = f(A_1, A_2, ..., A_n)$. Quantum Poirot could use quantum computer able to realize the co-gates for gates AND and OR (essentially time reversals) and write a quantum computer program solving the problem by constructing the Boolean co-function of Boolean function f.

What would happen in TGD Universe obeying zero enery ontology is following.

1. The statement C is represented as as positive energy part of zero energy state (analogous to initial state of physical event) and $A_1, ..A_n$ is represented as one state in the quantum superposition of final states representing various value combinations for $A_1, ..., A_n$. Zero energy states (rather than only their evolution) represents the arrow of time. The M-matrix characterizing time-like entanglement between positive and negative energy states generalizes generalizes S-matrix. S-matrix is such that initial states have well defined particle numbers and other quantum numbers whereas final states do not. They are analogous to the outcomes of quantum measurement in particle physics.

2. Negentropy Maximization Principle [2] maximizing the information contents of conscious experience (sic!) forces state function reduction to one particular $A_1, ..., A_n$ and one particular value combination consistent with C is found in each state function reduction. At the ensemble level one obtains probabilities for various outcomes and the most probable combination might represent the most plausible candidate for the murderer in quantum Poirot problem. Also in particle physics one can only speak about plausibility of the explanation and this leads to the endless n sigma talk. Note that it is absolutely essential that state function reduction occurs. Ironically, quantum problem solving causes dissipation at the level of ensemble but the ensemble

probabilities carry actually information! Second law of thermodynamics tells us that Nature is a pathological problem solver- just like my hero!

3. In TGD framework basic logical binary operations have a representation at the level of Boolean algebra realized in terms of fermionic oscillator operators. They have also space-time correlates realized topologically. $\wedge$ has a representation as the analog of three-vertex of Feynman graph for partonic 2-surfaces: partonic 2-surfaces are glued along the ends to form outgoing partonic 2-surface. $\vee$ has a representation as the analog of stringy trouser vertex in which partonic surfaces fuse together. Here TGD differs from string models in a profound manner.

To conclude, I am a Boolean dilettante and know practically nothing about what quantum computer theorists have done- in particular I do not know whether they have considered quantum inverse gages. My feeling is that only the gates with bits replaced with qubits are considered: very natural when one thinks in terms of Boolean logic. If this is really the case, quantal co-AND and co-OR having no classical counterparts would bring a totally new aspect to quantum computation in solving problems in which one cannot do without (quantum) Poirot and his little gray (quantum) brain cells.

3.5 How to understand transcendental numbers in terms of infinite integers?

Santeri Satama made in my blog a very interesting question about transcendental numbers. The reformulation of Santeri's question could be "How can one know that given number defined as a limit of rational number is genuinely algebraic or transcendental?". I answered to the question and since it inspired a long sequence of speculations during my morning walk on sands of Tullinniemi I decided to expand my hasty answer to a blog posting.

The basic outcome was the proposal that by bringing TGD based view about infinity based on infinite primes, integers, and rationals one could regard transcendental numbers as algebraic numbers by allowing genuinely infinite numbers in their definition.

1. In the definition of any transcendental as a limit of algebraic number (root of a polynomial and rational in special case) in which integer n approaches infinity one can replace n with any infinite integer. The transcendental would be an algebraic number in this generalized sense. Among other things this might allow polynomials with degree given by infinite integer if they have finite number of terms. Also mathematics would be generalized number theory, not only physics!

2. Each infinite integer would give a different variant of the transcendental: these variants would have different number theoretic anatomies but with respect to real norm they would be identical.

3. This would extend further the generalization of number concept obtained by allowing all infinite rationals which reduce to units in real sense and would further enrich the infinitely rich number theoretic anatomy of real point and also of space-time point. Space-time point would be the Platonia. One could call this number theoretic Brahman=Atman identity or algebraic holography.

3.5.1 How can one know that the real number is transcendental?

The difficulty of telling whether given real number defined as a limit of algebraic number boils down to the fact that there is no numerical method for telling whether this kind of number is rational, algebraic, or transcendental. This limitation of numerics would be also a restriction of cognition if p-adic view about it is correct. One can ask several questions. What about infinite-P p-adic numbers: if they make sense could it be possible to cognize also transcendentally? What can we conclude from the very fact that we cognize transcendentals? Transcendentality can be proven for some transcendentals such as π. How this is possible? What distinguishes "knowably transcendentals" like π and e from those, which are able to hide their real number theoretic identity?

1. Certainly for "knowably transcendentals" there must exist some process revealing their transcendental character. How π and e are proven to be transcendental? What in our mathematical cognition makes this possible? First of all one starts from the definitions of these numbers. e can be defined as the limit of the rational number $(1 + 1/n)^n$ and 2π could be defined as the

limit for the length of the circumference of a regular n-side polygon and is a limit of an algebraic number since Pythagoras law is involved in calculating the length of the side. The process of proving "knowable transcendentality" would be a demonstration that these numbers cannot be solutions of any polynomial equation.

2. Squaring of circle is not possible because π is transcendental. When I search Wikipedia for squaring of circle I find a link to Weierstrass theorem allowing to prove that π and e are transcendentals. In the formulation of Baker this theorem states the following: If α_1,...,α_n are distinct algebraic numbers then the numbers e^{α_1},...,e^{α_n} are linearly independent over algebraic numbers and therefore transcendentals. One says that the extension $Q(e^{\alpha_1}, ..., e^{\alpha_n})$ of rationals has transcendence degree n over Q. This is something extremely deep and unfortunately I do not know what is the gist of the proof. In any case the proof defines a procedure of demonstrating "knowable transcendentality" for these numbers. The number of these transcendentals is huge but countable and therefore vanishingly small as compared to the uncountable cardinality of all transcendentals.

3. This theorem allows to prove that π and e are transcendentals. Suppose on the contrary that π is algebraic number. Then also $i\pi$ would be algebraic and the previous theorem would imply that $e^{i\pi} = -1$ is transcendental. This is of course a contradiction. Theorem also implies that e is transcendental. But how do we know that $e^{i\pi} = -1$ holds true? Euler deduced this from the connection between exponential and trigonometric functions understood in terms of complex analysis and related number theory. Clearly, rational functions and exponential function and its inverse -logarithm- continued to complex plane are crucial for defining e and π and proving also $e^{i\pi} = -1$. Exponent function and logarithm appear everywhere in mathematics: in group theory for instance. All these considerations suggest that "knowably transcendental" is a very special mathematical property and deserves a careful analysis.

3.5.2 Exponentiation and formation of set of subsets as transcendence

What is so special in exponentiation? Why it sends algebraic numbers to "knowably transcendentals". One could try to understand this in terms of exponentiation which for natural numbers has also an interpretation in terms of power set just as product has interpretation in terms of Cartesian product.

1. In Cantor's approach to the notion of infinite ordinals exponentiation is involved besides sum and product. All three binary operations - sum, product, exponent are expressed set theoretically. Product and sum are "algebraic" operations. Exponentiation is "non-algebraic" binary operation defined in terms of power set (set of subsets). For m and n definining the cardinalities of sets X and Y, m^n defines the cardinality of the set Y^X defining the number of functions assigning to each point of Y a point of X. When X is two-element set (bits 0 and 1) the power set is just the set of all subsets of Y which bit 1 assigned to the subset and 0 with its complement. If X has more than two elements one can speak of decompositions of Y to subsets colored with different colors- one color for each point of X.

2. The formation of the power set (or of its analog for the number of colors larger than 2) means going to the next level of abstraction: considering instead of set the set of subsets or studying the set of functions from the set. In the case of Boolean algebras this means formation of statements about statements. This could be regarded as the set theoretic view about transcendence.

3. What is interesting that 2-adic integers would label the elements of the power set of integers (all possible subsets would be allowed, for finite subsets one would obtain just natural numbers) and p-adic numbers the elements in the set formed by coloring integers with p colors. One could thus say that p-adic numbers correspond naturally to the notion of cognition based on power sets and their finite field generalizations.

4. But can one naively transcend the set theoretic exponent function for natural numbers to that defined in complex plane? Could the "knowably transcendental" property of numbers like e and π reduce to the transcendence in this set theoretic sense? It is difficult to tell since this notion of power applies only to integers m, n rather than to a pair of transcendentals e, π. Concretization of $e^{i\pi}$ in terms of sets seems impossible: it is very difficult to imagine what sets with cardinality e and π could be.

3.5.3 Infinite primes and transcendence

TGD suggests also a different identification of transcendence not expressible as formation of a power set or its generalizations.

1. The notion of infinite primes replaces the set theoretic notion of infinity with purely number theoretic one.
   (a) The mathematical motivation could be the need to avoid problems like Russell's antinomy. In Cantorian world a given ordinal is identified as the ordered set of all ordinals smaller than it and the set of all ordinals would define an ordinal larger than every ordinal and at the same time member of all ordinals.
   (b) The physical motivation for infinite primes is that their construction corresponds to a repeated second quantization of an arithmetic supersymmetric quantum field theory such that the many particle states of the previous level become elementary particles of the new level. At the lowest level finite primes label fermionic and bosonic states. Besides free many-particle states also bound states are obtained and correspond at the first level of the hierarchy to genuinely algebraic roots of irreducible polynomials.
   (c) The allowance of infinite rationals which as real numbers reduce to real units implies that the points of real axes have infinitely rich number theoretical anatomy. Space-time point would become the Platonia. One could speak of number theoretic Brahman=Atman identity or algebraic holography. The great vision is that the World of Classical Worlds has a mathematical representation in terms of the number theoretical anatomy of space-time point.
2. Transcendence in purely number theoretic sense could mean a transition to a higher level in the hierarchy of infinite primes. The scale of new infinity defined as the product of all prime at the previous level of hierarchy would be infinitely larger than the previous one. Quantization would correspond to abstraction and transcendence.

This idea inspires some questions.

1. Could infinite integers allow the reduction of transcendentals to algebraic numbers when understood in general enough sense. Could real algebraic numbers be reduced to infinite rationals with finite real values (for complex algebraic numbers this is certainly not the case)? If so, then all real numbers would be rationals identified as ratios of possibly infinite integers and having finite value as real numbers? This turns out to be too strong a statement. The statement that all real numbers can be represented as finite or infinite algebraic numbers looks however sensible and would reduce mathematics to generalized number theory by reducing limiting procedure involved with the transition from rationals to reals to algebraic transcendence. This applies also to p-adic numbers.
2. p-Adic cognition for finite values of prime p does not explain why we have the notions of π and e and more generally, that of transcendental number. Could the replacement of finite-p p-adic number fields with infinite-P p-adic number fields allow us to understand our own mathematical cognition? Could the infinite-P p-adic number fields or at least integers and corresponding space-time sheets make possible mathematical cognition able to deduce analytic formulas in which transcendentals and transcendental functions appear making it possible to leave the extremely restricted realm of numerics and enter the realm of mathematics? Lie group theory would represent a basic example of this transcendental aspect of cognition. Maybe this framework might allow to understand why we can have the notion of transcendental number!

3.5.4 Identification of real transcendentals as infinite algebraic numbers with finite value as real numbers

The following observations suggests that it could be possible to reduce transcendentals to generalized algebraic numbers in the framework provided by infinite primes. This would mean that not only physics but also mathematics (or at least "physical mathematics") could be seen as generalized number theory.

1. In the definition of any transcendental as an $n \to \infty$ limit of algebraic number (root of a polynomial and rational in special case), one can replace n with any infinite integer if n appears as an argument of a function having well defined value at this limit. If n appears as the number of summands or factors of product, the replacement does not make sense. For instance, an algebraic number could be defined as a limit of Taylor series by solving the polynomial equation defining it. The replacement of the upper limit of the series with infinite integer does not however make sense. Only transcendentals (and possibly also some algebraic numbers) allowing a representation as $n \to \infty$ limit with n appearing as argument of expression involving a finite number of terms can have representation as infinite algebraic number. The rule would be simple.

   Transcendentals or algebraic numbers allowing an identification as infinite algebraic number must correspond to a term of a sequence with a fixed number of terms rather than sum of series or infinite product.

2. Each infinite integer gives a different variant of the transcendental: these variants would have different number theoretic anatomies but with respect to the real norm they would be identical.

3. The heuristic guess is that any genuine algebraic number has an expression as Taylor series obtained by writing the solution of the polynomial equation as Tarylor expansion. If so, algebraic numbers must be introduced in the standard manner and do not allow a representation as infinite rationals. Only transcendentals would allow a representation as infinite rationals or infinite algebraic numbers. The infinite variety of representation in terms of infinite integers would enormously expand the number theoretical anatomy of the real point. Do all transcendentals allow an expression containing a finite number of terms and N appearing as argument? Or is this the defining property of only "knowably transcendentals"?

One can consider some examples to illustrate the situation.

1. The transcendental π could be defined as $\pi_N = -iN(e^{i\pi/N} - 1)$, where $e^{i\pi/N}$ is N:th root of unity for infinite integer N and as a real number real unit. In real sense the limit however gives π. There are of course very many definitions of π as limits of algebraic numbers and each gives rise to infinite variety of number theoretic anatomies of π.

2. One can also consider the roots $exp(i2\pi n/N)$ of the algebraic equation $x^N = 1$ for infinite integer N. One might define the roots as limits of Taylor series for the exponent function but it does not make sense to define the limit when the cutoff for the Taylor series approaches some infinite integer. These roots would have similar multiplicative structure as finite roots of unity with p^n:th roots with p running over primes defining the generating roots. The presence of N^{th} roots of unity f or infinite N would further enrich the infinitely rich number theoretic anatomy of real point and therefore of space-time points.

3. There would be infinite variety of Neper numbers identified as $e_N = (1 + 1/N)^N$, N any infinite integer. Their number theoretic anatomies would be different but as real numbers they would be identical.

To conclude, the talk about infinite primes might sound weird in the ears of a layman but mathematicians do not lose their peace of mind when they here the word "infinity". The notion of infinity is relative. For instance, infinite integers are completely finite in p-adic sense. One can also imagine completely "real-worldish" realizations for infinite integers (say as states of repeatedly second quantized arithmetic quantum field theory and this realization might provide completely new insights about how to undestand bound states in ordinary QFT).

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