Dirac and Higher-Spin Equations of Negative Energies

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October 11, 2011

Abstract

It is easy to check that both algebraic equation $Det(\hat{p} - m) = 0$ and $Det(\hat{p} + m) = 0$ for 4-spinors $u-$ and $v-$ have solutions with $p_0 = \pm E_p = \pm\sqrt{\mathbf{p}^2 + m^2}$. The same is true for higher-spin equations. Meanwhile, every book considers the $p_0 = E_p$ only for both $u-$ and $v-$ spinors of the $(1/2, 0) \oplus (0, 1/2))$ representation, thus applying the Dirac-Feynman-Stueckelberg procedure for elimination of negative-energy solutions. Recent works of Ziino (and, independently, of several others) show that the Fock space can be doubled. We re-consider this possibility on the quantum field level for both $s = 1/2$ and higher spins particles.

The Dirac equation is:

$$[i\gamma^\mu \partial_\mu - m]\Psi(x) = 0\,. \tag{1}$$

At least, 3 methods of its derivation exist [1, 2, 3]:

- the Dirac one (the Hamiltonian should be linear in $\partial/\partial x^i$, and be compatible with $E_p^2 - \mathbf{p}^2 c^2 = m^2 c^4$);
- the Sakurai one (based on the equation $(E_p - \sigma \cdot \mathbf{p})(E_p + \sigma \cdot \mathbf{p})\phi = m^2\phi$);
- the Ryder one (the relation between 2-spinors at rest is $\phi_R(\mathbf{0}) = \pm\phi_L(\mathbf{0})$).

The γ^μ are the Clifford algebra matrices

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\,. \tag{2}$$

Usually, everybody uses the following definition of the field operator [4]:

$$\Psi(x) = \frac{1}{(2\pi)^3}\sum_h \int \frac{d^3\mathbf{p}}{2E_p}[u_h(\mathbf{p})a_h(\mathbf{p})e^{-ip\cdot x} + v_h(\mathbf{p})b_h^\dagger(\mathbf{p})]e^{+ip\cdot x}]\,, \tag{3}$$

as given *ab initio*. After introducing $\exp(\mp i p_\mu x^\mu)$ the 4-spinors ($u-$ and $v-$) satisfy the momentum-space equations: $(\hat{p}-m)u_h(p) = 0$ and $(\hat{p}+m)v_h(p) = 0$, respectively; the h is the polarization index. It is easy to prove from the characteristic equations $Det(\hat{p} \mp m) = (p_0^2 - \mathbf{p}^2 - m^2)^2 = 0$ that the solutions should satisfy the energy-momentum relation $p_0 = \pm E_p = \pm\sqrt{\mathbf{p}^2 + m^2}$.

The general scheme of construction of the field operator has been presented in [5]. In the case of the $(1/2,0)\oplus(0,1/2)$ representation we have:

$$\begin{aligned}
&\Psi(x) = \frac{1}{(2\pi)^3}\int d^4p\,\delta(p^2 - m^2)e^{-ip\cdot x}\Psi(p) = \\
&= \frac{1}{(2\pi)^3}\sum_h \int d^4p\,\delta(p_0^2 - E_p^2)e^{-ip\cdot x}u_h(p_0,\mathbf{p})a_h(p_0,\mathbf{p}) = \\
&= \frac{1}{(2\pi)^3}\int \frac{d^4p}{2E_p}[\delta(p_0 - E_p) + \delta(p_0 + E_p)][\theta(p_0) + \theta(-p_0)]e^{-ip\cdot x}\sum_h u_h(p)a_h(p) \\
&= \frac{1}{(2\pi)^3}\sum_h \int \frac{d^4p}{2E_p}[\delta(p_0 - E_p) + \delta(p_0 + E_p)]\left[\theta(p_0)u_h(p)a_h(p)e^{-ip\cdot x} + \theta(p_0)u_h(-p)a_h(-p)e^{+ip\cdot x}\right] \\
&= \frac{1}{(2\pi)^3}\sum_h \int \frac{d^3\mathbf{p}}{2E_p}\theta(p_0)\left[u_h(p)a_h(p)|_{p_0=E_p}e^{-i(E_pt-\mathbf{p}\cdot\mathbf{x})} + u_h(-p)a_h(-p)|_{p_0=E_p}e^{+i(E_pt-\mathbf{p}\cdot\mathbf{x})}\right]
\end{aligned} \tag{4}$$

During the calculations above we had to represent $1 = \theta(p_0) + \theta(-p_0)$ in order to get positive- and negative-frequency parts.[1] Moreover, during these calculations we

[1]See [6] for some discussion.

did not yet assumed, which equation this field operator (namely, the $u(p)$ spinor) satisfies, with negative- or positive- mass?

In general we should transform $u_h(-p)$ to the $v(p)$. The procedure is the following one [7]. In the Dirac case we should assume the following relation in the field operator:

$$\sum_h v_h(p) b_h^\dagger(p) = \sum_h u_h(-p) a_h(-p)\,. \tag{5}$$

We know that [3]

$$\begin{aligned} \bar{u}_\mu(p) u_\lambda(p) &= +m\delta_{\mu\lambda}\,, &(6)\\ \bar{u}_\mu(p) u_\lambda(-p) &= 0\,, &(7)\\ \bar{v}_\mu(p) v_\lambda(p) &= -m\delta_{\mu\lambda}\,, &(8)\\ \bar{v}_\mu(p) u_\lambda(p) &= 0\,, &(9)\end{aligned}$$

but we need $\Lambda_{\mu\lambda}(p) = \bar{v}_\mu(p) u_\lambda(-p)$. By direct calculations, we find

$$-m b_\mu^\dagger(p) = \sum_\lambda \Lambda_{\mu\lambda}(p) a_\lambda(-p)\,. \tag{10}$$

Hence, $\Lambda_{\mu\lambda} = -im(\sigma\cdot\mathbf{n})_{\mu\lambda}$ and

$$b_\mu^\dagger(p) = i\sum_\lambda (\sigma\cdot\mathbf{n})_{\mu\lambda} a_\lambda(-p)\,. \tag{11}$$

Multiplying (5) by $\bar{u}_\mu(-p)$ we obtain

$$a_\mu(-p) = -i\sum_\lambda (\sigma\cdot\mathbf{n})_{\mu\lambda} b_\lambda^\dagger(p)\,. \tag{12}$$

The equations are self-consistent.[2]

However, other ways of thinking are possible. First of all to mention, we have, in fact, $u_h(E_p,\mathbf{p})$ and $u_h(-E_p,\mathbf{p})$ originally, which satisfy the equations:[3]

$$\left[E_p(\pm\gamma^0) - \gamma\cdot\mathbf{p} - m\right] u_h(\pm E_p,\mathbf{p}) = 0\,. \tag{14}$$

Due to the properties $U^\dagger\gamma^0 U = -\gamma^0$, $U^\dagger\gamma^i U = +\gamma^i$ with the unitary matrix $U =$

[2]In the $(1,0)\oplus(0,1)$ representation the similar procedure leads to somewhat different situation:

$$a_\mu(p) = [1 - 2(\mathbf{S}\cdot\mathbf{n})^2]_{\mu\lambda} a_\lambda(-p)\,. \tag{13}$$

This signifies that in order to construct the Sankaranarayanan-Good field operator (which was used by Ahluwalia, Johnson and Goldman [Phys. Lett. B (1993)], it satisfies $[\gamma_{\mu\nu}\partial_\mu\partial_\nu - \frac{(i\partial/\partial t)}{E} m^2]\Psi(x) = 0$, we need additional postulates. For instance, one can try to construct the left- and the right-hand side of the field operator separately each other [6].

[3]Remember that, as before, we can always make the substitution $\mathbf{p}\rightarrow -\mathbf{p}$ in any of the integrands of (4).

$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \gamma^0\gamma^5$ in the Weyl basis,[4] we have

$$\Big[E_p\gamma^0 - \gamma\cdot\mathbf{p} - m\Big]\, U^\dagger u_h(-E_p,\mathbf{p}) = 0\,. \tag{15}$$

Thus, unless the unitary transformations do not change the physical content, we have that the negative-energy spinors $\gamma^5\gamma^0 u^-$ (see (15)) satisfy the accustomed "positive-energy" Dirac equation. Their explicite forms $\gamma^5\gamma^0 u^-$ are different from the textbook "positive-energy" Dirac spinors. They are the following ones:[5]

$$\tilde{u}(p) = \frac{N}{\sqrt{2m(-E_p+m)}} \begin{pmatrix} -p^+ + m \\ -p_r \\ p^- - m \\ -p_r \end{pmatrix}, \tag{16}$$

$$\tilde{\tilde{u}}(p) = \frac{N}{\sqrt{2m(-E_p+m)}} \begin{pmatrix} -p_l \\ -p^- + m \\ -p_l \\ p^+ - m \end{pmatrix}. \tag{17}$$

$E_p = \sqrt{\mathbf{p}^2 + m^2} > 0$, $p_0 = \pm E_p$, $p^\pm = E \pm p_z$, $p_{r,l} = p_x \pm i p_y$. Their normalization is to $-2N^2$.

What about the $\tilde{v}(p) = \gamma^0 u^-$ transformed with the γ_0 matrix? Are they equal to $v_h(p) = \gamma^5 u_h(p)$? The answer is NO. Obviously, they also do not have well-known forms of the usual $v-$ spinors in the Weyl basis differing by phase factor and in the sign at the mass term (!)

Next, one can prove that the matrix

$$P = e^{i\theta}\gamma^0 = e^{i\theta}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{18}$$

can be used in the parity operator as well as in the original Weyl basis. The parity-transformed function $\Psi'(t,-\mathbf{x}) = P\Psi(t,\mathbf{x})$ must satisfy

$$[i\gamma^\mu \partial'_\mu - m]\Psi'(t,-\mathbf{x}) = 0\,, \tag{19}$$

with $\partial'_\mu = (\partial/\partial t, -\nabla_i)$. This is possible when $P^{-1}\gamma^0 P = \gamma^0$ and $P^{-1}\gamma^i P = -\gamma^i$. The matrix (18) satisfies these requirements, as in the textbook case. However, if we would take the phase factor to be zero we obtain that while $u_h(p)$ have the eigenvalue $+1$, but

$$PR\tilde{u}(p) = PR\gamma^5\gamma^0 u(-E_p,\mathbf{p}) = -\tilde{u}(p)\,, \quad PR\tilde{\tilde{u}}(p) = PR\gamma^5\gamma^0 u(-E_p,\mathbf{p}) = -\tilde{\tilde{u}}(p)\,. \tag{20}$$

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[4] The properties of the $U-$ matrix are opposite to those of $P^\dagger\gamma^0 P = +\gamma^0$, $P^\dagger\gamma^i P = -\gamma^i$ with the usual $P = \gamma^0$, thus giving $\big[-E_p\gamma^0 + \gamma\cdot\mathbf{p} - m\big]\, P u_h(-E_p,\mathbf{p}) = -[\hat{p} + m]\,\tilde{v}_?(E_p,\mathbf{p}) = 0$. While, the relations of the spinors $v_h(E_p,\mathbf{p}) = \gamma_5 u_h(E_p,\mathbf{p})$ are well-known, it seems that the relations of the $v-$ spinors of the positive energy to $u-$ spinors of the negative energy are frequently forgotten, $\tilde{v}_?(E_p,\mathbf{p}) = \gamma^0 u_h(-E_p,\mathbf{p})$.

[5] We use tildes because we do not yet know their polarization properties.

Perhaps, one should choose the phase factor $\theta = \pi$. Thus, we again confirmed that the relative (particle-antiparticle) intrinsic parity has physical significance only.

Similar formulations have been presented by [8], and [9]. The group-theoretical basis for such doubling has been given in the papers by Gelfand, Tsetlin and Sokolik [10], who first presented the theory in the 2-dimensional representation of the inversion group in 1956 (later called as "the Bargmann-Wightman-Wigner-type quantum field theory" in 1993).

M. Markov wrote long ago *two* Dirac equations with the opposite signs at the mass term [8].

$$
\begin{aligned}
[i\gamma^\mu \partial_\mu - m]\,\Psi_1(x) &= 0\,, && (21)\\
[i\gamma^\mu \partial_\mu + m]\,\Psi_2(x) &= 0\,. && (22)
\end{aligned}
$$

In fact, he studied all properties of this relativistic quantum model (while he did not know yet the quantum field theory in 1937). Next, he added and subtracted these equations. What did he obtain?

$$
\begin{aligned}
i\gamma^\mu \partial_\mu \varphi(x) - m\chi(x) &= 0\,, && (23)\\
i\gamma^\mu \partial_\mu \chi(x) - m\varphi(x) &= 0\,, && (24)
\end{aligned}
$$

thus, φ and χ solutions can be presented as some superpositions of the Dirac 4-spinors $u-$ and $v-$. These equations, of course, can be identified with the equations for the Majorana-like $\lambda-$ and $\rho-$ we presented in ref. [11].[6]

$$
\begin{aligned}
i\gamma^\mu \partial_\mu \lambda^S(x) - m\rho^A(x) &= 0\,, && (25)\\
i\gamma^\mu \partial_\mu \rho^A(x) - m\lambda^S(x) &= 0\,, && (26)\\
i\gamma^\mu \partial_\mu \lambda^A(x) + m\rho^S(x) &= 0\,, && (27)\\
i\gamma^\mu \partial_\mu \rho^S(x) + m\lambda^A(x) &= 0\,. && (28)
\end{aligned}
$$

Neither of them can be regarded as the Dirac equation. However, they can be written in the 8-component form as follows:

$$
\begin{aligned}
[i\Gamma^\mu \partial_\mu - m]\,\Psi_{(+)}(x) &= 0\,, && (29)\\
[i\Gamma^\mu \partial_\mu + m]\,\Psi_{(-)}(x) &= 0\,, && (30)
\end{aligned}
$$

with

$$
\Psi_{(+)}(x) = \begin{pmatrix}\rho^A(x)\\ \lambda^S(x)\end{pmatrix}, \Psi_{(-)}(x) = \begin{pmatrix}\rho^S(x)\\ \lambda^A(x)\end{pmatrix}, \text{and}\, \Gamma^\mu = \begin{pmatrix}0 & \gamma^\mu\\ \gamma^\mu & 0\end{pmatrix} \tag{31}
$$

[6] Of course, the signs at the mass terms depend on, how do we associate the positive- or negative- frequency solutions with λ and ρ.

You may say that all this is just related to the basis rotation (unitary transformations). However, in the previous papers I explained: The connection with the Dirac spinors has been found [11, 13].[7] For instance,

$$\begin{pmatrix}\lambda^S_\uparrow(\mathbf{p})\\ \lambda^S_\downarrow(\mathbf{p})\\ \lambda^A_\uparrow(\mathbf{p})\\ \lambda^A_\downarrow(\mathbf{p})\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & i & -1 & i\\ -i & 1 & -i & -1\\ 1 & -i & -1 & -i\\ i & 1 & i & -1\end{pmatrix}\begin{pmatrix}u_{+1/2}(\mathbf{p})\\ u_{-1/2}(\mathbf{p})\\ v_{+1/2}(\mathbf{p})\\ v_{-1/2}(\mathbf{p})\end{pmatrix}. \tag{32}$$

Thus, we can see that the two 4-spinor systems are connected by the unitary transformations, and this represents itself the rotation of the spin-parity basis. However, the $\lambda-$ and $\rho-$ spinors describe the neutral particles, meanwhile $u-$ and $v-$ spinors describe the charged particles. Kirchbach [13] found the amplitudes for neutrinoless double beta decay $00\nu\beta$ in this scheme. It is obvious from (32) that there are some additional terms comparing with the standard formulation.

One can also re-write the above equations into the two-component form. Thus, one obtains the Feynman-Gell-Mann [12] equations. As Markov wrote himself, he was expecting "new physics" from these equations.

Barut and Ziino [9] proposed yet another model. They considered γ^5 operator as the operator of the charge conjugation. Thus, the charge-conjugated Dirac equation has the different sign comparing with the ordinary formulation:

$$[i\gamma^\mu \partial_\mu + m]\Psi^c_{BZ} = 0\,, \tag{33}$$

and the so-defined charge conjugation applies to the whole system, fermion+electromagnetic field, $e \to -e$ in the covariant derivative. The superpositions of the Ψ_{BZ} and Ψ^c_{BZ} also give us the "doubled Dirac equation", as the equations for $\lambda-$ and $\rho-$ spinors. The concept of the doubling of the Fock space has been developed in Ziino works (cf. [10, 14]) in the framework of the quantum field theory. In their case the charge conjugate states are simultaneously the eigenstates of the chirality. Next, it is interesting to note that for the Majorana-like field operators we have

$$\left[\nu^{ML}(x^\mu) + \mathcal{C}\nu^{ML\,\dagger}(x^\mu)\right]/2 = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{2E_p}\sum_\eta \left[\begin{pmatrix} i\Theta\phi^{*\,\eta}_L(p^\mu)\\ 0\end{pmatrix} a_\eta(p^\mu)e^{-ip\cdot x} + \right.$$
$$\left. + \begin{pmatrix}0\\ \phi^\eta_L(p^\mu)\end{pmatrix} a^\dagger_\eta(p^\mu)e^{ip\cdot x}\right], \tag{34}$$
$$\left[\nu^{ML}(x^\mu) - \mathcal{C}\nu^{ML\,\dagger}(x^\mu)\right]/2 = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{2E_p}\sum_\eta \left[\begin{pmatrix}0\\ \phi^\eta_L(p^\mu)\end{pmatrix} a_\eta(p^\mu)e^{-ip\cdot x} + \right.$$
$$\left. + \begin{pmatrix} -i\Theta\phi^{*\,\eta}_L(p^\mu)\\ 0\end{pmatrix} a^\dagger_\eta(p^\mu)e^{ip\cdot x}\right], \tag{35}$$

which, thus, naturally lead to the Ziino-Barut scheme of massive chiral fields, ref. [9].

[7] I also acknowledge personal communications from D. V. Ahluwalia on these matters.

Finally, I would like to mention that, in general, in the Weyl basis the $\gamma-$ matrices are *not* Hermitian, $\gamma^{\mu^\dagger} = \gamma^0\gamma^\mu\gamma^0$. The energy-momentum operator $i\partial_\mu$ is obviously Hermitian. So, the question, if the eigenvalues of the Dirac operator (the mass, in fact) would be always real, and the question of the complete system of the eigenvectors of the *non*-Hermitian operator deserve careful consideration [15]. Bogoliubov and Shirkov [5, p.55-56] used the scheme to construct the complete set of solutions of the relativistic equations, fixing the sign of $p_0 = +E_p$.

The conclusion is: the doubling of the Fock space and the corresponding solutions of the Dirac equation got additional mathematical bases in this talk presentation. Similar conclusion can be deduced for the higher-spin equations. I appreciate the discussions with participants of several recent Conferences.

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