Cover Letter:

Dear Editorial Board of the Israel Journal of Mathematics,

Contained below is the paper: *Analytic Gauge Functions, Invariances, and Modular Curves* that I, the author Thomas Evans, intend to submit for the consideration of publication in the Israel Journal of Mathematics. I am the sole author of this work, and any referenced works are contained as the last section of the aforementioned. The work is submitted in LaTeX format as a PDF file. Through the submission of this work in the form of the sending of an email, I certify that I agree to all stipulations contained in the copyright and all other clauses, located at http://www.ma.huji.ac.il/~ijmath/instructions.html. Thank you for your time and your reasonable consideration of my paper. Take care,

Sincerely,

Thomas Evans, New Jersey Mathematics Researchers, North New Jersey Division

Analytic Gauge Functions, Invariances, and Modular Curves

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Analytic Gauge Functions, Invariances, and Modular Curves Thomas Evans

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Abstract: The author introduces as an extension to the field of topology a sub-field Analytic Gauge Theory. The concepts of analytic numbers, the analytic field and analytic gauge functions are introduced and defined. The sub-field analytic gauge theory has an enormous application to the fields of topology, number theory, QFTs, amongst others, some of which are introduced. A rigorous examination and presentation will be contained in later works.

Note: This and all subsequent related papers are highly technical. Any reader should have a relatively advanced understanding of current mathematics, specifically the study of elliptic curves, topology, and the strictly mathematical applications of gauge theories.

Definition of terms: Gauge: By the term gauge the author means to represent either a) the normal definition or b) the representation of the quantity: $\ell = \alpha + z\beta$, where ℓ is a number in an analytic field, α and β are the sets of automorphisms of connective geometries, and z is the metric quaternion structure.

Analytic field: The field of analytic numbers.

Analytic number: A number $\ell = \alpha + z\beta$, where ℓ is a number in an analytic field, α and β are the sets of automorphisms of connective geometries, and z is the metric quaternion structure.

Gauge function: $\ell(s)$, a function whose range is in the analytic numbers is a gauge function.

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Introduction:

It is the underlying purpose of the author throughout this and subsequent related papers to consider the examination of conjectures such as the Birch-Swinnerton-Dyer conjecture, the Riemann Hypotheses, as well as a number of other misunderstood or unacknowledged phenomena. It is the author's hope that through such considerations, both autonomous and presented herein, that it may become evident that the introduction of fundamental, new practices is a necessity to any advancement in the directions of the aforementioned. This represents the first in a series of eight (8) papers regarding these materials. Throughout the remaining 7 the author presents, to a much greater degree of rigor, the basic theory of analytic gauge functions, associated phenomenology, and there from a solution to the (two) above conjectures. This paper facilitates an introduction to the theory of analytic gauges. In the first section the author presents a re-examination of the concepts of geometries of connections. Very briefly introduced are the basic concepts of analytic numbers, analytic fields, analytic gauge functions, etc.

1) Geometries of Connections

1.1) Quaternion line bundles. Three useful ways of thinking of a quaternion line bundle E over a space M.

-1 E is a 1-dimensional quaternion vector bundle.

-2 E is a 2-dimensional complex vector bundle with a C-antilinear bundle map $J:E\to E$ such that $J^2=-1$,

-3 E is a 4-dimensional real vector bundle with three real bundle maps I,J,K, where $I^2=J^2=K^2=-1$, and $IJ=-JI=K$.

Given the quaternions H the usual basis {1,i,j,k}, then in 3, the bundle maps I,J,K correspond to scalar multiplication by i,j,k, respectively. In 2, i acts by complex scalar multiplication and j acts by J. A quaternion line bundle can be presented in terms of transition functions for an open cover $(o_a)_{a\in A}$ of M. For each $\alpha,\beta\in A$ there is a map

(1.1) $$g_{\beta\alpha}:O_a\cap O_\beta\to H^x,$$

satisfying the computability or "cocycle" condition

(1.2) $$g_{\alpha\gamma}g_{\gamma\beta}g_{\beta\alpha}\equiv 1 in O_\alpha\cap O_\beta\cap O_\gamma.$$

The bundle is then constructed by gluing $O_\alpha xH$ to $O_\beta xH$ along $(O_\alpha\cap O_\beta)xH$ by the map $(x,v)\mapsto(x,v:g_{\beta\alpha}(x)^{-1})$. The multiplicative group of unit quaternions is the Lie group $S_{p_1}=[v\in H:v\overline{v}=1]$. The standard inner product $\langle v,w\rangle=\mathrm{Re}(v\overline{w})$ on H is $S_{p_1}-$invariant.

Definition 1.1.: A quaternion line bundle is called metric if it carries a bundle inner product which is $S_{p_1}-$invariant, i.e., for which scalar multiplication by i,j, and k is orthogonal in each fiber. Every quaternion line bundle over a manifold admits $S_{p_1}-$invariant metrics. The transition functions for a metric bundle E can be chosen in the form

(1.3) $$g_{\alpha\beta}:O_\alpha\cap O_\beta\to S_{p_1}.$$

The principal $S_{p_1}-$bundle, (obtained by gluing $O_a xS_{p_1}$ to $O_\beta xS_{p_1}$ as above) is just the unit sphere bundle in E. The (left) action of S_{p_1} is just scalar multiplication. This is analogous to metric complex line bundles, where the principal U_1-bundle is just the unit circle bundle. The basic example of a metric quaternion line bundle is the tautological line bundle E over quaternion projective space $P^n(H)$. Recall that $P^n(H)$ is the set of 1-dimensional subspaces of H^{n+1}. Then E= $\{(1,v)\in P^n(H)xH^{n+1}: r\in 1\}$ with projection $P:E\to P^n(H)$ given by P(1,r)=1. The fibers inherit an invariant inner product from the standard one on H^{n+1}. The unit sphere bundle of $E\to P^n(H)$ is just the Hopf maps $4n+3\to P^n(H)$. The bundle $E\to P^n(H)$ is 4n-classifying. Of interest here is the case n=1, since $P^1(H)\cong S^n$. The unit sphere bundle, or principal $S_{p_1}-$bundle, is just the Hopf map $S^T\to S^4$. It can also be viewed as follows. Write $S^n = Spin_s / Spin_n = Spin_s S_{p_1} x S_{P_1} = S_{p_2}/S_{p_1} x S_{p_1}$. Then $S^7 = S_{p_2}/S_{p_1}$ and the map $S^7\to S^n$ can be reexpressed as $Spin_s / S_{p_1}\to Spin_s / Spin_n$. From here one easily sees that E is just the (positive) spinor bundle of S^n. In terms of transition functions the bundle $E\to S^n$ is quite simple. Write $S^n = D_{+-}{}^n \cup D_{+-}{}^n$ where $D_{+-}{}^n$ are neighborhoods of the "upper and lower hemispheres". Then $D_{+-}{}^n \cup D_{+-}{}^n \cong S^3 x(-1,1)$ and the map

(1.4) $$g_{+-}: S^3 X(-1,1)\to S_{p_1} = S^3$$

is just the projection $g_{+-}(x,x)=X$. The set of equivalence classes $L_H(M)$ of quaternion line bundles over a 4-manifold is simple to describe.

Theorem 1.1: Let $L_H(M^n)\overset{m}{\rightarrow}(M^n,S^n)\overset{m}{\rightarrow}H^n(M^n;\square)=\square$ (where $[M^n,S^n]$ denotes the set of homotopy classes of maps from M^n to S^n). Given $f:M^n\to S^n$, the corresponding element in $L_H(M^n)$ is the induced bundle f^*E. The corresponding element in $H^n(M^n;\square)=\square$ is the degree of f. The composition of these maps is the Euler class of the bundle.

Proof.: The bundle $E\to S^n$ is a n-classifying and has Euler class -1. For a quaternion line bundle E over M^n, we have $\chi(E)=c_2(E)=-\frac{1}{2}P_1(E)$, where χ=Euler class, c_2=the 2nd Chern class, and p_1 the first Pontrjagin class. To see this recall that $=c_1^2-2c_2$ and $c_2=\chi$. Using the antiautomorphism J of E as a complex bundle, we see that E=$\bar{E}$ and so $c_1=0$.

Definition 1.2.: The instanton number of a quaternion line bundle E over a compact oriented n-manifold is - $\chi(E)$. The bundles of instanton number 1 on M can be obtained by pulling back E via a particularly simple map.

1.2) Connections. Let E be any smooth real vector bundle over a differentiable manifold M, and denote by $\Gamma(E)$ the space of smooth cross sections of E. We denote the space of smooth p-forms with values in E by $\Omega P(E)\equiv\Gamma(\Lambda^P\Gamma^*M\otimes E)$. At any point

$x \in M$, an element $\phi \in \Omega P(E)$ is just an alternating p-linear map $\phi x : T_x M \to E_x . \Omega^0(E) = \Gamma(E)$.

Definition 2.2.1) A connection on E is a linear map $\nabla : \Omega^0(E) \to \Omega^I(E)$ such that $\nabla(f\phi) = df \otimes \phi + f\nabla\phi$ for any $f \in C^\infty(M)$ and any $\phi \in \Omega^0(E)$. A connection is just a rule which allows us to take derivatives of smooth cross sections of E. Given a section $\phi \in \Omega^0(E)$, we have given, for any tangent vector v at any point of M, the covariant derivative $\nabla_{v\phi} \in E_x$ of ϕ in the direction of V at x. If V is a globally defined smooth vector field, then $\nabla_{v\phi}$ is again a smooth section of E, i.e., ∇_v gives a linear map $\nabla_v : \Omega^0(E)$. Property (2.3) can be rephrased by saying that $\nabla_v(f\phi) = (vf)\phi + f\nabla_{v\phi}$ for all smooth vector fields v and all ϕ, f as before. If E is trivialized, then $E \cong MxR^m$, the cross sections become R^m - valued functions, and we can define a connection by taking derivatives in the usual way. In our notation this connection $\nabla : \Omega^0(R^m)$ is just (m copies of) the de Rham exterior derivative d. (Thus, for a coordinate vector field $\partial / \partial x_i$ and an R^m - valued function ϕ, we have $\nabla\partial / \partial x_i \phi = d\phi(\partial / \partial x_i) = (\partial\phi / \partial x_i.)$ This connection depends on the choice of trivialization. Given two connections ∇^1 and ∇^2 on a bundle E, and given a smooth real-valued function f , the "convex combination" $\nabla = f\nabla^1 + (1-f)\nabla^2$ is again a connection on E. The connections $\nabla = d$ on local trivializations of E can be spliced together by a partition of unity to give a connection on all of E.

Given a connection ∇ on a bundle E, there are connections naturally induced on E*, $\otimes^s, E, \Lambda^p E, etc.$, in a canonical way. We shall only need the case $Hom(E,E)$ (i.e., $L : E \to E$ is a smooth bundle map), then $\nabla(L) \equiv [\nabla, L]$; that is, $\nabla(L)(\phi) = \nabla(L\phi) - L(\nabla\phi)$ for any $\phi \in \Omega^0(E)$. Given connections ∇ and ∇' on bundles E and E' over M, there are naturally defined connections $\nabla \oplus \nabla'$ on $E \oplus E'$ and $\nabla \otimes \nabla'$ on $E \otimes E'$. The first is obvious; the second is given by the rule $(\nabla \otimes \nabla')(\phi \otimes \phi') = (\nabla\phi) \otimes \phi' + \phi \otimes (\nabla'\phi')$. Of fundamental importance for any connection is

Definition 2.2.2.: The curvature of a connection ∇ is the 2-form $R\nabla \in \Omega^2(Hom(E,E))$, with values in Hom(E,E) defined for smooth vector fields V, W by the rule

(2.7) $$R_{v,w} = \nabla_v \nabla_w - \nabla_w \nabla_v - \nabla_{[v,w]} .$$

Note that (2.7) actually defines a second-order differential operator on E. However, we can easily compute that, for $f \in C^\infty(M)$, $\phi \in \Omega^0(E)$, $R_{v,w}(f\phi) = fR_{v,w}(\phi)$, so $R_{v,w}$ is in fact zero-order. , i.e., a section of Hom(E,E). A similar easy exercise shows $R_{fv,w} = fR_{v,w} = R_{v,fw}$, so R is tensorial in v and w as claimed. The curvature clearly measures the lack of "commutativity" of second covariant derivatives.

Given a connection on E, the map $\nabla : \Omega^0(E) \to \Omega^I(E)$ can be extended to a general de Rham sequence $\Omega^0(E) \overset{d^\nabla - \nabla}{\to} \Omega^I(E) \overset{d^\nabla}{\to} \Omega^2(E) \overset{d\nabla}{\to} ...$, where d^∇ is defined on $\phi \in \Omega^p(E)$ by setting

$$
\begin{aligned}
d^\nabla \phi(V_0,...,V_p) &= \sum_{u=0}^{p} (-1)^J \nabla_{V_J} (\phi(V_0,...,V_J,...,V_p)) \\
&+ \sum_{i<j} (-1)^{itj} \phi([v_i, v_j]), v_0,...,v_i,...,V_J,...,V_p)
\end{aligned}
.
$$

It is not generally true $d^\nabla o d^\nabla = 0$. In fact, $d^\nabla o d^\nabla = R^\nabla$ on $\Omega^0(E)$. However, considering $R^\nabla \in \Omega^2(Hom(E,E,))$, we always have

$$d^\nabla R^\nabla = a .$$

This is called the Bianchi identity for R^∇. (Note that here we are using the induced connection (2.4) on $Hom(E,E)$.) The proof of (2.10) is straightforward and reduces essentially to the Jacobi identity.

1.3) We now consider metrics on both E and M.

Definition 1.3.1) Suppose E carries a metric, i.e., an inner product $\langle .,. \rangle$ smoothly defined in the fibers. A connection ∇ on E is said to be riemannian if, for all sections $\phi_1 \phi_2 \in \Omega^0(E)$,

$$d \langle \phi_1, \phi_2 \rangle = \langle \nabla \phi_1 \phi_2 \rangle + \langle \phi_1, \nabla \phi_2 \rangle .$$

This simply means that the covariant derivative of the inner product, as a section $E^* \otimes E^*$, is identically zero. Convex combinations of riemannian connections are riemannian connections, s a straightforward partition of unity arguments show that Riemannian connections always exist.

Given a Riemannian metric on M, there is a unique connection ∇ on T_m such that $\nabla_{vw} - \nabla_{wv} \equiv [v,w]$ for all vector fields v, w. This is called the canonical Riemannian connection. We shall always use this connection on M. Given metrics on M and E, there are naturally induced metrics on all the associated bundles, such as $\Lambda^P T^* M \otimes E$. Riemannian connections on E and M give riemannian connections in these bundles. In particular, the pointwise inner product gives an L_2-norm in $\Omega^P(E)$ by setting $(\Psi, \Psi) = \int_m \langle \Phi, \Psi \rangle$. The maps d^∇ given in (2.8) then have formal adjoints

$$\Omega^0(E) \overset{d^\nabla}{\leftarrow} \Omega^1(E) \overset{d^\nabla}{\leftarrow} \Omega^2(E) \overset{d^\nabla}{\leftarrow}$$

with the property that

$$(d^\nabla \Phi, \Psi) = (\Psi, \delta^\nabla \Psi)$$

for all $\Phi \in \Omega^P(E), \Psi \in \Omega^{P+1}(E)$ with compact support. Using the riemannian connection ∇ on $\Lambda^P T^* M \otimes E$, we can write these operations as

$$(d^{\nabla}\Phi)(v_0,...,v_p)=\sum_{j=0}^{p}(-1)^j(\nabla_v,\Phi)(v_0,...,\hat{v}_j,...,v_A),$$

$$(\delta^{\nabla}\Phi)(v_2,...,v_p)=\sum_{j=1}^{n}(\nabla_{e_j}\Phi)(e_j,v_2,...,v_p)_g$$

where $\{e_1,...,e_n\}$ is any orthonormal basis of T_xM at the point x in question. It follows from property (3.2) that the curvature of a riemannian connection ∇ satisfies

$$\left\langle R_{v,w}{}^{\nabla}\phi_1,\phi_2\right\rangle+\left\langle \phi_1,R^{\nabla}{}_{v,w}\phi_2\right\rangle=0.$$

for $\phi_1,\phi_2\in E$. Hence, R^{∇} has its value in the subbundle $\wp_D(E)(Hom(E,E))$ of skew-symmetric endomorphisms of E.

1.4) $S_{p_1}-$connections.: Suppose now that E is a metric quaternion bundle over a manifold M.

Definition 1.4.1) An $S_{p_1}-$connection on E is a riemannian connection ∇ that is h-linear, i.e., it commutes with scalar multiplication by quaternions. In terms of the real picture of E, this means that $\nabla(I)=[\nabla,I]=0,\nabla(J)=[\nabla,J]=0,\text{and}\nabla(k)=[\nabla,k]=0$. Since ∇ is H-linear, so is R^{∇}. Therefore, R^{∇} has values in $Hom_H(E,E)$. As we previously pointed out, $R_{v,w}{}^{\nabla}$ is also skew-symmetric. This leads us to consider the two very important bundles:

$$\wp_E\equiv\{L\in Hom_H(E,E):L'=-L\},$$
$$G_E\equiv\{L\in Hom_H(E,E):L'=L^{-1}\}.$$

Under the bracket $[L_1,L_2]=L_1\circ L_2-L_2\circ L_1$, $\wp_E$ becomes a bundle of Lie algebras, each isomorphic to $\wp_{p_1}$. Under composition, G_E becomes a bundle of Lie groups, each isomorphic to S_{p_1}. There is a pointwise exponential map

$$\exp:\wp_E\to G_E$$

given by the usual infinite series in each fiber. G_E is just the bundle of automorphisms of E, i.e., the bundle of H-linear bundle isometries. Thus $\wp_E$ is the infinitesimal automorphism bundle. The bundle $\wp_E$ is preserved by the covariant differentiation on $Hom(E,E)$ induced by any S_{p_1}-connection on E. Thus, if M is riemannian, any S_{p_1}-connection ∇ induces maps

$$\Omega^0(\wp_E)\underset{\delta^{\nabla}}{\overset{d^{\nabla}}{\rightleftarrows}}\Omega^1(\wp_E)\underset{\delta^{\nabla}}{\overset{d^{\nabla}}{\rightleftarrows}}\Omega^2(\wp_E)\underset{\delta^{\nabla}}{\overset{d^{\nabla}}{\rightleftarrows}}\cdots.$$

We have $R^{\nabla}\in\Omega^2(\wp_E)$ and the Bianchi identity says that $d^{\nabla}R^{\nabla}=0$.The geometry of the space of S_{p_1}-connections is based on the fundamental sequence 94.5).

1.5) Change of connections. Fix a metric quaternion line bundle E over a manifold M, and let $\wp$ denote the space of all S_{p_1}-connections on E. It is elementary to see that $\ell\neq 0$.

Given two connections $\nabla',\nabla_{\dagger}\in\ell$, we consider the difference $A=\nabla'-\nabla$. It follows from (2.3) that $A(f\phi)=fA(\phi)$ for any $f\in C^{\infty}(M)$;hence, A is a zero-order operator; i.e., it is tensorial. Given a tangent vector $V \in T_x M$, the map $Av : E_x \to E_x$ is H-linear and skew-symmetric (by(4.2)). Hence, $Av \in (\wp_E)$, the operator $\nabla' \equiv \nabla + A$ again satisfies the axioms for an S_{p_1}-connection. Hence, we have

Proposition 1.5.1) The space ℓ of S_{p_1}-connections E is an affine space having $\Omega^1(\wp_E)$ as the vector group of translations. This, at any connection $\nabla \in \ell$ we have natural identifications

$$\Omega^1(\wp_E) \equiv T_\nabla \ell \equiv \ell .$$

If we fix ∇, then any other connection ∇' is uniquely expressed as $\nabla' = \nabla + A$. The curvature of $R^{\nabla'} = R^\nabla + d^\nabla A + [A, A]$, where [A,A] is the $\wp_E$-valued 2-form defined by setting $[A, A]_{v,w} = [A_v, A_w]$. To prove this we choose $v, w \in T_x M$ and extend them to local fields so that $[v, w] = 0$. Then

$$\begin{aligned} R_{v,w}{}^{\nabla'} &= (\nabla_v + A_v)(\nabla_w + A_w) - (\nabla_w + A_w)(\nabla_v + A_v) \\ &= R_{v,w}{}^{\nabla} + [\nabla_v, A_w] - [\nabla_w, A_v] + [A_v, A_w] \\ &= R^{\nabla}{}_{v,w} + \nabla_v(A_w) - \nabla_w(A_v) + [A, A]_{v,w} \\ &= R^{\nabla}_{v,w} + (dA)_{v,w} + [A, A]_{v,w} \end{aligned} ,$$

as claimed.

Example 5.4. Let $M = S^n$. Fix a pair of antipodal points $\overset{+}{-}P \in S^n$, and let $u\overset{+}{-} = S^n - \{\overset{+}{-}P\}$. We can choose coordinate charts $x\overset{+}{-} : U\overset{+}{-} \to \mathbb{R}^n$ with coordinate transformation $\psi : R^n - (0) \to R^n - \{0\}$ given by $\psi(x) = 1/x = \overline{x}/|x|^2$,

where we identify $\mathbb{R}^n \cong H$. The bundle $E \to S^n$ of instanton number 1 can be presented by two trivializations, $E/U \cong R^n x H$ and $e/\underline{u} \cong R^n x H$ joined together by the map

$$\psi : (R^n - \{0\})xH \to (R^n - \{0\}xH$$

given by

$$\psi(x, v) = (1/x, v \cdot x/|x|.) .$$

The transition function $g_{+-}(x)(v) = v \cdot x/|x|$ is H-linear under left scalar multiplication and orthogonal metric $\langle u, v \rangle \equiv R_c(u \cdot \nabla)$. Under the trivialization the sections of E over U_+ become simply smooth functions $f : R^n \to H$. Each such function can be written as

$$f(x) = f_1(x) + if_2(x) + if_s(x) + kf_n(x)$$

where $f_1, ..., f_n$ are real-valued. A connection ∇ on E can be expressed over U_L as $\nabla = d + A$, where $A_a : R^n \to \mathrm{Im}(H)$ is a smooth function for each a. This we write

$$\nabla \frac{\partial}{\partial x_a} f = \frac{\partial f}{\partial x_a} + f \cdot A_a \text{ for } 1 \le a \le n$$

It is convenient to decompose the patching transformation into two parts: the coordinate transformation $y = 1/x$, and the bundle transformation given by right multiplication by the function $u = x/|x| = \overline{y}/|\underline{y}|$. Under the transformation a section f becomes $\overline{f}$, where

$$\overline{f} = f \cdot u .$$

Thus, $\overline{f}(x) = f(x)(x/|x|)$ in x-coordinates and $\overline{f}(y) = f(1/y)(\overline{y}/|y|)$ in y-coordinates. Under the transformation a connection 1-form becomes

$$\overline{A} = -\overline{u}du + uAu .$$

Again this can be expressed in either x-or y-coordinates. Note that, under the coordinate change $y = 1/x$, A transforms like a 1-form. One must express the $dt's$ in terms of the $dy^{\beta}'s$.

1.6.1)Automorphisms(the gauge group). By the gauge group of a metric quaternion line bundle E, we mean the group ℓ of smooth bundle automorphisms preserving the metric and quaternion structure. This is exactly the space of smooth sections of G_E, i.e.,

$$\ell \equiv \Gamma(G_B).$$

There is an associated gauge algebra defined by

$$\Im \equiv \Gamma(\wp_E),$$

and, from (4.4), an exponential map

$$\exp : \wp \to \Im .$$

Under pointwise bracket, $\Im$ is a Lie algebra. The group $\Im$ plays a role in the study of connections similar to the role played by the diffeomorphism group in the study of manifolds. There is a natural action of $\Im$ on the space of S_{p_1}-connections. For $g \in \Im$ and $\nabla \in \Im$, the transformed connection is

$$\boxed{\nabla^{s} \equiv g \circ \nabla \circ g^{-1}} .$$

This means that $\nabla_{\xi}\phi = g(\nabla v(g^{-1}\phi))$ .It follows immediately that

$$\boxed{R^{\nabla^1} = g \circ R_{\circ g^{-1}}^{\nabla}} .$$

Metrics on E and M induce metrics on $\Lambda^{P} T^{*} M \otimes E$, which can be written as

$$\langle \Phi, \Psi \rangle = \sum_{I_1 < \ldots < I_p} traceR(\Phi'_{e_1 \ldots \in_{1_p}} \circ \Psi_{e_1 \ldots e_{1_p}}),$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal basis of $T_x M$ at the point in question. Since g is pointwise orthogonal, it is clear that

$$\|R^{\nabla t}\| = \|R^{\nabla}\| \text{ for all } g \in \Im .$$

Given $g \in \Im$, the difference of the connections $\nabla^{n} - \nabla$ can clearly be written as

$$\nabla^{s} - \nabla = g(\nabla g^{-1} 0 = -(\nabla g) g^{-1},$$

where the last equality follows by differentiating the identity $gg^{-1} \equiv 1$. More generally, if we express another connection $\overline{\nabla}$ as $\overline{\nabla} = \nabla + A$, then

$$\overline{\nabla^{s}} = g(\nabla + A)g^{-1} = \nabla + g(\nabla g^{-1}) + gAg^{-1},$$

hence, writing $\overline{\nabla} s = \nabla + As$, we have

$$As = -(\nabla g)g^{-1} + gAg^{-1}.$$

The corresponding curvature formula follows directly from (6.5) and (5.3). Now recall that at any connection there is a canonical identification $T_{\nabla}\ell \cong \Omega^1(\wp_E)$. In this picture we now examine the tangent space to the orbit $\ell(\nabla)$ of ∇ under the gauge group. The tangent space $T_z\ell$ is just the gauge algebra $\mathsf{A} = \Omega^0(\wp_E)$. Given $y \in \mathsf{A}$, we consider the curve $g_E = e / y$ and its corresponding curve of connections $\nabla \pm \nabla^{g_t} = \nabla - (\nabla g_t)g_t^{-1}$ (cf.(6.7)). Taking derivatives (fiber by fiber) gives

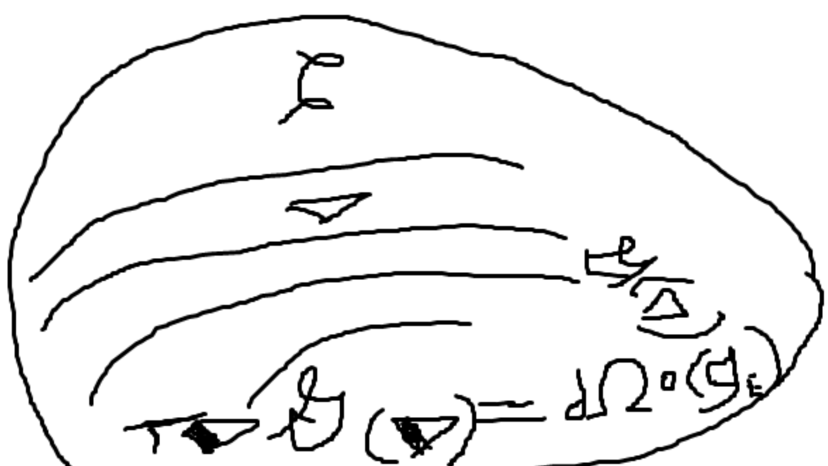
$$\frac{d}{dt}\nabla \eta t = 0 = -\nabla(\gamma) = -d^{\nabla\gamma},$$

so we have

Proposition 1.6.1) Under the canonical identifications $T_1\ell = \Omega^0(\wp_E)$ and $T_{\nabla} = \Omega^1(\wp_E)$, the differential at 1 of the action of ℓ on ∇ is just the map

$$\begin{array}{ccc} \Omega^0(\wp_E) & \xrightarrow{-d^{\nabla}} & \Omega^0(\wp_E) \\ \cong & & \cong \\ T_1\Im & \rightarrow & T_{\nabla}\Im \end{array}.$$

In particular, the subspace $\mathrm{Im}(d^{\nabla})(\Omega^1(\wp_E))$ represents the tangent space to the orbit $\Im(\nabla)$ of the gauge group on ∇.



1.7)Sobolev completions. Let E be any riemannian bundle with connection over a compact riemannian manifold M. To each positive integer $k \geq 0$ we can define a Sobolev k-form $\|.\|_A$ on $\Omega^0(E)$ by setting

$$\|\phi\|_A^2 = \int_M \{\|\phi\|^2 + \|\nabla\phi\|^2 + \ldots + \left\|\nabla\nabla\ldots\overset{k}{\nabla}\phi\right\|^2\}.$$

Making a different choice of metrics and connections gives an equivalent norm. The completion of $\Omega^0(E)$ in this norm is a Hilbert space that we denote $\Omega^0_k(E)$. The space $\Omega^P(E)$ is just sections of the bundle $\Lambda^P T^*M \otimes E$, and its completion in the Sobolev k-norm is denoted $\Omega^p_k(E)$. Each map in the sequence

$$\Omega^0_k(E)\underrightarrow{d^\nabla}\Omega^1_{k-1}(E)\underrightarrow{d^\nabla}\Omega^2_{k-2}(E)\underrightarrow{d^\nabla}\dots.$$

2)Analytic Gauge Functions.

2.1)Analytic numbers: An analytic number is defined

$$\ell = \alpha + z\beta,$$

where α and β

$$\alpha = \mathrm{Im}\begin{pmatrix} g_a, & g_b, & g_c, & g_i \\ g_a, & g_b, & g_c,\dots, g_i \\ g_a, & g_b, & g_c, & g_i \end{pmatrix}\otimes \log(G)$$

$$\beta = \mathrm{Im}\begin{pmatrix} g_a, & g_b, & g_c, & g_i \\ g_a, & g_b, & g_c,\dots, g_i \\ g_a, & g_b, & g_c, & g_i \end{pmatrix}\otimes \log(G)$$

and z is the metric quaternion structure.

2.1.1)Analytic field: We define the analytic field as the field of analytic numbers.

2.2)Analytic gauge functions. We define a gauge function $\ell(s)$, a function whose range is in the analytic numbers,

$$\ell(s) := \frac{\Theta^{\square}}{(\alpha,\beta)^{\in\partial_{n+1}}}(1(r-1))^{2n+1(d(sds^2))}\Theta^{\square}$$
$$= \frac{\Theta^{\square}}{B}(1(r-1))^{2n+1(d(sds^2))}\Theta^{\square},$$

where $B = (\alpha,\beta)^{\in\partial_{n+1}}$.

Proposition 2.2.1) 1) There is a system of gauge functions, denoted a gauge system,

2)Any gauge system is always constrained by an Evans formulation,

3)This is called a gauge invariance,

4)For any gauge invariance $\wp$ for a gauge system G we can obtain from solutions of G lesser invariances,

5)We can obtain results for solutions to a function $\ell(s)$ defined in G,

6)For any gauge invariance $\wp$ for a gauge system G defining a function $\ell(s)$ with solutions containing some rational point on a real algebraic curve C defined on a riemannian surface M we can establish an abelian gauge invariance defining the principal bundle connections over C,

7)We can define an L-series from finite abelian groups for any rational point of a non-singular projective model for which there exists a solution in the form of a singularity $L(C,1,...,s)$,for which all nontrivial finitely generated solutions are defined sequentially,

8)For an holomorphic continuation of L(C,s) into the complex plane there is an input for which all solutions lie sequentially at a representation of a discrete quantity.

Proof:

1)There is a system of gauge functions, denoted a gauge system,

2)Any gauge system is always constrained by an Evans formulation.

2.)Theorem 2.2.1)We can define a surface ϕ_m $\phi_m = ([w(r)](r-sds^2)^{(r-1)^2})$,

Theorem 2.2.2)We can define a smooth phase space function G on $\phi_m = 0$,

Theorem 2.2.3) If the smooth phase space function vanishes on ϕ_m , $G = g^m \phi_m$ for some functions g^m .

Theorem 2.2.4) If $\lambda_n \partial_{q^n} + \mu^n \partial p_n = 0$ for arbitrary variations $\partial_{q^n} \partial p_n$ tangent to the constraint surface, then

$$\lambda_n = u^m \frac{\partial \phi_m}{\partial_{q^n}},$$
$$\mu^m = u^m \frac{\partial \phi_m}{\partial p_n},$$

for some u^m . The equalities here are the equalities on the surface.

The proofs of theorems 2.2.1)-2.2.3) are based on the fact that one can locally choose the independent constraint functions ϕ_m ' as first coordinates of a regular coordinate system $(y_m ', x_\alpha)$, with $y_m ' \equiv \phi_m '$. In these coordinates one has, since $G(0,x) = 0$,

$$G(y,x) = \int_0^1 \frac{d}{dt} G(ty,x)dt$$
$$= y_m ' \int_0^1 G,_{m'}(ty,x)dt ,$$

so that

$$G = g^m \phi_m ,$$

with $g^{m'} = \int_0^1 G,_{m'}(ty,x)dt$ and $g^{\bar{m}} = 0$ .The proof of the second theorem is based on the fact that the constraint surface is of dimension $2N - M'$, and therefore the tangent variations $\partial_{q^n} \partial p_n$ at a point form a 2(N-M')-dimensional vector space. Hence, there exist exact by M' independent solutions of $\lambda_n \partial_{q^n} + \mu^n \partial p_n = 0$. By the regularity assumptions, the M' gradients $(\partial\phi_{m'} / \partial_{q^n}, \partial\phi_{m'} / \partial p_n)$ of the independent constraints are linearly independent.

We define relations, the first of which enables us to recover the action in from the knowledge of the related action p_n and of extra parameters u^m . No two different sets of u's can yield the same actions. The u's are expressed as functions of the coordinates and point actions, in principle, as functions of the coordinates and the actions by solving the equations

$$\dot{q}^n = \frac{\partial H}{\partial p_n}(q, p(q, \dot{p})) + u^m(q, \dot{q}) \frac{\partial \phi_m}{\partial p_n}(q, p(q, \dot{q})) .$$

We define the transformation from $(q, \dot{q})$ – space to the surface $\phi_m(q, p) = 0$ of (q, p, v) – space by

$$q^n = q^n,$$

$$\{P_n = \frac{\partial L}{\partial_{q^n}}(q, \dot{q}) .$$

$$u^m = u^m(q, \dot{q})$$

This transformation between spaces of the same directionality 2N is invertible, since one has

$$q^n = q^n,$$

$$\{ \ \dot{q}^n = \frac{\partial H}{\partial p_n} \quad .$$

$$\phi_m(q, p) = 0$$

These relations construct the Evans formulation

$$\dot{q}_n = \frac{\partial H}{\partial p_n} + u^m \frac{\partial \phi_m}{\partial p_n},$$

$$\dot{p}_n = \frac{\partial H}{\partial_{q^n}} - u^m \frac{\partial \phi_m}{\partial_{q^n}},,$$

$$\phi_m(q, p) = 0$$

from

$$\partial \int_{t_1}^{t_2} (\dot{q}^n p_n - H - u^m \phi_m) = 0 ,$$

for arbitrary variations $\partial_{q^n}, \partial p_n, \partial u_m$ subject to the conditions

$$\phi_m = d, \partial \phi_m = 0 .$$

4) For any gauge invariance $\wp$ for a gauge system G we can obtain from solutions of G lesser invariances,

5)We can obtain results for solutions to a function $\ell(s)$ defined in G ,

5) $\ell(s) := \frac{\Theta^{\square}}{B}(1(r-1))^{2n+1(d(sds^2))}\Theta^{\square}$,

where r, n, s, and d are all analytic numbers

$$\begin{aligned} r &= \alpha + z\beta \\ n &= \alpha + z\beta \\ s &= \alpha + z\beta \\ d &= \alpha + z\beta \end{aligned}.$$

B is the analytic field

$$B = (\alpha, \beta)^{\in \partial_{n+1}},$$

and $\Theta^{\square}$ is the action field. We define

$$r,n,s,d,= \begin{matrix} \operatorname{Im}\begin{pmatrix} g_a, & g_b, & g_c, & g_i \\ g_a, & g_b, & g_c,\ldots, g_i \\ g_a, & g_b, & g_c, & g_i \end{pmatrix} \otimes \log(G) \\ +z\operatorname{Im}\begin{pmatrix} g_a, & g_b, & g_c, & g_i \\ g_a, & g_b, & g_c,\ldots, g_i \\ g_a, & g_b, & g_c, & g_i \end{pmatrix} \otimes \log(G) \end{matrix}.$$

We define z: Let (M, h) be a compact, connected Riemannian 4-manifold with covariantly constant almost complex structures $\{I_1, I_2, I_3\}$ satisfying $I_1 I_2 = I_2 I_1 = I_3$. This is a covariantly constant quaternion structure. Each almost complex structure I, given on the base space M defines a 2-form θ_1 on M which is covariantly constant $j\theta_\xi(x,y) = h(I,x,y), i = 1,2,3$. The manifold M carries the canonical orientation compatible with the quaternion structure. The base metric h together with this orientation gives the Hodge operator*; $\Lambda^2(M) \to \Lambda^2(M)$, which is involutive. So the bundle $\Lambda^2 = \Lambda^2(M)$ split into $\Lambda^2 = \Lambda^+ + \Lambda^-$ ($\Lambda^+ and \Lambda^{-1}$ are subbundles of self-dual 2-forms and of anti-self-dual 2-forms, respectively). Then over the manifold M Λ^+ becomes trivial. We have the decomposition

$$\Lambda^+ = R\theta_1 \oplus R\theta_2 \oplus R\theta_3.$$

Let P be a smooth principal bundle over the manifold M with a compact simple Lie group G. Fix a positive number l>4 in order that analysis on gauge fields works well and denote by $A = \mathsf{A}_p$ the set of all $L\underset{T}{2}$ connections on P. The set A is an affine space with nodal vector space $\Omega^1(g_p)_t$, the space of $L\underset{T}{2}$ 1-forms over M taking values in the adjoint bundle $g_p = PxA_\alpha g$ (g is the Lie algebra of G). Then $\mathsf{A} = A + \Omega^1(g_p)_t$ for some fixed smooth connection A. The subset A_{1T} in A consisting of irreducible connections is dense and open relative to the $L\underset{T}{2}$-topology. A connection is said to be irreducible if the centralizer of its holonomy group in G reduces to the center Z_a of G. The group $G = G_p$ of $L\underset{T+1}{2}$ gauge transformations of P acts on A smoothly

as $g(A) = g^{-1}dg + g\cdot^{-1} A \cdot g_0$. $G/_{1_G}$ acts freely on A_{1T} so that by the slice argument A_{1T} has a fibration over the orbit space $B_{1T} = \mathsf{A}_{2T} / cG/_{1_G}$, with fiber $G/_{1G}$. The Pontrjagin number $p_1 = p_1(g_p \otimes C)[M]$ is calculated for each simple Lie group as follows; $\begin{matrix} p_1 = 4nk, G = SU(n); 16k, G = G_1; 36k, G = F_2; 48k, G = E_6; \\ 72k, G = E_T; 120k, G = E_8 \end{matrix}$,
where k is an integer called the index of the bundle. On the moduli space M of ASD connections a Riemannian metric is defined by a gauge invariant L_2 inner product. We define the metric quaternion structure as the above covariantly constant quaternion structure and the associated results and the Riemannian metric on the moduli space M .We define a natural Riemannian metric on the moduli space as follows. Since A is affine, the tangent space $T_A\mathsf{A}_{1T}$ is isomorphic to $\Omega^1(g_p)_t$. On this tangent space an inner product is well-defined by

$$\begin{aligned} \langle \beta, \gamma \rangle &= \int_m (-tr \otimes n)(\beta, \gamma) dv = \int_m (-tr)(\beta \Lambda * \gamma) \\ \beta, \gamma &\in \Omega^1(g_p), \qquad \text{for } G = SU(n) \end{aligned}.$$

For general G we replace -tr by some adjoint invariant inner product. This inner product is gauge invariant. Hence the inner product descends to $B_{1T} = \mathsf{A}_{1T} / (G/_{1G})$, the orbit space of irreducible connections on P and its restriction on the generic part of M *gen* of M lying smoothly in B_{1T} provides a Riemannian metric there.

6,7 and 8 will be addressed in future works.

3) Conclusions: We can establish an analytic gauge theory and define basic concepts analytic fields, analytic numbers, and analytic gauge functions. Analytic gauge functions have solutions that allow us to obtain lesser invariances. From this paper in future works the author will address:

1)Analytic Scale and Tate-Shafarevich Invariances,

2)Generalized form of the Evans Holomorphicity Conjecture for all Modular Curves,

3) Analytic Gauge Functions, Scale, Tate-Shafarevich Invariances and the Integrality of the Tate-Shafarevich group for a complete L-series

$$C^* = |\mathrm{X}_c| R_\infty W_\infty \prod_{p|-2\nabla} w_p / |C(\square)^{tors}|^2,$$

4)Special form of the Evans Holomorphicity Conjecture for Specific Number-Theoretic Applications

5)Analytic Gauge Functions and Evans Generators

6)Conclusions regarding Analytic Gauge Theory and a Solution to the Birch-Swinnerton-Dyer Conjecture,

7)Conclusions regarding Analytic Gauge Theory and its' Number-Theoretic Applications.

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