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Abstract: Presented is a new determination of conditions proving the Riemann Hypothesis of any global L-function, drawing heavily on conceptual and mathematical parallels from quantum theory, specifically those summarized by Bohm in his 1951 text. We present a proof of this for a special case concerning the function $\zeta(s)$, defined by Riemann in his seminal 1859 paper, "On the number of primes less than a given number". A new method of defining a system of inverted concatenations at the simple pole(s) of a global L-function is introduced and used to finalize our proof.

Introduction: The theory regarding the Riemann Hypothesis, the statement that of

$$\xi(t) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$$

with $s = \frac{1}{2} + it$, the nontrivial zeros of $\zeta(s)$ have real part equal to $\frac{1}{2}$, i.e. the prototypification of general classes of functions, L-functions, Dirichlet series with a suitable Euler product satisfying an appropriate functional equation and a Riemann Hypothesis, are of central and significant relevance. The author refers the reader to [1] for a history of development. It is the purpose of this paper to show, a new determination of solubility. The proof of the Riemann Hypothesis follows. Applications exist to general global L-functions. Some future work is discussed.

1) We begin with a co-ordinate transformation. We can verify the Riemann Hypothesis for a given range. The number $N(T)$ of zeroes of $\zeta(s)$ in the rectangle $\Re$ with vertices at $-1-iT, 2-iT, 2+iT, -1+it$ is given by Cauchy's integral

$$N(T) - 1 = \frac{1}{2\pi}\int_{\partial R} \frac{-\zeta'}{\zeta}(S)ds$$

provided T is not the imaginary part of a zero. It is our purpose to determine a transition of dimensionality, from d=2 to d=3. This can be done simply, by introducing a system of inverted concatenations at the simple pole of $\zeta(s)$ defining $N(T)-1$ in that range. However as we will show this is not a sufficient method for determining a proof and as such further steps must be taken. We then have

Proposition 1.1) For $g(x)$ continuous and real for real t defined in $f(x)$, we define components of arbitrary x defined in□ $\xi_1, \xi_2, ..., \xi_n$ with respect to the basis $\{e\} = \{e_1, e_2, ..., e_n\}$ of the n-dimensional space k_n, and let the quantities $\eta_1, \eta_2, ..., \eta_n$ be defined by the formulas

$$(1.1)\quad \begin{aligned} \eta_1 &= s_{11}\xi_1 + s_{12}\xi_2 + ... + s_{1n}\xi_n, \\ \eta_2 &= s_{21}\xi_1 + s_{22}\xi_2 + ... + s_{2n}\xi_n, \\ &\dots\dots\dots\dots\dots\dots \\ \eta_n &= s_{n1}\xi_1 + s_{n2}\xi_2 + ... + s_{nn}\xi_n, \end{aligned}$$

where $\det\|s_{jk}\| \neq 0$. Then a new basis $\{f\} = \{f_1, f_2, ..., f_n\}$ can be found in the space k_n such that the numbers $\eta_1, \eta_2, ..., \eta_n$ are the components of a vector x with respect to the basis $\{f\}$.

Proof: Introduce the matrix $s = \|s_{jk}\|$ and the matrix $P = (S')^{-1}$ with elements denoted by $P_i^{(j)}$. Substituting these elements into the formulas (1.1), we get a new basis $\{f\} = \{f_1, f_2, ..., f_n\}$. We assert that this is the desired basis. In fact, consider the transformation formulas

(1.2) $$\eta_k = \sum_{j=1}^{n} q_k^{(j)} \xi_j \quad (k = 1, 2, ..., n),$$

which give the components of the vector x with respect to the new basis. These formulas can be written in terms of the matrix $(P^{-1})'$. But in the present case, $(P^{-1})'$ coincides with S, since

$$(P^{-1})' = ([(S')^{-1}]^{-1})' = (S')' = S.$$

Hence, given any vector x, the quantities $\eta_1, \eta_2, ..., \eta_n$ are just the components of x with respect to the basis $\{f\}$.

Proposition 1.2) For the arbitrary vector x defined in □ with components $\eta_1, \eta_2, ..., \eta_n$, setting d=2 yields

$$\eta_1, \eta_2 \equiv \eta_{(1-d)}.$$

The proof of this is evidenced by [2], [3], and [4].

In the presentation of these propositions it is our purpose to demonstrate first the providing of the transformation of components of a vector x=s, where $s = \sigma + \frac{1}{2}t$, then the representation of these components given varying dimensionalities, third the introduction of an orthogonality component in x, and lastly the complete transformation of dimensionality of □. From the third of these we can define orthogonality relations that, yielding certain valuations, proves a theorem of locality for general global L-functions and a corresponding Riemann Hypothesis.

Proposition 1.3) For the arbitrary vector x defined in □ with d=2, we can introduce an orthogonality component a, such that $a \cdot x = a_{ixy}$. The proof of this is evidenced by [3] and [5].

Theorem 1.3.1) By proposition 1.3) we van define orthogonality relations that, given certain values, proves a theorem of locality of the Riemann hypothesis of any global L-function. We begin by defining boundary conditions to simplify the necessary calculations. Consider the field □. Let us imagine a cube of side L. We then imagine that this structure is repeated periodically throughout □ for all values, so that for some rectangle $\Re$, definition in the complex plane yields in some way association to the cubes L. Let us suppose, further, that the half-planes are the same at corresponding points of every cube. We now assert that these boundary conditions will yield the same phenomena as will any other boundary conditions at the walls. To prove this, we need only ask why the phenomena is independent of the type of boundary. The answer is that, from a theoretical viewpoint, the wall merely serves to prevent deviation from constancy. Making the values periodic must have the same effect because each cube cannot deviate from constancy. If this were not so, the system would cease to be periodic. Thus, we have a boundary condition that serves the essential function of keeping the value in any individual cube constant. Although artificial, it must give the right answer, and it will make the calculations easier by simplifying the Fourier analysis of the half-planes.

Let $a(z, y, z, t)$ be any conceivable solution of a global L-function, we will use $\zeta(s)$, where

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

$\zeta(s)$ is the function of the complex variable s, defined in the half-plane $\Re(s) > 1$ and in the whole complex plane by analytic continuation. As shown by Riemann, $\zeta(s)$ extends to as a meromorphic function with only a simple pole at s=1, with residue 1, and satisfies the functional equation

$$\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s) = \pi^{-(1-s)/2}\Gamma(\frac{1-s}{2})\zeta(1-s).$$

It will be the purpose of one of our later propositions to present a system of inverted concatenations at s=1. a(x) is imposed by our boundary conditions, that it must be periodic in space with period L/n, where n is an integer. It is a well-known mathematical theorem that an arbitrary periodic function $f(x, y, z, t)$ can be represented by means of a Fourier series in the following manner:

(1.3)

$$\sum_{l,m,n}^{f(x,y,z,t)} [a_{l,m,n}(t)\cos\frac{2\pi}{L}((x + my + nz)) + b_{l,m,n}(t)\sin\frac{2\pi}{L}(lx + my + nz)]$$

where l, m, n are complex numbers. Any choice of a's and b's leading to a convergent series defines a function, $f(x, y, z, t)$, which is periodic in the sense that is takes on the same value each time x, y, or z changes by L. For a given function, $f(x, y, z, t)$ it can be shown that the $a_{l,m,n}(t)$ and the $b_{l,m,n}(t)$ are given by the following formulas:

(1.4)

$$a_{l,m,n}(t) + a_{-l,-m,-n}(t) = \frac{2}{L^3}\int_0^L\int_0^L\int_0^L dxdydz\cos\frac{2\pi}{L}(lx + my + nz)f(x, y, z, t)$$

$$b_{l,m,n}(t) - b_{-l,-m,-n}(t) = \frac{2}{L^3}\int_0^L\int_0^L\int_0^L dxdydz\sin\frac{2\pi}{L}(lx + my + nz)f(x, y, z, t).$$

These formulas illustrate the fact that only the sum of the a's and the difference of the b's are determined by the function f. From the above, we may conclude that f may be specified completely in terms of the quantities $a_{l,m,n} + a_{-l,-m,-n}$ and $b_{l,m,n} - b_{-l,-m,-n}$, but we prefer to retain the specification in terms of the $a_{l,m,n}$ and $b_{l,m,n}$ because of the simpler expressions to which they lead. Equations (1.4) are obtained by the aforementioned orthogonality relations:

(1.5a)
$$\int_0^L\int_0^L\int_0^L dxdydz \cos\frac{2\pi}{L}(lx+my+nz)\sin\frac{2\pi}{L}(l'x+m'y+n'z)=0$$

$$\int_0^L\int_0^L\int_0^L dxdydz \cos\frac{2\pi}{L}(lx+my+nz)\cos\frac{2\pi}{L}(l'x+m'y+n'z)=0$$

unless

$$\begin{pmatrix} l=l' \\ m=m' \\ n=n' \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} l=-l' \\ m=-m' \\ n=-n' \end{pmatrix}$$

in which case it is $L^3/2$, except when $l=m=n=0$, in which case it is L^3.

(1.5b)
$$\int_0^L\int_0^L\int_0^L \sin\frac{2\pi}{L}(lx+my+nz)\sin\frac{2\pi}{L}(l'x+m'y+n'z)=0$$

unless

$$\begin{pmatrix} l=l' \\ m=m' \\ n=n' \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} l=-l' \\ m=-m' \\ n=-n' \end{pmatrix}$$

in which case it is $L^3/2$. Fourier analysis, in the preceding form, enables us to represent an arbitrary function as a sum of standing plane waves of all possible wavelengths and amplitudes. The entire treatment is essentially the same as that used with waves in strings and organ pipes, except that it is three-dimensional. Let us now expand the vector potential in a Fourier series. Because a is a vector, involving three components, each $a_{l,m,n}$ and $b_{l,m,n}$ also has three components and, hence, must be represented as a vector:

$$a=\sum_{l,m,n}\left[a_{l,m,n}(t)\cos\frac{2\pi}{L}(lx+my+nz)+b_{l,m,n}(t)\sin\frac{2\pi}{L}(lx+my+nz)\right].$$

We assume that $a_{0,0,0}$ is zero in the above series. We now introduce the propagation vector k, defined as follows:

(1.6)
$$k_x=\frac{2\pi l}{L} \qquad k_y=\frac{2\pi m}{L} \qquad k_z=\frac{2\pi n}{L}$$

$$k^2=\left(\frac{2\pi}{L}\right)^2(l^2+m^2+n^2).$$

By orienting our co-ordinate axes in such a way that the z axis is directed along the k vector, we obtain $l=m=0$, and $k=2\pi/L$. From the definition of k, it follows that $k/2\pi$ is the number of waves in the distance L; hence the wavelength is $\lambda=2\pi/k$, or

$k = 2\pi / \lambda$. In this co-ordinate system a typical wave takes the form $\cos 2\pi nz / L$. Thus, the vector k is in the direction in which the phase of the wave changes. Going back to arbitrary co-ordinate axes, we conclude that k is a vector in the direction of the propagation of the wave. Its magnitude is $2\pi / \lambda$, and it is allowed to take on only the values permitted by integral l, m, and n in eq. (1.6). With this simplification of notation, we obtain

$$a = \sum_{k} \left[a_k(t) \cos k \cdot r + b_k(t) \sin k \cdot r \right]. \tag{1.7}$$

where the summation extends over all permissible values of k.

Let us now apply the condition $\operatorname{div} a = 0$ to (1.7). We have

$$\operatorname{div} a = \sum_{k} \left(-k \cdot a_k \sin k \cdot r + k \cdot b_k \cos k \cdot r \right) = 0.$$

It is a well-known theorem that if a Fourier series is identically zero, then all of the coefficients, a_k and b_k , must vanish. From the above it follows that $k \cdot a_k(t) = k \cdot b_k(t) = 0$. Thus, $a_k(t)$ and $b_k(t)$ are perpendicular to k , as are the half-planes belonging to the *kth* wave. Since the vibrations are normal to the direction of propagation, the waves are transverse. The direction of the half-plane $\Re[s] > 1$ is also called the direction of polarization. To describe the orientation of a_k let us return to the set of co-ordinate axes in which the z axis is in the direction of k. The vector a_k can have only x and y components, and if we specify the value of these, we shall have specified both the magnitude and the direction of a_k . We designate the direction of the vector a_k by the subscript μ , writing $a_{k,\mu}$ where μ is allowed to take on the values 1 and 2. For $\mu = 1$, $a_{k,\mu}$ is in the x direction; but for $\mu = 2$, it is in the y direction. All possible a_k vectors can then be represented as a sum of some $a_{k,1}$ vector, and some other $a_{k,2}$ vector. Hence, the most general vector potential, subject to the condition that $\operatorname{div} a = 0$, is given by

$$a = \sum_{k,\mu} \left[a_{k,\mu}(t) \cos k \cdot r + b_{k,\mu}(t) \sin k \cdot r \right]. \tag{1.8}$$

Here the summation extends over all permissible k vectors and over the two possible values of μ . It can be verified from (1.8) and the partial differential equation defining a in complex space

$$\nabla^2 a - \frac{1}{c^2} \frac{\partial^2 a}{\partial t^2} = 0 \tag{1.9}$$

that the $a_{k,\mu}$ satisfies the following differential equation:

$$\frac{d^2 a_{k,\mu}}{dt^2} + k^2 c^2 a_{k,\mu} = 0 \tag{1.10}$$

which shows that the $a_{k,\mu}$ terms oscillate with simple harmonic motion and with angular frequency, $\omega = kc$. The first step in evaluating the half-plane value is to express $\Re[s] > 1$

and $\Re[s]>0$ in terms of the Fourier series for a. These expressions are:

$$\Re[s]>1=\frac{1}{c}\sum_{k,\mu}(\dot{a}_{k,\mu}\cos k\cdot r+\dot{b}_{k,\mu}\sin k\cdot r)$$
$$\Re[s]>0=\sum_{k,\mu}(-k\times a_{k,\mu}\sin k\cdot r+k\times b_{k,\mu}\cos k\cdot r).$$

Let us now evaluate the k,μ following over the cube of side L:

$$\frac{1}{8\pi}\int(\Re[s]>1)^2 d\tau=\frac{1}{8\pi c^2}\sum_{k,\mu}\sum_{k,\mu}\int_0^L\int_0^L\int_0^L dxdydz \begin{pmatrix} \dot{a}_{k,\mu}\cdot\dot{a}_{k',\mu'}\cos k\cdot r\cos k'\cdot r+\dot{b}_{k,\mu}\cdot\dot{b}_{k',\mu'}\sin k\cdot r \\ \sin k'\cdot r+\dot{a}_{k,\mu}\cdot\dot{b}_{k',\mu'}\cos k\cdot r\sin k'\cdot r+\dot{b}_{k,\mu}\cdot\dot{a}_{k',\mu'} \\ \sin k\cdot r\cos k'\cdot r \end{pmatrix}.$$

With the aid of eqs. (1.5) we see that all integrals vanish except when $k=k'$, and that all terms involving $\dot{a}_{k,\mu}\cdot\dot{b}_{k',\mu'}$ are zero. Furthermore, $\dot{a}_{k,\mu}\cdot\dot{a}_{k,\mu'}=0$ unless $\mu=\mu'$. When $\mu\neq\mu'$, the two vectors are, by definition, perpendicular to each other. Thus, the above expression reduces to

$$\int\frac{(\Re[s]>1)^2 dt}{8\pi}=\frac{L^3}{8\pi c^2}\sum_{k,\mu}[\frac{1}{2}(\dot{a}_{k,\mu})^2+\frac{1}{2}(\dot{b}_{k,\mu})^2].$$

With a similar method, which involves somewhat more algebra, we obtain

$$\int\frac{(\Re[s]>0)^2}{8\pi}=\frac{L^3}{8\pi}\sum_{k,\mu}[\frac{1}{2}(a_{k,\mu})^2+\frac{1}{2}(b_{k,\mu})^2].$$

Thus, the constancy in the half-plane is (with $L^3=V$)

(1.11)

$$C=\frac{V}{8\pi c^2}\sum_{k,\mu}\{\frac{1}{2}[(\dot{a}_{k,\mu})^2+c^2k^2(a_{k,\mu})^2]+\frac{1}{2}[(\dot{b}_{k,\mu})^2+c^2k^2(b_{k,\mu})^2]\}$$
$$=\infty \text{ for all } \Re[s]<0<\Re[s]>1 \text{ defined in } \square .$$

By continued analysis, we obtain constancy

(1.12) $$C=nDO$$

which reduces to $\Re[s]>0<\Re[s]<1$. We obtain the probability that a zero lies on the line $\Re[s]=\frac{1}{2}$

(1.13) $$N(critical)\ \square\ 2\pi e^{-Cn/\ln T}=e^{-nDO/\ln T},$$

corresponding with the density by the Riemann-von Mangoldt formula

$$N(T)=\frac{T}{2\pi}\ln\left(\frac{T}{2\pi e}\right)+O(\ln T).$$

Thus, we may complete our proof as follows, beginning with the obvious justification of our arguments. For the line $\Re[s]=\frac{1}{2}$, we have frequencies

$$x=D\cos\omega t,\qquad y=D\cos(\omega t-\delta). \tag{1.14}$$

It is specified that the distributions x and y of N(T) have the same frequency. We assume $\delta=0$, so that any value, constant and residual, in the sense that we are only concerned with the N(T) of the present, unless other values of N(T) are specified, of s lies on the straight line

$$\frac{y}{x}=\frac{b}{a}=\frac{1}{2}.$$

It is the basis of this proof to yield an exact determination of the wavelength as utilized in our above propositions, and how we may heuristically observe a proof in this way. The following interpretation is noted: each coordinate s as in $s\equiv\sigma+it$ is, a periodic function of the constancy as obtained by the oscillator in our above propositions. We define the following:

$$Dv=v:$$

$$\text{Wavelength}\times\text{Frequency}=\text{velocity of propagation of the phase.}$$

Now, it is evident that we have obtained a method to observe the proof or disproof of our assertions. For the value s, there must be constant t for any plane perpendicular to the direction of propagation. This translates to constant t given real part perpendicular to the direction of propagation. It is known that there are an infinite number of zeros on the line $\Re[s]=\frac{1}{2}$, therefore we assume the direction of propagation to approach ∞, iff we have constancy of value equivalent to $\Re[s]$. Thus, to determine a value in □ with real part perpendicular to $\frac{1}{2}$, we obtain a value $s=\sigma+-2t$. To determine the L-function yielding this value, we need only look at the inverse of the Riemann zeta function, with $\Re[s]=-2$. Thus,

$$\sum_{n=1}^{\infty}-n^{k},\quad \Re[s]=-2, \tag{1.15}$$

so that

$$\lambda(s)=\sum_{n=1}^{\infty}-n^{k}=\frac{1}{\Gamma(x)}\int_{-\infty}^{\infty}\frac{-u^{x}}{-e^{u}}du, \tag{1.16}$$

where $\Gamma(x)$ is the gamma function. We define the satisfied functional equation, and then obtain our value of t,

$$\Gamma(-2s)-\pi^{2s}\lambda(s)=\Gamma(2+s)-\pi^{s-2}\lambda(-s). \tag{1.17}$$

Then, we define our value of t as constant for which t remains trivial and consistently perpendicular iff $t=-2,-4,-6,\ldots$, etc. We have finalized this part of our approach. It is evident that the density of the distribution of zeros yields the continuity of propagation of the line $\Re[s] = \frac{1}{2}$. Call the angle between the wave-normal and the z-axis a. If we let the wave normal n fall in reflected non-values of $\Re[s] = \frac{1}{2}$, as in Fig. 1, then, the normal to the reflected wave, n', will be reflected similarly, and will also make the angle a with the z-axis.

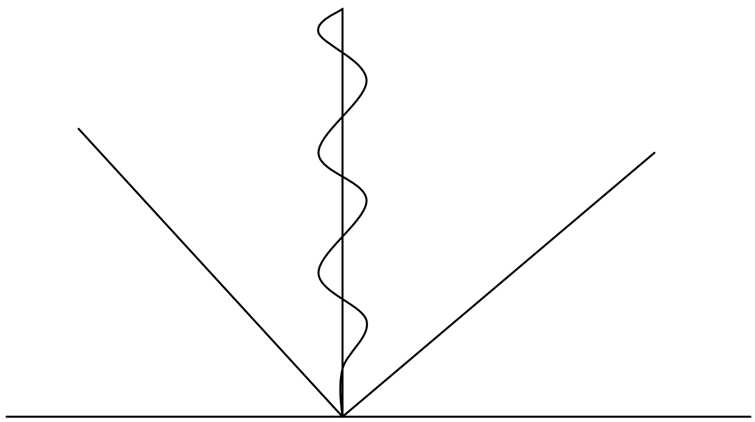

Fig 1., with the angle closest to the y axis (horizontal) denoted y, the corresponding angle denoted a, and the corresponding angle denoted $\frac{y}{\cos a}$.

Thus, to prove the Riemann hypothesis of any global L-function, we need only determine the values of the standing wave corresponding to the density of the distribution of the zeros of the L-function, as the locality corresponds exactly to the values and propagation of a standing wave, such that the corresponding values, propagation, and assumed direction of propagation yield a determination of the density of the distribution of zeros. In the case of the line $\Re[s] = \frac{1}{2}$, we have demonstrated the direction of propagation to be ∞, and thus, in addition to the above arguments, we have our proof. We may determine the Riemann hypothesis of any global L-function in this way, and we have yielded a more effective representation of the phenomena concerning the density of the distribution of zeros.

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