A superluminal effect with oscillating neutrinos

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Abstract

A simple quantum relativistic model of $\nu_\mu - \nu_\tau$ neutrino oscillations in the OPERA experiment is presented. This model suggests that the two components in the neutrino beam are separated in space. After being created in a meson decay, the μ-neutrino moves 18 meters ahead of the beam's center of energy, while the τ-neutrino is behind. Both neutrinos have subluminal speeds, however the advanced start of the ν_μ explains why it arrives in the detector 60 ns earlier than expected. Our model does violate the special-relativistic ban on superluminal signals. However, usual arguments about violation of causality are not applicable here. The invalidity of standard special-relativistic arguments is related to the interaction-dependence of the boost operator, which implies that boost-transformed space-time coordinates of events with interacting particles do not obey linear and universal Lorentz formulas.

1 Introduction

A recent preprint [1] published by the OPERA collaboration claims observation of a superluminal effect in neutrino propagation. Muon-type neutrinos (ν_μ) with energies of about 17 GeV were produced by the CERN accelerator and captured by the OPERA neutrino detector 730 kilometers away. It is believed that in the course of propagation the muon neutrinos partially converted to tau neutrinos (ν_τ) due to the effect of *neutrino oscillations*. The carefully measured propagation time of the ν_μ beam was 60 ns shorter than the one expected from the usual assumption about subluminal propagation speeds. There is a great deal of scepticism in the scientific community regarding this remarkable result. However, in this paper we will assume that the superliminal OPERA effect is valid, and offer an explanation, which, on one hand, is fully within mainstream quantum relativistic physics, and on the other hand, challenges the traditional interpretation of Einstein's relativity theory.

In section 2 we consider a simple but realistic model of $\nu_\mu - \nu_\tau$ oscillations. The model is formulated in one spatial dimension, but its generalization for the real 3D world is not expected to bring any significant changes. The model is fully relativistic, meaning that commutation relations of the Poincaré Lie algebra are explicitly satisfied by operators of the total momentum, total energy and boost. The interaction responsible for oscillations is fully controlled by a momentum-dependent function $f(p) \equiv |f(p)|e^{i\alpha(p)}$. The modulus $|f(p)|$ of this function determines the frequency of oscillations of neutrinos with momentum p. The phase factor $e^{i\alpha(p)}$ plays a different physical role: If the phase $\alpha(p)$ changes rapidly with p, then the two components (ν_μ and ν_τ) of the neutrino beam are separated by a certain distance χ. The theory does not specify the behavior of $\alpha(p)$ distance and the distance χ. Then it is possible to assume that the separation between neutrino components amounts to several meters, so that the ν_μ neutrino moves ahead of the beam's center-of-energy, while the ν_τ neutrino lags behind. Note that both components are subluminal, as expected for massive particles.

In section 3 we use this theory to explain the OPERA experiment. When the initial μ-neutrino is created in a meson decay at the CERN site, this particle emerges not at the interaction vertex, but 18 meters in the forward direction. This advanced start explains the early arrival of muon neutrinos in the OPERA detector in spite of the subluminal propagation speed.

Ourproposed explanation requires instantaneous appearance of a decay product (ν_μ) 18 meters away from the interaction vertex. This is in sharp disagreement with traditional special relativity, which claims that superluminal propagation of any physical signal is inconsistent with the principle of causality. In section 4 we will argue that our model does not violate causality even in the moving reference frame. The key idea is that transition to the moving frame should be performed by using a boost operator, which depends on interaction. Therefore, transformations of observables (including positions of particles) in the relevant interacting system (unstable meson plus muon plus two oscillating neutrinos) are different from simple and universal Lorentz formulas of special relativity. This allows us to reject the special-relativistic ban on superluminal velocities and, at the same time, obey the causality principle. Finally, we formulate a few predictions for future neutrino experiments, which follow from our model.

2 Theory

We would like to describe a free neutrino system oscillating between two states: μ-neutrino and τ-neutrino. For simplicity, we will ignore the possible effect of the third (electronic) e-neutrino species. Then the Hilbert space can be constructed as a direct sum of two one-particle subspaces

$$\mathcal{H} = \mathcal{H}_\mu \oplus \mathcal{H}_\tau \tag{1}$$

This Hilbert space will be used for both non-interacting and interacting neutrino systems considered in this section.

2.1 Non-interacting system

Both $\mathcal{H}_\mu$ and $\mathcal{H}_\tau$ are Hilbert spaces carrying unitary irreducible spinless representations of the Poincaré group characterized by free neutrino masses m_μ and m_τ, respectively.[1] If the interaction responsible for neutrino oscillations is turned off then the noninteracting representation of the Poincaré group acting in the Hilbert space $\mathcal{H}$ can be built as a direct sum of these two irreducible representations. To write this representation in an explicit form we will choose a convenient basis set in (1). At each momentum p we can select two orthonormal basis states of definite flavor:

$$\begin{aligned} |\nu_\mu\rangle &\equiv \left[\begin{array}{c} 1 \\ 0 \end{array}\right] \\ |\nu_\tau\rangle &\equiv \left[\begin{array}{c} 0 \\ 1 \end{array}\right] \end{aligned}$$

Then each state vector $|\psi\rangle$ can be represented as a 2-component momentum-dependent vector in this basis

$$|\psi\rangle \equiv \left[\begin{array}{c} \Phi_\mu(p) \\ \Phi_\tau(p) \end{array}\right]$$

Finite transformations from the Poincaré group (space translation, time translations and boosts) can be written as exponential functions of generators. They have simple expressions in the flavor basis (see section 2.5 in [2] and section 5.1 in [3])[2]

$$e^{\frac{i}{\hbar}P_0 a}|\psi\rangle = \left[\begin{array}{c} e^{\frac{i}{\hbar}pa}\Phi_\mu(p) \\ e^{\frac{i}{\hbar}pa}\Phi_\tau(p) \end{array}\right]$$

[1] In our 1-dimensional world neutrinos are spinless. For definiteness we will assume that $m_\tau > m_\mu$, though this is not critical for our results.

[2] In this paper we adopt Schrödinger representation: Any inertial change of the observer is reflected in the change of system's state vector (or wave function). Different observers use the same Hermitian operator to describe a given observable.

$$
\begin{aligned}
e^{-\frac{i}{\hbar}H_0 t}\left[\begin{array}{c}\Phi_\mu(p)\\ \Phi_\tau(p)\end{array}\right] &= \left[\begin{array}{c}e^{-\frac{i}{\hbar}\omega_\mu(p)t}\Phi_\mu(p)\\ e^{-\frac{i}{\hbar}\omega_\tau(p)t}\Phi_\tau(p)\end{array}\right]\\
e^{\frac{i}{\hbar}K_0 c\theta}|\psi\rangle &= \left[\begin{array}{c}\sqrt{\frac{\omega_\mu(\Lambda_\mu p)}{\omega_\mu(p)}}\Phi_\mu(\Lambda_\mu p)\\ \sqrt{\frac{\omega_\tau(\Lambda_\tau p)}{\omega_\tau(p)}}\Phi_\tau(\Lambda_\tau p)\end{array}\right]
\end{aligned}
$$

where

$$
\begin{aligned}
\omega_{\mu,\tau}(p) &\equiv \sqrt{m_{\mu,\tau}^2 c^4 + p^2 c^2}\\
\Lambda_{\mu,\tau}p &\equiv p\cosh\theta - \frac{\omega_{\mu,\tau}}{c}\sinh\theta
\end{aligned}
$$

and parameter θ is related to the boost velocity by formula $v = c\tanh\theta$.

The basis of the corresponding representation of the Poincaré Lie algebra is provided by Hermitian operators of total momentum P_0, total energy H_0 and boost K_0. The explicit form of these generators can be obtained by differentiation

$$
P_0 = -i\hbar\lim_{a\to 0}\frac{d}{da}e^{\frac{i}{\hbar}P_0 a} = \left[\begin{array}{cc}p & 0\\ 0 & p\end{array}\right] \tag{2}
$$

$$
H_0 = \left[\begin{array}{cc}\omega_\mu(p) & 0\\ 0 & \omega_\tau(p)\end{array}\right] \tag{3}
$$

$$
K_0 = -i\hbar\left[\begin{array}{cc}\frac{\omega_\mu(p)}{c^2}\frac{d}{dp}+\frac{p}{2\omega_\mu(p)} & 0\\ 0 & \frac{\omega_\tau(p)}{c^2}\frac{d}{dp}+\frac{p}{2\omega_\tau(p)}\end{array}\right] \tag{4}
$$

2.2 Interaction

In the Dirac's instant form of dynamics [2, 4], relativistically invariant description of interaction is achieved by adding extra terms to both the energy operator $H = H_0 + V$ and the boost operator $K = K_0 + Z$, while keeping the total momentum P_0 unchanged. The choice of interactions V and Z must ensure that Poincaré commutators remain the same as in the non-interacting case

$$
[H, P_0] = 0 \tag{5}
$$

$$
[K, P_0] = -\frac{i\hbar}{c^2}H \tag{6}
$$

$$
[K, H] = -i\hbar P_0 \tag{7}
$$

In the flavor basis we can write the full Hamiltonian as a 2×2 momentum-dependent matrix

$$H = H_0 + V = \left[\begin{array}{cc} \omega_\mu(p) & f(p) \\ f^*(p) & \omega_\tau(p) \end{array} \right] \qquad (8)$$

where $f(p)$ is a complex function. For future use it will be convenient to write $f(p) = |f(p)|e^{i\alpha(p)}$, where $\alpha(p)$ is a real *phase function.*

2.3 Mass (energy) eigenstates

Our primary goal in this paper is to calculate the time evolution of neutrino states. This can be done most easily if we find eigenvalues $E_{1,2}$ and eigenstates of H. So, we need to solve equation

$$0 = \left[\begin{array}{cc} \omega_\mu(p) - E_{1,2}(p) & f(p) \\ f^*(p) & \omega_\tau(p) - E_{1,2}(p) \end{array} \right] \left[\begin{array}{c} \Phi_\mu^{1,2}(p) \\ \Phi_\tau^{1,2}(p) \end{array} \right] \qquad (9)$$

together with normalization conditions ($i = 1, 2$)

$$|\Phi_\mu^i(p)|^2 + |\Phi_\tau^i(p)|^2 = 1 \qquad (10)$$

For the eigenvalues E_1, E_2 we obtain

$$|f(p)| = \sqrt{(\omega_\mu(p) - E_1(p))\,(\omega_\tau(p) - E_1(p))} = \sqrt{(\omega_\mu(p) - E_2(p))\,(\omega_\tau(p) - E_2(p))} \qquad (11)$$

A necessary requirement for this theory to be relativistically invariant is that energy eigenvalues have the standard momentum dependence

$$E_{1,2}(p) = \sqrt{m_{1,2}^2 c^4 + p^2 c^2} \qquad (12)$$

where $m_{1,2}$ are neutrino mass eigenvalues.[3] We suppose that these eigenvalues are known and consider (11) as a definition of the modulus $|f(p)|$ of the interaction function.[4] Note that energy eigenvalues do not depend on the phase function $\alpha(p)$. So, we are free to choose any real function $\alpha(p)$ in our study. In the next subsection we will build an interacting representation of the Poincaré group explicitly, thus showing that (11) - (12) are also sufficient conditions for the relativistic invariance.

[3]The operator of mass is defined as $M = +\sqrt{H^2 - P_0^2 c^2}/c^2$. Here we assume that $m_2 > m_1 > 0$.

[4]A useful property $\omega_\mu(p) + \omega_\tau(p) = E_1(p) + E_2(p)$ also follows from the definition (11).

As can be verified by direct substitution in (9) - (10), the eigenvectors of the full Hamiltonian are

$$
\begin{aligned}
|1, p\rangle &= \begin{bmatrix} A(p) \\ -B(p)e^{-i\alpha(p)} \end{bmatrix} && (13)\\
|2, p\rangle &= \begin{bmatrix} B(p)e^{i\alpha(p)} \\ A(p) \end{bmatrix} && (14)
\end{aligned}
$$

where we introduced notation

$$
\begin{aligned}
A(p) &\equiv +\sqrt{\frac{\omega_\tau(p) - E_1(p)}{E_2(p) - E_1(p)}} \\
B(p) &\equiv +\sqrt{\frac{\omega_\mu(p) - E_1(p)}{E_2(p) - E_1(p)}} \\
A^2(p) + B^2(p) &= 1
\end{aligned}
$$

Note also that (13) - (14) are eigenvectors of the total momentum P_0 and mass M.

Next we need to find a connection between the flavor and mass-energy bases. If $(\Psi_1(p), \Psi_2(p))$ is a state vector written in the basis of mass eigenstates,[5] then the corresponding expansion in the flavor basis is obtained by a unitary transformation

$$
\begin{bmatrix} \Phi_\mu(p) \\ \Phi_\tau(p) \end{bmatrix} = \begin{pmatrix} A(p) & B(p)e^{i\alpha(p)} \\ -B(p)e^{-i\alpha(p)} & A(p) \end{pmatrix} \begin{pmatrix} \Psi_1(p) \\ \Psi_2(p) \end{pmatrix} \tag{15}
$$

The transformation from the flavor basis to the mass basis is provided by the inverse matrix

$$
\begin{pmatrix} \Psi_1(p) \\ \Psi_2(p) \end{pmatrix} = \begin{bmatrix} A(p) & -B(p)e^{i\alpha(p)} \\ Be^{-i\alpha(p)} & A \end{bmatrix} \begin{bmatrix} \Phi_\mu(p) \\ \Phi_\tau(p) \end{bmatrix} \tag{16}
$$

2.4 Interacting representation of the Poincaré group

The mass basis is useful because the interacting representation of the Poincaré group takes especially simple form there

[5] Here we use round parentheses to indicate that expansion coefficients refer to the mass basis. Square brackets are used for the flavor basis.

$$
\begin{aligned}
e^{-\frac{i}{\hbar}Ht}\left(\begin{array}{c}\Psi_1(p)\\ \Psi_2(p)\end{array}\right) &= \left(\begin{array}{c}e^{-\frac{i}{\hbar}E_1(p)t}\Psi_1(p)\\ e^{-\frac{i}{\hbar}E_2(p)t}\Psi_2(p)\end{array}\right) \qquad (17)\\
e^{\frac{i}{\hbar}Kc\theta}\left(\begin{array}{c}\Psi_1(p)\\ \Psi_2(p)\end{array}\right) &= \left(\begin{array}{c}\sqrt{\frac{E_1(\Lambda_1 p)}{E_1(p)}}\Psi_1(\Lambda_1 p)\\ \sqrt{\frac{E_2(\Lambda_2 p)}{E_2(p)}}\Psi_2(\Lambda_2 p)\end{array}\right)
\end{aligned}
$$

where

$$
\Lambda_i p \equiv p\cosh\theta - \frac{E_i}{c}\sinh\theta
$$

is the usual boost transformation of momentum.

The interacting generators in the mass basis can be obtained by differentiation similar to (3) - (4)

$$
\begin{aligned}
H &= i\hbar\lim_{t\to 0}\frac{d}{dt}e^{-\frac{i}{\hbar}Ht} = \left(\begin{array}{cc}E_1(p) & 0\\ 0 & E_2(p)\end{array}\right)\\
K &= -i\hbar\left(\begin{array}{cc}\frac{E_1(p)}{c^2}\frac{d}{dp}+\frac{p}{2E_1(p)} & 0\\ 0 & \frac{E_2(p)}{c^2}\frac{d}{dp}+\frac{p}{2E_2(p)}\end{array}\right) \qquad (18)
\end{aligned}
$$

By noticing the analogy of these formulas with the non-interacting representation in subsection 2.1 one can convince oneself that commutators (5) - (7) are, indeed, satisfied. So, our theory is relativistically invariant.

2.5 Time evolution

Obviously, the state vector with one μ-neutrino having a normalized momentum-space wave function $\psi(p)$

$$
\begin{aligned}
|\psi\rangle &\equiv \left[\begin{array}{c}\psi(p)\\ 0\end{array}\right] \qquad (19)\\
\int dp|\psi(p)| &= 1
\end{aligned}
$$

is not an eigenstate of the Hamiltonian (8). So, neutrino states with definite flavor are not stationary. Our goal in this subsection is to calculate the time evolution of these states.

Let us now make further simplifications by assuming that the initial wave function $\psi(p)$ is localized in a narrow region Δp of the momentum space. We will also assume that in this range function $|f(p)|$ varies slowly, while "interaction phase" grows linearly $\alpha(p) \approx \frac{1}{\hbar}\chi p$.[6] Moreover, it is reasonable to assume that neutrinos are ultrarelativistic, so we can set

$$\begin{aligned}
p &\gg m_{1,2}c \\
E_1(p) &= \sqrt{m_1^2 c^4 + p^2 c^2} \approx cp \\
E_2(p) &= \sqrt{m_2^2 c^4 + p^2 c^2} \approx cp + \gamma
\end{aligned}$$

Next we use (16) to expand the initial state vector (19) in the basis of eigenvectors of the full Hamiltonian

$$|\psi\rangle = \begin{pmatrix} \psi(p)A \\ \psi(p)Be^{-\frac{i}{\hbar}\chi p} \end{pmatrix}$$

The time evolution of this state vector is obtained from (17)

$$|\psi(t)\rangle \equiv e^{-\frac{i}{\hbar}Ht}|\psi\rangle = \begin{pmatrix} \psi(p)Ae^{-\frac{i}{\hbar}E_1(p)t} \\ \psi(p)Be^{-\frac{i}{\hbar}\chi p}e^{-\frac{i}{\hbar}E_2(p)t} \end{pmatrix}$$

Its components in the flavor basis can be found using transformation (15)

$$\begin{aligned}
|\psi(t)\rangle &= \begin{pmatrix} A & Be^{\frac{i}{\hbar}\chi p} \\ -Be^{-\frac{i}{\hbar}\chi p} & A \end{pmatrix} \begin{pmatrix} \psi(p)Ae^{-\frac{i}{\hbar}E_1(p)t} \\ \psi(p)Be^{-\frac{i}{\hbar}\chi p}e^{-\frac{i}{\hbar}E_2(p)t} \end{pmatrix} \\
&= \psi(p) \begin{bmatrix} A^2 e^{-\frac{i}{\hbar}E_1(p)t} + B^2 e^{-\frac{i}{\hbar}E_2(p)t} \\ ABe^{-\frac{i}{\hbar}\chi p}\left(e^{-\frac{i}{\hbar}E_2(p)t} - e^{-\frac{i}{\hbar}E_1(p)t}\right) \end{bmatrix} \\
&\approx \psi(p) \begin{bmatrix} e^{-\frac{i}{\hbar}cpt}\left(A^2 + B^2 e^{-\frac{i}{\hbar}\gamma t}\right) \\ ABe^{-\frac{i}{\hbar}\chi p}e^{-\frac{i}{\hbar}cpt}\left(e^{-\frac{i}{\hbar}\gamma t} - 1\right) \end{bmatrix}
\end{aligned}$$

To switch to the position representation we perform a Fourier transform

$$\frac{1}{2\pi\hbar}\int dp e^{\frac{i}{\hbar}px}\psi(p)e^{-\frac{i}{\hbar}cpt} \begin{bmatrix} A^2 + B^2 e^{-\frac{i}{\hbar}\gamma t} \\ ABe^{-\frac{i}{\hbar}\chi p}\left(e^{-\frac{i}{\hbar}\gamma t} - 1\right) \end{bmatrix}$$

[6] More generally, one can write $\alpha(p) \approx \frac{1}{\hbar}(\beta + \chi p)$, where β is a real constant, but the constant unimodular factor $\exp(\frac{i}{\hbar}\beta)$ is irrelevant for our discussion, so we set $\beta = 0$.

$$
\begin{aligned}
&\approx \frac{\overline{\psi}}{2\pi\hbar}\int dp e^{\frac{i}{\hbar}px}e^{-\frac{i}{\hbar}cpt}\left[\begin{array}{c} A^2+B^2e^{-\frac{i}{\hbar}\gamma t} \\ ABe^{-\frac{i}{\hbar}\chi p}\left(e^{-\frac{i}{\hbar}\gamma t}-1\right)\end{array}\right] \\
&\approx \overline{\psi}\left[\begin{array}{c} \left(A^2+B^2e^{-\frac{i}{\hbar}\gamma t}\right)\delta(x-ct) \\ AB\left(e^{-\frac{i}{\hbar}\gamma t}-1\right)\delta(x-\chi-ct)\end{array}\right]
\end{aligned} \tag{20}
$$

Here we took into account that the range of $\psi(p)$ is much larger than the period of oscillations of imaginary exponents, so we can simply move out of the integral some average value $\overline{\psi}$. Due to the normalization of $\psi(p)$, this value has to be unimodular $|\overline{\psi}|^2 = 1$. By doing these approximations, we also neglected the wave function "spreading" effect, which is known to be superluminal, but negligibly small [5, 6, 7, 8, 9, 10, 11].

2.6 Oscillations and the neutrino "size"

Equation (20) is our main result, and in this subsection we will analyze physical implications of this formula. The probabilities for finding μ-neutrino and τ-neutrino change with time as

$$
\begin{aligned}
\rho_\mu(t) &= \left|A^2+B^2e^{-\frac{i}{\hbar}\Delta t}\right|^2 = A^4+B^4+2A^2B^2\cos\left(\frac{\gamma t}{\hbar}\right) \\
\rho_\tau(t) &= A^2B^2\left|e^{-\frac{i}{\hbar}\Delta t}-1\right|^2 = 2A^2B^2\left(1-\cos\left(\frac{\gamma t}{\hbar}\right)\right) \\
\rho_\mu(t)+\rho_\tau(t) &= 1
\end{aligned}
$$

In the ultrarelativistic limit the oscillation period is[7]

$$
T = \frac{\hbar}{\gamma} = \hbar\left(E_2(p)-E_1(p)\right)^{-1} \approx \frac{\hbar E}{(m_2^2-m_1^2)c^4} \tag{21}
$$

where $E \approx E_1(p) \approx E_2(p) \approx cp$. In the particular case of "full mixing" ($A^2 = B^2 = 1/2$) both probabilities oscillate between two extremes 0% and 100%

$$
\begin{aligned}
\rho_\mu(t) &= \frac{1}{2}\left(1+\cos\left(\frac{\gamma t}{\hbar}\right)\right) \\
\rho_\tau(t) &= \frac{1}{2}\left(1-\cos\left(\frac{\gamma t}{\hbar}\right)\right)
\end{aligned}
$$

This example is shown in Fig. 1.

From arguments of delta functions in (20) we can find classical trajectories of the two neutrino species

[7] For a review of neutrino oscillations see [12].

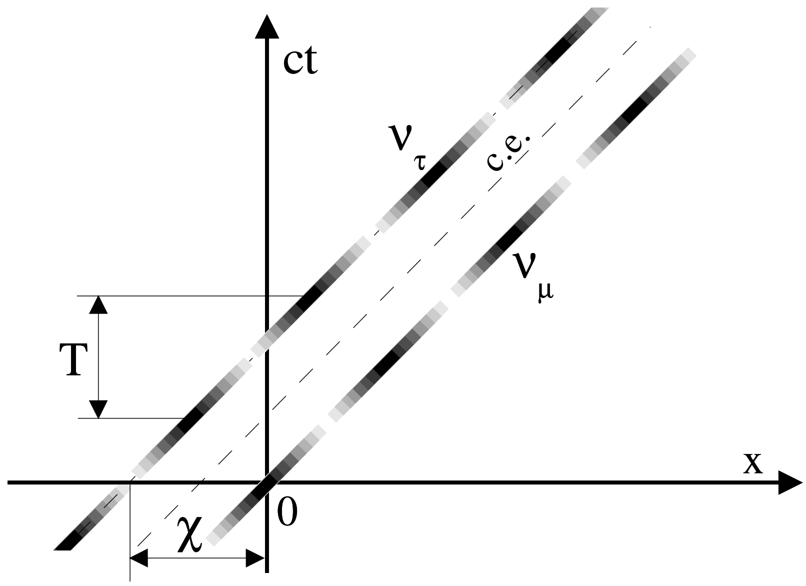

Figure 1: Space-time diagram for a free oscillating neutrino system. The two components ν_μ and ν_τ have different trajectories separated by the distance χ. Varying line densities indicate the oscillating probabilities $\rho_{\mu,\tau}(t)$ for finding the two particles. "c.e." is the center-of-energy trajectory.

$$x_\mu(t) = ct \tag{22}$$

$$x_\tau(t) = \chi + ct \tag{23}$$

We see that both neutrinos move with (almost) the speed of light, as expected. The remarkable property is the presence of parameter χ in (23). This means that the two neutrino components do not overlap in space. They have different trajectories separated by the distance χ. Recall that χ is a free and unrestricted real parameter in our theory. In the example shown in Fig. 1 this parameter has been chosen negative.

2.7 Conservation laws

The behavior of the two-neutrino system described above is rather peculiar: The system oscillates not only between two flavor states, but also between two different trajectories. In a sense, this object has a non-vanishing size χ, and nothing in the theory forbids this size to be macroscopically large, e.g., several meters. In order to convince ourselves in the validity of this solution, let us check that conservation laws have not been violated. Our solution is not an eigenvalue of any physical observable (like flavor number, momentum, energy, position, etc.), so, we can only verify the conservation of certain expectation values.

First, let us check that the total momentum of the system is conserved. In the mass basis the operator of total momentum P_0 is "p times unity operator", i.e., the same as in the flavor basis (2). Then it is easy to show that the expectation value of P_0 does not depend on time

$$
\begin{aligned}
\langle P_0(t)\rangle &\equiv \langle\psi(t)|P_0|\psi(t)\rangle \\
&= \int dp \left(A\psi^*(p)e^{\frac{i}{\hbar}E_1(p)t}, Be^{\frac{i}{\hbar}\chi p}\psi^*(p)e^{\frac{i}{\hbar}E_2(p)t}\right)\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}\begin{pmatrix} A\psi(p)e^{-\frac{i}{\hbar}E_1(p)t} \\ Be^{-\frac{i}{\hbar}\chi p}\psi(p)e^{-\frac{i}{\hbar}E_2(p)t} \end{pmatrix} \\
&= \int dpp\left(A^2|\psi(p)|^2 + B^2|\psi(p)|^2\right) = \int dpp|\psi(p)|^2 = \langle p\rangle
\end{aligned}
$$

Similarly, we demonstrate the time independence of the total energy[8]

$$
\begin{aligned}
\langle H(t)\rangle &\equiv \langle\psi(t)|H|\psi(t)\rangle \\
&= \int dp \left(A\psi^*(p)e^{\frac{i}{\hbar}E_1(p)t}, Be^{\frac{i}{\hbar}\chi p}\psi^*(p)e^{\frac{i}{\hbar}E_2(p)t}\right)\begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}\begin{pmatrix} A\psi(p)e^{-\frac{i}{\hbar}E_1(p)t} \\ Be^{-\frac{i}{\hbar}\chi p}\psi(p)e^{-\frac{i}{\hbar}E_2(p)t} \end{pmatrix} \\
&= \int dp(E_1A^2 + E_2B^2)|\psi(p)|^2 \approx E
\end{aligned}
$$

Another less known conservation law says that the center of energy of any isolated physical system must move with constant velocity along a straight line. This law follows from the definition of the center-of-energy position ([13] and section 4.3 in [3])

$$R = -\frac{c^2}{2}(KH^{-1} + H^{-1}K)$$

and the relationship[9]

$$K(t) \equiv e^{\frac{i}{\hbar}Ht}Ke^{-\frac{i}{\hbar}Ht} = K - P_0t$$

which is a direct result of the basic commutators (6) - (7). Using the matrix form of the boost operator (18) and taking into account that[10]

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[8] which includes both kinetic energies of the particles and the potential energy of their interaction

[9] This formula is written in the Heisenberg representation

[10] Here we assume that $\phi(p)$ is a real function and perform integration by parts.

$$\int dp \frac{E_1(p)}{c^2}\psi^*(p)\frac{d\psi(p)}{dp} = \int dp \frac{E_1(p)}{2c^2}\frac{d}{dp}|\psi(p)|^2 = -\int dp \frac{d}{dp}\left(\frac{E_1(p)}{2c^2}\right)|\psi(p)|^2$$
$$\approx \quad -\int dp \frac{1}{2c}|\psi(p)|^2 = -\frac{1}{2c}$$

we obtain

$$\begin{aligned}
\langle K(t)\rangle &\equiv \langle\psi(t)|K|\psi(t)\rangle \\
&= -i\hbar\int dp \left(A\psi^*(p)e^{\frac{i}{\hbar}E_1(p)t}, Be^{\frac{i}{\hbar}\chi p}\psi^*(p)e^{\frac{i}{\hbar}E_2(p)t}\right)\begin{pmatrix} \frac{E_1(p)}{c^2}\frac{d}{dp}+\frac{p}{2E_1(p)} & 0 \\ 0 & \frac{E_2(p)}{c^2}\frac{d}{dp}+\frac{p}{2E_2(p)}\end{pmatrix} \\
&\begin{pmatrix} A\psi(p)e^{-\frac{i}{\hbar}E_1(p)t} \\ Be^{-\frac{i}{\hbar}\chi p}\psi(p)e^{-\frac{i}{\hbar}E_2(p)t}\end{pmatrix} \\
&= -i\hbar\int dp\Big[A^2\left(\frac{E_1(p)}{c^2}\psi^*(p)\frac{d\psi(p)}{dp} - \frac{i}{\hbar}pt|\psi(p)|^2 + \frac{p}{2E_1(p)}|\psi(p)|^2\right) \\
&+ B^2\left(-\frac{i}{\hbar}\chi\frac{E_2(p)}{c^2}|\psi(p)|^2 - \frac{i}{\hbar}pt|\psi(p)|^2 + \frac{E_2(p)}{c^2}\psi^*(p)\frac{d\psi(p)}{dp} + \frac{p}{2E_2(p)}|\psi(p)|^2\right)\Big] \\
&\approx -i\hbar\left(-\frac{iA^2}{\hbar}pt - \frac{i}{\hbar}\chi\frac{B^2E_2(p)}{c^2} - \frac{iB^2}{\hbar}pt\right) = -\chi\frac{B^2E_2}{c^2} - pt = \langle K\rangle - \langle p\rangle t
\end{aligned}$$

The center-of-energy trajectory is then obtained as

$$\langle R(t)\rangle \quad = \quad -\frac{c^2\langle K(t)\rangle}{\langle H(t)\rangle} \approx \frac{\chi B^2 E_2}{(E_1A^2+E_2B^2)} + ct \approx \chi B^2 + ct$$

This means that the center-of-energy moves with the light speed c, as expected. This imaginary trajectory lies between real trajectories (22) - (23) of the two neutrino components. In the case of full mixing ($B^2 = 1/2$) the center of energy is right in the middle between two neutrinos, as shown in Fig. 1.

3 OPERA experiment

3.1 Neutrino creation reaction

In the OPERA experiment, CERN accelerator supplies high energy protons, which fall on a graphite target and produce multiple secondary particles, including charged $\pi^{\pm}$ and $K^{\pm}$ mesons. The mesons decay in-flight and emit muon neutrinos, which are

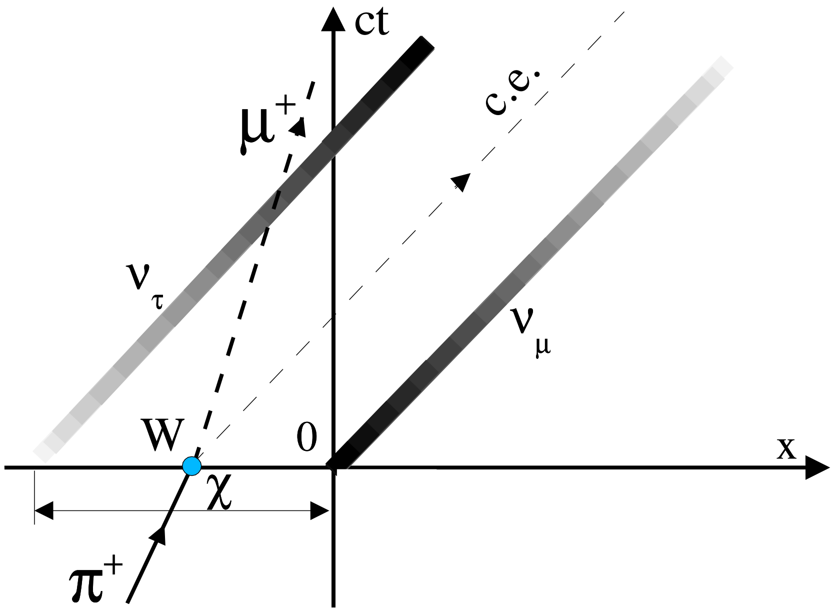

Figure 2: Space-time diagram for the neutrino creation reaction $\pi^+ \to \mu^+ + \nu_\mu$. The center-of-energy trajectory emerges directly from the decay interaction vertex W, while ν_μ and ν_τ trajectories are displaced.

eventually captured by the OPERA detector. In Fig. 2 we sketch a space-time diagram for the $\pi^+ \to \mu^+ + \nu_\mu$ decay process.

In subsection 2.6 we have established that neutrino system may have a large size ($|\chi|$ = several meters). So, it is important to understand the location of this object at the point of its creation. Here we will be helped by the law of continuity of the center-of-energy trajectory mentioned above. This law should remain valid even in the pion decay process. But it cannot be satisfied if ν_μ is emitted directly from the decay interaction vertex. As shown in Fig. 2, the decay point (marked "W" on the figure) should lie on the imaginary line representing the neutrino center-of-energy trajectory (the thin dashed line in the figure). In this case, the μ-neutrino component is displaced by the distance of $|\chi|B^2$ in the forward direction, while ν_τ is $|\chi|A^2$ meters behind.

3.2 Neutrino detection

The currently accepted value for the difference of squared masses is $m_2^2 - m_1^2 = 2.7 \cdot 10^{-3} eV$ [14]. Then from (21) we obtain the oscillation period for the muon-tau neutrino pair with energy $E = 17$ GeV

$$T = \frac{\hbar E}{(m_2^2 - m_1^2)c^4} \approx 0.0041 s$$

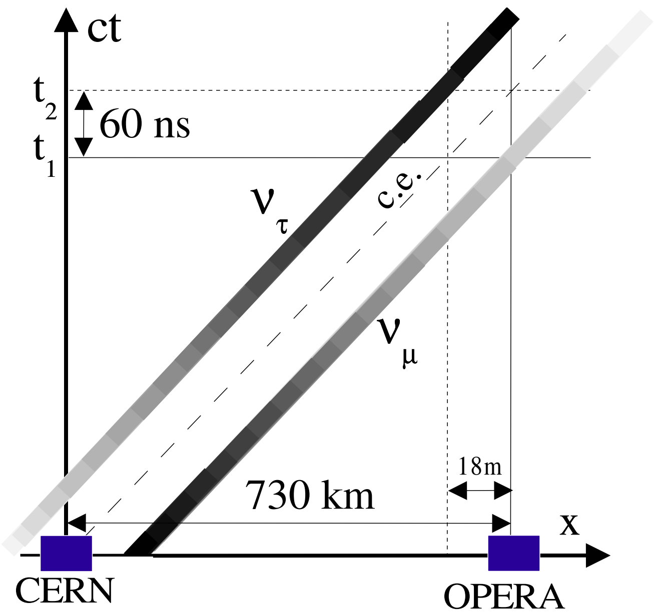

Figure 3: Schematic representation of the OPERA neutrino experiment. The 60 ns advance in the μ-neutrino arrival time is explained by their creation 18 meters away from the meson decay point at time $t = 0$.

The time of neutrino flight between CERN and OPERA with the speed of light is 0.0024 s. This means that the beam arriving in the detector has completed one half of one oscillation cycle, which maximizes the chance of detecting ν_τ and serves the original purpose of this experiment, which is the study of $\nu_\mu - \nu_\tau$ oscillations.

Now we can collect all the results obtained so far in order to suggest a realistic picture of the OPERA experiment and explain the superluminal behavior of the neutrinos. We will use our theory described above and assume full mixing (which is not essential) and the value $\chi = -36m$ (which is essential). According to this model, the imaginary trajectory of the neutrino center-of-energy is directly attached to the decay interaction vertex W, as shown in Fig. 2. This imaginary trajectory arrives in OPERA "on schedule" without superluminal suprises. The μ-neutrino emitted in the meson decay event has a position, which is advanced by $|\chi|/2 = 18m$ with respect to the center of energy. On the other hand, the τ-neutrino component of the beam moves $18m$ behind the center of energy. The speed of all three points is very close to the speed of light. So, naturally, μ-neutrinos arrive in the detector 60 ns ahead of schedule, while τ-neutrinos are 60 ns late. This is illustrated in Fig. 3.

4 Discussion

In this article we have formulated a simple model of oscillating neutrinos. This model satisfies all requirements of relativistic quantum theory: An unitary representation of the Poincaré group is constructed explicitly in the two-neutrino Hilbert space, and this representation takes into account interaction responsible for neutrino oscillations. Surprisingly, this simple model predicts an effect, which, to the best of author's knowledge, has not been noticed before: The two components of the neutrino beam may not overlap in space. They can be separated from each other by a macroscopically large distance $|\chi|$, and this separation does not violate any conservation law. This property can naturally explain the superluminal effect seen in the OPERA experiment if we assume that $\chi = -36$ meters for neutrinos with energies about 17 GeV. Unfortunately, our simple model cannot predict how the parameter χ depends on neutrino energy, but it is reasonable to assume that macroscopic separations between different neutrino flavors can be found at other energies too. The observed independence of the neutrino arrival time on its energy [1] suggests that χ remains nearly constant within the energy interval 13.6 - 42.9 GeV. This confirms the validity of our approximation $\alpha(p) \approx \frac{\chi}{\hbar}p$, i.e., the relative insignificance of terms nonlinear in p.

4.1 Comments on causality

According to our model, the OPERA result does not mean that neutrinos move faster than light. Nevertheless, they violate the special-relativistic ban on superluminal propagation in a different manner. The model presented above can be interpreted as a statement that the $\nu_\mu - \nu_\tau$ system has a large radius ($\approx$18 meters). The violation of special relativity occurs already at time $t = 0$, when such a big system is created *instantaneously* in a meson decay, while according to the traditional concepts, its creation must take at least 60 ns. So, our model implies that, indeed, there is a superluminal signal propagation. According to usual ideas, this is impossible, because the principle of causality would be violated. The traditional argument invokes Lorentz transformations of special relativity. They say that if (x, t) are space-time coordinates of a physical event in the reference frame at rest, then in the inertial frame moving with velocity $v \equiv c \tanh\theta$ space-time coordinates of the same event are given by formulas

$$x' = x\cosh\theta - ct\sinh\theta \tag{24}$$

$$t' = t\cosh\theta - (x/c)\sinh\theta \tag{25}$$

Special relativity postulates that these formulas remain valid in all circumstances, independent on the physical nature of the event occurring at (x, t) and interactions responsible for this event. The claim is that (24) - (25) express fundamental universal properties of the space-time.[11] The tacit or explicit assumption used in most discussions of quantum superluminal effects is that space-time arguments of wave functions must transform by the same formulas, i.e., that the position-space wave function transforms to the moving frame as

$$\psi(x,t) \to \psi(x\cosh\theta - ct\sinh\theta, t\cosh\theta - (x/c)\sinh\theta) \tag{26}$$

If this were true, then the appearance of ν_μ at point 0 in Fig. 2 would be scandalous, because, according to (26), one would be able to find a moving reference frame in which event 0 (creation of the μ-neutrino) has happened before event W (decay of the π-meson). So, in this moving frame the effect would occur *before* its cause, which is impossible.

However, there is absolutely no reason to believe in the transformation law (26) if we use the Newton-Wigner's definition of the particle's position [13] and Wigner-Dirac formulation of quantum dynamics [4]. In this theory, formula (26) is not valid even in the case of a single non-interacting particle. The correct transformation of the position-space wave function to the moving frame is

$$\psi'(x,t) = \langle x| e^{-\frac{i}{\hbar}H_0 t} e^{\frac{i}{\hbar}K_0 c\theta} |\psi\rangle$$

which is not the same as (26). This fundamental difference is demonstrated by the well-known effects of superluminal spreading of wave packets and the loss of particle localization in the moving frame [5, 6, 7, 8, 9].

In the interacting case the picture is even more complicated as one needs to use *interacting* energy and boost operators to find the wave function transformation

$$\psi'(x,t) = \langle x| e^{-\frac{i}{\hbar}Ht} e^{\frac{i}{\hbar}Kc\theta} |\psi\rangle$$

Let us consider the time evolution of the initial state (19) seen from the moving reference frame in the case of full mixing $A = B = 1/\sqrt{2}$

$$|\psi(\theta,t)\rangle = e^{-\frac{i}{\hbar}Ht} e^{\frac{i}{\hbar}Kc\theta}|\psi\rangle = \frac{1}{\sqrt{2}} \left(\begin{array}{c} e^{-\frac{i}{\hbar}E_1(\Lambda_1 p)t} \sqrt{\frac{E_1(\Lambda_1 p)}{E_1(p)}} \psi(\Lambda_1 p) \\ e^{-\frac{i}{\hbar}E_2(\Lambda_2 p)t} \sqrt{\frac{E_2(\Lambda_2 p)}{E_2(p)}} e^{-\frac{i}{\hbar}\chi\Lambda_2 p} \psi(\Lambda_2 p) \end{array} \right)$$

Switching to the flavor basis by usual formula (15) we obtain

[11] Here we intentionally avoid discussion of gravity, space-time curvature, etc.

$$
\begin{aligned}
|\psi(\theta,t)\rangle &= \frac{1}{2}\begin{pmatrix} 1 & e^{\frac{i}{\hbar}\chi p} \\ -e^{-\frac{i}{\hbar}\chi p} & 1 \end{pmatrix}\begin{pmatrix} e^{-\frac{i}{\hbar}E_1(\Lambda_1 p)t}\sqrt{\frac{E_1(\Lambda_1 p)}{E_1(p)}}\psi(\Lambda_1 p) \\ e^{-\frac{i}{\hbar}E_2(\Lambda_2 p)t}\sqrt{\frac{E_2(\Lambda_2 p)}{E_2(p)}}e^{-\frac{i}{\hbar}\chi\Lambda_2 p}\psi(\Lambda_2 p) \end{pmatrix} \\
&= \begin{bmatrix} e^{-\frac{i}{\hbar}E_1(\Lambda_1 p)t}\sqrt{\frac{E_1(\Lambda_1 p)}{E_1(p)}}\psi(\Lambda_1 p) + e^{\frac{i}{\hbar}\chi p}e^{-\frac{i}{\hbar}\chi\Lambda_2 p}e^{-\frac{i}{\hbar}E_2(\Lambda_2 p)t}\sqrt{\frac{E_2(\Lambda_2 p)}{E_2(p)}}\psi(\Lambda_2 p) \\ -e^{-\frac{i}{\hbar}\chi p}e^{-\frac{i}{\hbar}E_1(\Lambda_1 p)t}\sqrt{\frac{E_1(\Lambda_1 p)}{E_1(p)}}\psi(\Lambda_1 p) + e^{-\frac{i}{\hbar}\chi\Lambda_2 p}e^{-\frac{i}{\hbar}E_2(\Lambda_2 p)t}\sqrt{\frac{E_2(\Lambda_2 p)}{E_2(p)}}\psi(\Lambda_2 p) \end{bmatrix}
\end{aligned}
\tag{27}
$$

We will not analyze this result in detail here, just mention two remarkable features, which disagree with traditional interpretations of special relativity. First, the oscillation period observed from the moving frame *does not* scale with velocity according to the usual Einstein's time dilation formula: $T' \neq T\cosh\theta$ [15]. Second, even at $t = 0$, the probability of finding μ-neutrino is less than 1 and the probability of finding τ-neutrino is greater than 0. This means that definitions of the ν_μ state and ν_τ state are different for different observers. So, this oscillating system lacks clearly identified local events, whose space-time coordinates could be used in a rigorous discussion of causality. These two unusual features are very similar to the properties of unstable particles discussed in [16, 17, 18, 19].

Even if these difficulties are resolved, formula (27) cannot provide the definitive answer about causality, because in the real experiment we are not dealing with free neutrinos. The crucial superluminal effect (an instantaneous creation of the macroscopic two-neutrino system) occurs at the point of meson decay. Then, for a meaningful discussion, we need to include in our description the unstable meson and its decay products as well as interactions responsible for the meson decay and neutrino oscillations. These interactions are fundamentally different from "normal" interactions, e.g., between two charges. Nevertheless, it is instructive to note that boost transformations of space-time locations of events in relativistic Hamiltonian systems of interacting particles are different from Lorentz formulas (24) - (25) even in the classical (non-quantum) limit ([20, 21, 22] and section 11.2 in [3]). This fact is essential for the proof that instantaneous action-at-a-distance potentials remain instantaneous in all reference frames, so that causality is preserved (see section 11.4 in [3]). If we assume that similar arguments hold for decay/oscillation interactions as well, the no conflict with causality will be found in the OPERA superluminal results.

These arguments lead us to the conclusion that the system of oscillating neutrinos does not behave in a way expected from a naïve application of special relativity. However, this does not mean that the causality postulate is violated. A proper discussion of causality requires more realistic modeling of the neutrino preparation event. Such a modeling would be a promising line of further research, but it is beyond the scope of the present paper.

4.2 Other experiments and predictions

When the OPERA results are discussed, two other neutrino observations are usually mentioned. One of them is the MINOS experiment [14], which saw a hint of advanced propagation of μ-neutrinos, however, large experimental uncertainties did not allow the authors to make a definitive conclusion about superluminality. This experiment is different from OPERA[12] in the sense that the propagation time was measured between two neutrino detectors. In this case, according to our model, no superluminal effects can be observed as neutrino's speed is not different from c. The other experiment concerns observation of neutrinos arriving to the Earth from supernova SN1987A [23, 24, 25]. This observation confirmed that neutrino's speed coincides with the speed of light to a high precision, which is also consistent with our model.

Based on our study, three predictions can be formulated, which may be useful for those designing future experiments measuring neutrino propagation speed:

1. We predict that a more thorough remake of the MINOS experiment will confirm that the speed of neutrinos is not higher than the speed of light.

2. The observed superluminal effect in the OPERA setup is independent on the distance traveled by the neutrino beam. If the neutrino energy is kept at 17 GeV, then for any source-detector distance μ-neutrinos will arrive to the detector by 60 ns "too early".

3. If τ-neutrinos (instead of ν_μ) are detected in the OPERA setup, then the superluminal effect will disappear: ν_τ will be found in the detector later than expected. In the case of full mixing, the delay time is going to be 60 ns (i.e., 120 ns later than ν_μ).

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[12] where the point of neutrino emission has not been actually observed, but inferred (incorrectly, in our opinion) from the timing of the accelerated proton beam

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