The structure of Fourier series

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Fourier series is constructed basing on the idea to model the elementary oscillation $(-1,+1)$ by the exponential function with negative base, viz. $(-1)^n$.

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We inspect the dependence, say, on time of a bounded quantity f expanding it into the sum of periodic processes w_k. First of all, consider a discrete process f_j, rendered by the vector comprised of n values, $\mathbf{f} = (f_0, ..., f_j, ..., f_k, ..., f_{n-1})$, and, accordingly, expand it into the sum of vectors $\mathbf{w}_k = (w_{0k}, ..., w_{jk}, ..., w_{kk}, ..., w_{n-1.k})$:

$$\mathbf{f} = \sum_{k=0}^{n-1} c_k \mathbf{w}_k = c_0 \begin{pmatrix} w_{00} \\ \vdots \\ w_{j0} \\ \vdots \\ w_{k0} \\ \vdots \\ w_{n-10} \end{pmatrix} + \cdots + c_j \begin{pmatrix} w_{0j} \\ \vdots \\ w_{jj} \\ \vdots \\ w_{kj} \\ \vdots \\ w_{n-1j} \end{pmatrix} + \cdots + c_k \begin{pmatrix} w_{0k} \\ \vdots \\ w_{jk} \\ \vdots \\ w_{kk} \\ \vdots \\ w_{n-1k} \end{pmatrix} + \cdots + c_{n-1} \begin{pmatrix} w_{0n-1} \\ \vdots \\ w_{jn-1} \\ \vdots \\ w_{kn-1} \\ \vdots \\ w_{n-1n-1} \end{pmatrix}. \quad (1)$$

We seek for the coefficients c_k of the expansion (1).

The simplest discrete periodic process is described by the vector

$$\mathbf{w} = (1, q, q^2, ..., q^j, ..., q^k, ..., q^{n-1}), \quad \text{where } q = -1. \quad (2)$$

Process (2) is the oscillation $w_j = (-1)^j$ between 1 and -1 depending on integer argument j and has the period 2. We will generalize this configuration from $q = -1$ to

$$q_k = (-1)^{\frac{2k}{n}}. \quad (3)$$

Then process $\mathbf{w}_k$,

$$\mathbf{w}_k = (1, q_k, q_k^2, ..., q_k^j, ..., q_k^k, ..., q_k^{n-1}), \quad (4)$$

will have period n/k: starting from $w_{0k} = 1$ through $j = n/k$ points there will be again $w_{jk} = q_k^j = 1$, and at the intermediate value $j = n/(2k)$ the function is $w_{jk} = q_k^j = -1$. At $k = 0$ the period of the oscillation is infinite, i.e. the quantity is constant. At $k = n/2$ the process is identical to (2). When $k > n/2$ the period of the oscillation will be less than 2. In the whole, the frequency of the processes varies in the range $0 \le k/n < 1$.

The set of vectors $\mathbf{w}_k$ is orthogonal in the sense that

$$\mathbf{w}_j^* \cdot \mathbf{w}_k = \sum_{m=0}^{n-1} w_{mj}^{-1} w_{mk} = \sum_{m=0}^{n-1} q_j^{-m} q_k^m = \sum_{m=0}^{n-1} (-1)^{\frac{2(k-j)m}{n}} = n\delta_{jk} \quad (5)$$

where δ_{jk} is the Kronecker delta. Equality (5) follows from the formula of geometric progression

$$1 + p + p^2 + ... + p^{n-1} = (1 - p^n)/(1 - p) \quad (6)$$

with the denominator $p = (-1)^{2(k-j)/n}$.

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The property of orthogonality of vectors $\mathbf{w}_k$ is convenient by that multiplying the expansion (1) by vector $\mathbf{w}_k^*$, there can be immediately, making use of (5), found coefficient c_k:

$$\mathbf{w}_k^* \cdot \mathbf{f} = \sum_{m=0}^{n-1} w_{mk}^* f_m = nc_k \, . \tag{7}$$

In the result we obtain

$$f_j = \sum_{k=0}^{n-1} c_k (-1)^{\frac{2k}{n}j} , \tag{8}$$

$$c_k = \frac{1}{n} \sum_{m=0}^{n-1} f_m (-1)^{-\frac{2k}{n}m} . \tag{9}$$

The fractional power of -1, as in (8) and (9), can be reduced to $\sqrt{-1}$ and rendered in the exponential form with a positive base.
Theorem:

$$e^{\sqrt{-1}\varphi} = \cos\varphi + \sqrt{-1}\sin\varphi. \tag{10}$$

Proof (for definition of the Euler's number e see Appendix).
Indeed, on the one side we have:

$$e^{\sqrt{-1}\varphi_1} \cdot e^{\sqrt{-1}\varphi_2} = e^{\sqrt{-1}(\varphi_1 + \varphi_2)}.$$

On the other, by the trigonometry:

$$(\cos\varphi_1 + \sqrt{-1}\sin\varphi_1)(\cos\varphi_2 + \sqrt{-1}\sin\varphi_2) = \cos(\varphi_1 + \varphi_2) + \sqrt{-1}\sin(\varphi_1 + \varphi_2).$$

Besides, differentiating the left-hand side of (10) we obtain

$$\frac{d}{d\varphi} e^{\sqrt{-1}\varphi} = \sqrt{-1} e^{\sqrt{-1}\varphi}.$$

While the differential of the right-hand side (10) is

$$\frac{d}{d\varphi}(\cos\varphi + \sqrt{-1}\sin\varphi) = -\sin\varphi + \sqrt{-1}\cos\varphi = \sqrt{-1}(\cos\varphi + \sqrt{-1}\sin\varphi).$$

This is sufficient in order to substantiate equality (10).
Consider the change of $(-1)^x$ in dependence on x:

$$(-1)^0 = 1, \ \ (-1)^{1/2} = \sqrt{-1}, \ \ (-1)^1 = -1, \ \ (-1)^{3/2} = -\sqrt{-1}, \ \ (-1)^2 = 1.$$

According to (10), we have $\exp(\sqrt{-1}\pi) = -1$ and, consequently, $\exp(\sqrt{-1}\pi x)$ changes with x as

$$e^{\sqrt{-1}\pi \cdot 0} = 1, \ \ e^{\sqrt{-1}\pi \cdot 1/2} = \sqrt{-1}, \ \ e^{\sqrt{-1}\pi \cdot 1} = -1, \ \ e^{\sqrt{-1}\pi \cdot 3/2} = -\sqrt{-1}, \ \ e^{\sqrt{-1}\pi \cdot 2} = 1.$$

Thus we have shown that

$$(-1)^x = e^{\sqrt{-1}\pi x}. \tag{11}$$

Now we have the convenient form (11), (10) which visually demonstrates the oscillation as rotation of a unit vector in coordinates $(1, \sqrt{-1})$. Fig.1 may illustrate the distribution on the plane in these coordinates of components of a vector (4) with the denominator (3).

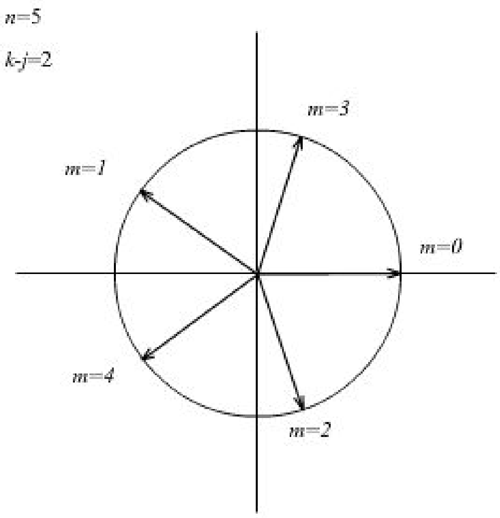

Figure 1: The products $\exp[i2\pi(k-j)m/n]$ of the components of orthogonal vectors in relation (5) for the case of the basis consisting of $n=5$ vectors are shown on the complex plane: m-th component of j-th and k-th vectors, $k-j=2$, $m=0,1,2,3,4$.

Using (11) in (8) and (9), we find the standard form of the Fourier expansion

$$f_j \;=\; \sum_{k=0}^{n-1} c_k e^{i\frac{2\pi k}{n}j}, \tag{12}$$

$$c_k \;=\; \frac{1}{n}\sum_{m=0}^{n-1} f_m e^{-i\frac{2\pi k}{n}m} \tag{13}$$

where the well-known designation for the imaginary unit $\sqrt{-1}=i$ is assumed.

Next we proceed to the continuous presentation supposing

$$t=j\Delta t, \quad T=n\Delta t=\texttt{const}, \quad \Delta t\to 0. \tag{14}$$

Using (14) in (12), (13):

$$f(t) \;=\; \sum_{k=0}^{n-1} c_k e^{i\frac{2\pi k}{n\Delta t}j\Delta t} \to \sum_{k=0}^{\infty} c_k e^{i\frac{2\pi k}{T}t}, \tag{15}$$

$$c_k \;=\; \frac{1}{n\Delta t}\sum_{m=0}^{n-1} f_m e^{-\frac{2\pi k}{n\Delta t}m\Delta t}\Delta t \to \frac{1}{T}\int_o^T f(t)e^{-i\frac{2\pi k}{T}t}dt. \tag{16}$$

In (15) and (16) T is the time duration of the process, and T/k period of the k-th harmonics.

Replacing in (15) and (16) $f(t)$ by $f(t+t_0)$ we obtain formulae for expansion of the function at any finite interval (t_0, t_0+T).

Notice that we may extend the geometric progression (6) at the same length n into the region of negative powers:

$$p^{-n}+p^{-n+1}+...+p^{-1}+1+p+p^2+...+p^{n-1}=(p^{-n}-p^n)/(1-p). \tag{17}$$

In this event, for the extended vectors $\mathbf{w}_k$ there holds the orthogonality with $q_k=(-1)^{k/n}$, so that we will have instead of (5)

$$\mathbf{w}_j^*\cdot\mathbf{w}_k=\sum_{m=-n}^{n-1} q_j^{-m}q_k^m=\sum_{m=-n}^{n-1}(-1)^{\frac{(k-j)m}{n}}=2n\delta_{jk}. \tag{18}$$

The symmetrical expansion has a more regular character: the periods of the discrete processes in question never become less than 2. At a given k the period equals to $2n/k$, i.e. two times more long than it is at the same k in the one-sided distribution (4). At $k = -n$ the period equals to 2, then with the increase of k it grows up to infinity at $k = 0$, and further drops gradually to almost 2 at $k = n - 1$. So that the frequency varies in the range $-1/2 \leq k/(2n) < 1/2$. In accord with (18), we recast relations (15) and (16) putting $2n$ in place of n:

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{i\frac{\pi k}{T}t}, \tag{19}$$

$$c_k = \frac{1}{2T} \int_{-T}^{T} f(t) e^{-i\frac{\pi k}{T}t} dt \tag{20}$$

where the function is taken at a finite interval $(-T, +T)$. If $f(t)$ is defined on the interval (t_1, t_2), then we have in (19), (20) $T = (t_2 - t_1)/2$, and $f(t)$ should be substituted by $f(t + (t_1 + t_2)/2)$.

Notice that if the function f is real, then (20) entails $c_{-k} = c_k^*$, where * is the sign of complex conjugation. We will represent the expansion of the real function in the real form. From (19), using Euler formula (10):

$$f(t) = c_0 + \sum_{k=1}^{\infty} \left(c_k e^{i\frac{\pi k}{T}t} + c_{-k} e^{-i\frac{\pi k}{T}t} \right) = c_0 + \sum_{k=1}^{\infty} \left[(c_k + c_k^*) \cos\frac{\pi k}{T}t + i(c_k - c_k^*) \sin\frac{\pi k}{T}t \right]. \tag{21}$$

Coefficients c_0, $c_k + c_k^*$, $i(c_k - c_k^*)$ of this expansion are real and can be determined from (20) as

$$c_k + c_k^* = \frac{1}{T} \int_{-T}^{T} f(t) \cos\frac{\pi k}{T} t dt, \tag{22}$$

$$i(c_k - c_k^*) = \frac{1}{T} \int_{-T}^{T} f(t) \sin\frac{\pi k}{T} t dt \tag{23}$$

where $c_0^* = c_0$.

Thus, at a finite interval a function can be expanded into the Fourier series.

Let the time interval be infinite: $T \to \infty$. We will redefine variables:

$$\frac{\pi k}{T} = \omega, \quad \frac{T c_k}{\pi} \to c(\omega). \tag{24}$$

Using (24) in (19) yields

$$f(t) = \sum_{k=-\infty}^{\infty} c(\omega) \frac{\pi}{T} e^{i\omega t} \to \int_{-\infty}^{\infty} c(\omega) e^{i\omega t} d\omega \tag{25}$$

since varying k by one ω changes by π/T, i.e. $\delta\omega = \pi/T$. Using (24) in (20):

$$c(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt. \tag{26}$$

So, we deal with the Fourier integral on the entire number axis.

There can be deduced a formula for the concise rendering and remembering of the Fourier expansion. First, refashion (25) as

$$f(t) = \int_{-\infty}^{\infty} d\omega' c(\omega') e^{i\omega' t}. \tag{27}$$

Substituting (27) into (26):

$$c(\omega) = \int_{-\infty}^{\infty} d\omega' c(\omega') \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} dt e^{i(\omega' - \omega)t} \right]. \tag{28}$$

From (28) we have

$$\delta(\omega' - \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt e^{i(\omega' - \omega)t}. \tag{29}$$

Similarly to (29), there can be written δ-function for t:

$$\delta(t' - t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{i\omega(t' - t)}. \tag{30}$$

Using the representation (30), we may easily obtain formula for the Fourier transform

$$f(t) = \int_{-\infty}^{\infty} dt' f(t') \delta(t' - t) = \int_{-\infty}^{\infty} d\omega \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} dt' f(t') e^{-i\omega t'} \right] e^{i\omega t}. \tag{31}$$

Compare (31) with (25) and (26).

Let

$$f(t) = \sum_k c_k e^{i\omega_k t} \tag{32}$$

i.e. we have a set of harmonic oscillators. Substituting (32) in (26):

$$c(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_k c_k e^{i(\omega_k - \omega)t} dt = \sum_k c_k \delta(\omega - \omega_k) \tag{33}$$

where the definition (29) is used. In reality, (32) is blurred, and the discrete spectrum (33) degrades into the sum of Gauss components

$$c(\omega) = \sum_k \frac{c_k}{\sqrt{2\pi\sigma_k^2}} \exp\left[-\frac{(\omega - \omega_k)^2}{2\sigma_k^2} \right] \tag{34}$$

that is shown in Fig.2.

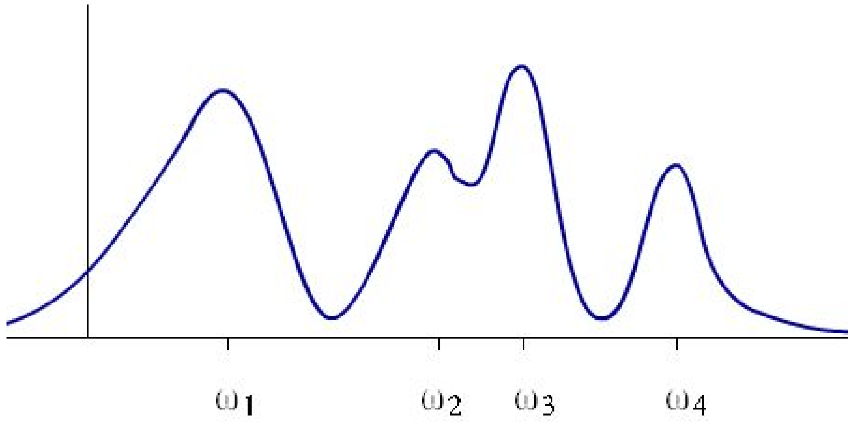

Figure 2: A realistic spectrum of the composite signal.

Appendix A: THE FIRST REMARKABLE LIMIT

Binomial $1+1/n$ raised to the power n at $n \to \infty$ is bounded by excess and deficiency in the following way:

$$2\frac{1}{2} < \left(1+\frac{1}{n}\right)^n = 1 + n\cdot\frac{1}{n} + \frac{n(n-1)}{2!}\frac{1}{n^2} + + \frac{n(n-1)(n-2)}{3!}\frac{1}{n^3} + ... \xrightarrow{n\to\infty}$$
$$1 + 1 + \frac{1}{2!} + \frac{1}{3!} + ... < 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + ... = 1 + \frac{1}{1-\frac{1}{2}} = 3\,.$$

Denoting

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \tag{A1}$$

we are seeking for

$$\frac{d}{dx}e^x = \lim_{\Delta x\to 0}\frac{e^{x+\Delta x} - e^x}{\Delta x} = e^x \lim_{\Delta x\to 0}\frac{e^{\Delta x}-1}{\Delta x}.$$

Supposing $\Delta x = 1/n$ gives

$$\lim_{\Delta x\to 0}\frac{e^{\Delta x}-1}{\Delta x} = \lim_{n\to\infty} n\left[\left(1+\frac{1}{n}\right)^{n\cdot\frac{1}{n}} - 1\right] = 1.$$

Hence:

$$\frac{d}{dx}e^x = e^x.$$

Relationship (A1) can be generalized. We have

$$S_1 = \left(1+\frac{m}{n}\right)^n = 1 + n\cdot\frac{m}{n} + \frac{n(n-1)}{2!}\left(\frac{m}{n}\right)^2 + ... \xrightarrow{n\to\infty} 1 + m + \frac{m^2}{2!} + \frac{m^3}{3!} + ...\,,$$
$$S_2 = \left(1+\frac{1}{n}\right)^{nm} = 1 + mn\cdot\frac{1}{n} + \frac{mn(mn-1)}{2!}\left(\frac{1}{n}\right)^2 + ... \xrightarrow{n\to\infty} S_1.$$

Therefore:

$$e^m = \left[\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right]^m = \lim_{n\to\infty}\left(1+\frac{m}{n}\right)^n.$$