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THE BERRY-KEATING HAMILTONIAN H=xp AND ITS SQUARE WITH BOUNDARY CONDITIONS $f(nx) = f(x) \quad n \in N$

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- *ABSTRACT: We present a Berry-Keating model with 'periodic' conditions in the dilation group so boundary conditions* $f(nx) = f(x) \quad n \in N$ *applied to the operator* $H_{bk}{}^2$ *with* $-i\left(x\frac{df}{dx} + \frac{f}{2}\right) = H_{bk}$*, for the square of the Berry-Keating operator with these boundary conditions we manage to prove that the Eigenvalues of* $H_{bk}{}^2$ *are approximately (in the semiclassical approximation)* $E_n = \frac{1}{4} + \left(\frac{2\pi n}{\log n}\right)^2$*, also we study the case of the Eigenvalues for the operator* $x\frac{df}{dx} + \frac{f}{2} = \lambda_n f$ *which are asymptotic to* $\frac{1}{2} + i\gamma_n$ *as* $n \to \infty$*,* $\zeta\left(\frac{1}{2} + i\gamma_n\right) = 0$*. For simplicity in this paper we will use units so* $2m = 1 = \hbar$ *also log means the natural logarithm*
- *Keywords: = Riemann Zeros, Quantum mechanics, WKB approach*

One of the first Hilbert-Polya operators proposed to solve RH was the Berry-Keating operator $H_{BK} = -i\left(x\frac{d}{dx} + \frac{1}{2}\right) = H_{BK}^{\dagger}$ This operator comes from the Quantization of the Hamiltonian $H_{BK} = xp$ with the Canonical Quantization conditions involving commutators $[x, p] = i$, the idea is that if we apply the Quantizatio rules to the Hamiltonian $H_{BK} = xp$ and apply certain regularization conditions , then we recover the asymptotics of the smooth term in the Riemann-Von Mangoldt formula $N(E) \approx N_{SC}(E) = \frac{Area}{2\pi} = \frac{E}{2\pi}\left(\log\left(\frac{E}{2\pi}\right) - 1\right) + \frac{7}{8} + ...$, with the semiclassical 'regularization' of the Planck unit-cell in the phase space as $|x| > l_x$, $|p| > l_p$ and the product $l_p . l_x = 2\pi$ for some cut-offs l_x, l_p

Now, We can can go an step further and consider the square of the Berry-Keating Hamiltonian in the form of an Eigenvalue problem

$$-x^2 \frac{d^2\Psi(x)}{dx^2} - 2x\frac{d\Psi(x)}{dx} - \frac{\Psi(x)}{4} = i\left(x\frac{d}{dx}+\frac{1}{2}\right)i\left(x\frac{d}{dx}+\frac{1}{2}\right)\Psi(x) = E_n\Psi(x) \quad (1)$$

This Hamiltonian defined in (1) is self-adjoint so its Eigenvalues will be real and also the expected value of the Hamiltonian will be positive $E_n \geq 0$ $\langle \Psi | H_{BK}^2 | \Psi \rangle = \langle H_{BK}\Psi | H_{BK}\Psi \rangle = \| H_{BK} \| \geq 0$. Equation (B.1) is of Euler-Cauchy type, its general solution will be given by $\Psi(x) = A_+ x^{-\frac{1}{2}+i\sqrt{E_n}} + A_- x^{-\frac{1}{2}-i\sqrt{E_n}}$ $A_\pm \in R$

Wihtout further boundary conditions then any positive Real number will be an eigenvalue of (1) with the Eigenfunctions given above, we will impose the restriction that the Eigenfunctions satisfy the 'periodic' conditions $\Psi(nx) = \Psi(x)$ for any positive integer 'n' different from 0 , this is motivated from the solution of the Eigenvalue problem in Quantum Mechanics

$$-\frac{d^2\Psi(x)}{dx^2} = p^2\Psi(x) = E_n\Psi(x) \qquad E_n = 4n^2\pi^2 \qquad \Psi(x+n) = \Psi(x) \qquad n \in Z \qquad (2)$$

The Hamiltonian inside (2) is the one corresponding to a free particle so $E = p^2$, here 'p' is the generator of the translations $\hat{p} \to -i\frac{\partial}{\partial x}$, the allowed values of the Hamiltonian in the WKB approximation are $p = 2n\pi$ for natural 'n' taking into account the boundary conditions for the eigenvalue problem $-i\frac{\partial \varphi}{\partial x} = \lambda_n \varphi \quad \varphi(x+n) = \varphi(x)$

For the case of Berry-Keating Hamiltonian we have used the Dilation operator $\Theta \to -i\left(x\frac{d}{dx}+\frac{1}{2}\right)$, so it seems quite plausible that we can impose boundary conditions inside our Hamiltonian that must be 'periodic' in dilations $\Psi(xn) = \Psi(x)$ for every natural number 'n' , in fact our Hamiltonian is invariant under the group of dilations $y = ax$. then from this symmetry and using the Solution of the Cauchy-Euler differential equation (1) we have that our Wave function must satisfy

$$A_+ x^{-\frac{1}{2}+i\sqrt{E_n}} + A_- x^{-\frac{1}{2}-i\sqrt{E_n}} = A_+ x^{\frac{2\pi ni}{\log n}} + A_- x^{-\frac{2\pi ni}{\log n}} \qquad x^{\pm\frac{2\pi in}{\log n}} . n^{\pm\frac{2\pi in}{\log n}} = (xn)^{\pm\frac{2\pi in}{\log n}} = x^{\pm\frac{2\pi in}{\log n}} \quad (3)$$

Since $n^{\pm\frac{2\pi in}{\log n}} = 1 = e^{2\pi in}$, here 'n' is a quantum number that labells the different energy levels , now if we make the exponents in both sides equal then we have three different equations (all equivalents) for the energy levels

$$s = i\sqrt{E_n} = \frac{1}{2} + \frac{2\pi i n}{\log n} \qquad \bar{s} = -i\sqrt{E_n} = \frac{1}{2} - \frac{2\pi i n}{\log n} \qquad s.\bar{s} = s.(1-s) = E_n = \frac{1}{4} + \left(\frac{2\pi n}{\log n}\right)^2 \quad (4)$$

Then we have managed to give a finite 'spectrum' for the square of the Berry-Keating Hamiltonian

$$\left(\frac{1}{4} + \left(\frac{2\pi n}{\log n}\right)^2\right)\Psi_n(x) = -\frac{d}{dx}\left(x^2 \frac{d}{dx}\right)\Psi_n(x) - \frac{\Psi_n(x)}{4} \qquad \Psi(xn) = \Psi(x) \quad (5)$$

From the Riemann-Von Mangoldt formula for the formula N(E) we find that the imaginary parts of the Riemann Zeros obey the following asymptotics $\gamma_n \approx \frac{2\pi n}{\log n}$, this means that perhaps the symmetry $\Psi(xn) = \Psi(x)$ is just an approximation and that will become only valid in the limit $n \to \infty$

A similar problem that ones encounters using the WKB approximation is the following

$$E_n \Psi_n(x) = -\frac{d^2\Psi_n(x)}{dx^2} + x\Psi_n(x) \qquad E_n = \left(\frac{3\pi n}{2}\right)^{2/3} \qquad (6)$$

In this case the EXACT energies should be the zeros of the Airy function $Ai(x)$, the semiclassical approximation gives that these roots are asymptotic to $x_n \to Dn^{2/3}$ as $n \to \infty$ $D = 1.31037..$,with $Ai(x_n) = 0$.

We can apply the same reasoning to the operator $x\frac{df}{dx} + \frac{f}{2} = \lambda_n f$, the solutions to this differential equation can be written as $\Psi(x) = H(x)C_+ x^{-\frac{1}{2}+i\lambda_n} + H(x)C_- x^{-\frac{1}{2}-i\lambda_n}$ with the aid of the Heaviside step function (also invariant under dilations $x \to nx$)
$H(x) = \begin{cases} x > 0 & 1 \\ x < 0 & 0 \end{cases}$, in this case imposing the boundary condition $f(nx) = f(x)$ for every natural number 'n' then the Eigenvalues $\lambda_n = \frac{1}{2} \pm \frac{2\pi n i}{\log n}$, in the limit of big 'n' the Eigenvalues will tend to the TRUE Riemann zeros $\lambda_n = \frac{1}{2} \pm \frac{2\pi n i}{\log n} \to \rho_n \quad \zeta(\rho_n) = 0$

For the differential operator $x\frac{df}{dx} + \frac{f}{2} = \lambda_n f$ if we writte the Eigenfunctions as

$f_n(x) = x^{-\frac{1}{2}+iE_n}$ and we want the Eigenvalues E_n to be equal to the imaginary part of the Riemann zeros, then the following approximate identity (explicit formula for the Chebyhsev function) must hold

$$\sum_{n=-\infty}^{\infty} f_n(x) \approx 1 - \sum_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} \delta(x-n) + \frac{1}{x^3 - x} \qquad \Lambda(n) \begin{cases} \log p & n = p^k \\ 0 & \text{otherwise} \end{cases} \qquad (7)$$

This identity is analogue to the sum proved by Euler and others $1 + 2\sum_{n=-\infty}^{\infty} \cos\left(2\pi nx\right) = 0$ which holds for every 'x' different from an integer, as it can be seen every term in the series satisfy the functional equation $f(x+n) = f(x)$ for every integer 'n' , in the case of the Berry operator, the Eigenfunctions satisfy (by application of iterative dilations) the functional identity $f(nx) = f(n^k x) = f(x)$ with $k \in Z$ and $n \in N$.

If we take $x \to \infty$ and $x \neq p^k$ (different from a prime or prime power) inside (7) then it becomes the boundary condition. $\sum_{n=-\infty}^{\infty} f_n(x) \approx 1$

Conclusions and Final Remarks:

- We have applied the Quantization rules for the Berry-Keating Hamiltonian $H = xp$ and its square $H^2 = (xp)^2$, we have obtained the Eigenvalues and Energies $\lambda_n = \frac{i}{2} - \frac{2\pi in}{\log n}$ (for 'xp') and $E_n = \frac{1}{4} + \left(\frac{2\pi in}{\log n}\right)^2$ using boundary conditions of the type $\Psi(nx) = \Psi(x)$
- The general non-constant function that satisfy $\Psi(nx) = \Psi(x)$ is given by a linear combination of $x^{\pm i\eta}$ with $\eta = \eta(n) = \frac{2\pi n}{\log n}$, if we take logarithm in both sides of $n^{\pm \frac{2\pi in}{\log n}} = 1$ with $\log x = \log|x| + i\theta + 2i\pi n$, we find $2\pi in = 2\pi in$, here 'n' plays the role of a quantum number labelling the Energies of the system
- In order to get a 'discrete' spectrum we need to impose the boundary conditions $\Psi(nx) = \Psi(x)$ otherwise we would have a continuum spectrum
- In both cases the factor $\frac{2\pi n}{\log n} \approx \gamma_n$ appears, for big n $n \to \infty$, this term is the first term in the asymptotic expansion of the imaginary part for the Riemann zeros $\zeta\left(\frac{1}{2} + i\gamma_n\right) = 0 \quad n \to \infty$, ther is no known closed expression for the imaginary parts of the zeros $\gamma_n \neq g(n)$.

- For the operator $H_{bk}^2 = \Delta$, the allowed values of the mometum operator are given by $-i\left(x\frac{d}{dx}+\frac{1}{2}\right) \to \frac{2\pi n}{\log n} = \frac{2\pi}{\lambda}$, this means that for $n \to \infty$ the Eigenvalues of the momentum operator are the imaginary parts of the Riemann zeros
- The differential operator $i\left(x\frac{d}{dx}+\frac{1}{2}\right)$ is invariant under the group of dilations so the natural boundary conditions to be imposed would be $\Psi(nx) = \Psi(x)$, or in a more general case $\Psi(n^k x) = \Psi(x)$ with n a natural number and $k \in Z$

- Through the paper we have used the semiclassical approach, we have ignored the oscillating term of the number of zeros $\frac{1}{\pi}\Im m \log \zeta\left(\frac{1}{2}+is\right)$, in the case of the semiclassical approximation the Energies are obtained from solving the equation $N(E_n) \approx \int_\Gamma \frac{dqdp}{2\pi} H(E-qp) \approx \frac{E}{2\pi}\left(\log\left(\frac{E}{2\pi}\right)-1\right)+\frac{7}{8} = n+\frac{1}{2} \approx n$ (here 'H' stands for the Heaviside's step function not the Hamiltonian operator)

- Using the boundary condition $\Psi(nx) = \Psi(x)$ we have obtained the asymptotic (smooth) expansion (only the first term) for the imaginary parts of the Zeros, however we do not know if this symmetry (invariance under dilations) will be an exact symmetry or only an initial approximation to the true symmetry, since we are dealing with the semiclassical approximation we believe that our symmetry will be exact in the limit $n \to \infty$

- In both cases for the roots of the Airy function $Ai(x)$ and the Riemann zeta $\zeta\left(\frac{1}{2}+is\right)$ we do not know how to compute the zeros exactly, if we use the semiclassical approximation in QM for the Hamiltonians $H = p^2 + x$ and $H = xp$, we obtain the first term in the asymptotic expansion as $n \to \infty$ of the Airy $x_n \approx \left(\frac{3\pi n}{2}\right)^{2/3}$ and Riemann zeros $\gamma_n \approx \frac{1}{2}+\frac{2\pi n i}{\log n}$, this is because in the number of zeros less than a given 'E' we have ignored the oscillating term $\frac{1}{\pi}\Im m \log \zeta\left(\frac{1}{2}+is\right)$ and use the Stirling's approximation for the logarithmic derivative of the Gamma function

- The infinite potential well model $-\Delta = -\frac{d^2}{dx^2}$ with periodic boundary conditions $\Psi(x+n) = \Psi(x)$ can be easily generalized to several dimension by using the Laplace operator $-\Delta f = -\sum_i \frac{\partial^2 f}{\partial x_i^2}$, then the phase space is the one of a n-dimensional torus, for the case of the Berry-Keating operator we do not know how to generalize these results for more than one dimension

Although we have made a superficial introduction to the Berry-Keating Hamiltonian and its motives, the reader can find more references in the paper [9] ,[10] from sierra Townsed and Laguna, the Riemann-Von Mangodlt formula is described in the book [2] from Apostol, also a good introduction to Quantum mechanics and semiclassical approximations is given in [4] by Griffits or in the book of Statistical Mechanics [6] by Kittel

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