The Fine Structure Constant Interpretations of Quantum Theory

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Abstract: The fine structure constant give a simple derivation of the localized wavefunction, Schrödinger equation and the uncertainty principle in Quantum theories.

Keywords: Fine structure constant, Wave function, Uncertainty Principle, Quantum Theory.

The fine-structure constant α and proton-to-electron mass ratio β are deeply involved in the Quantum theory. [1, 2] Pauli considered quantum mechanics to be inconclusive without understanding of the fine structure constant. [2] Feynman also said that nobody truly understands quantum mechanics. [3] In this short paper, we discuss the fine structure constant interpretation of Quantum Theory. [4]

(1) Wavefunction in Quantum Theory

One basic problem in the quantum interpretation is the wavefunction. From 1926 to 1928, there are some proposed quantum wave equations $\boldsymbol{\hat{H}}\Psi = \boldsymbol{i\hbar\frac{\partial}{\partial t}}\Psi$, for example

$$\begin{aligned}
[\tfrac{-\hbar}{2\mu}\nabla^2 - V(r)]\Psi(\mathbf{r},t) &= i\hbar\tfrac{\partial}{\partial t}\Psi(\mathbf{r},t) && (a)\ \textit{Schrödinger} && (1)\\
[\tfrac{1}{2m}(\vec{\sigma}\cdot(\vec{p}-q\vec{A}))^2 + q\phi]\Psi_{1,2} &= i\hbar\tfrac{\partial}{\partial t}\Psi_{1,2} && (b)\ \textit{Pauli}\\
[\nabla^2 - (\tfrac{mc}{\hbar})^2]\Psi(\mathbf{r},t) &= \tfrac{1}{c^2}\tfrac{\partial}{\partial t}\Psi(\mathbf{r},t) && (c)\ \textit{Klein-Gordon}\\
[\beta mc^2 + \sum_{k=1}^{3}\alpha_k P_k c]\Psi_{1,2,3,4} &= i\hbar\tfrac{\partial}{\partial t}\Psi_{1,2,3,4} && (d)\ \textit{Dirac}
\end{aligned}$$

where the *non*-relativistic equations are (a) for the atomic electron configuration and (b) for the Dirac limitation of spin-1/2 particles, and the other two relativistic quantum equations are (c) for the spin-0 free particle and (d) for the spin-1/2 particle-antiparticles. The different Hamiltonian $\hat{H}$ on the left-side are proposed based on the imitations of various classical physical process, such as, the Schrodinger's Hamilton is imitating the optical wave equation.

The eigenvalues of a negatively charged electron orbiting the positively charged nucleus is determined by the time-independent Schrödinger equation $\boldsymbol{\hat{H}}\Psi(\mathbf{r}) = \boldsymbol{E}\Psi(\mathbf{r})$. [5] Wavefunction $\Psi_{nlm} = \mathrm{R}_{nl}(\mathbf{r})\,\mathrm{Y}_{lm}(\theta,\phi)$ (n, l, m are quantum numbers) is geometrically quantized in a 3D cavity. According to Born, Ψ is the probability amplitude and $\Psi\Psi^*$ is the probability density. [6] Each state only allows two counter-spin electrons ($\boldsymbol{m_s} = \pm\frac{1}{2}$) as the Pauli principle, $\mathbf{e}^2\left|\Psi_{nlm}\right|^2 = \alpha\hbar c\left|\Psi_{nlm}\right|^2$ is the probability density of the paired charge.

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The Schrödinger wavefunction Ψ_{nlm} indeed describes the *twins* in a box. Dirac relativistic quantum equation solves the spin-1/2 problem. Then, $\mathbf{e}\left|\Psi_{nlmm_s}\right|^2 = \sqrt{\alpha\hbar c}\left|\Psi_{nlmm_s}\right|^2$ is the probability density of the charge. The fine structure constant can be defined as the conservation of angular momentum

$$\frac{\mathbf{e}^2}{c} = \pm\alpha\hbar \tag{2}$$

The wavefunction had an entropy format $S = k\ln\Psi$ for $\hat{\boldsymbol{H}}\Psi = \boldsymbol{E}\Psi$ in Schrödinger's first paper in 1926. [5] The Boltzmann constant k is linked to α by the dimensionless blackbody radiation constant α_R and primes [7]

$$\alpha_R = \mathbf{e}^2(\tfrac{4\sigma}{ck^4})^{1/3} = (\prod\tfrac{\mathbf{p}^2}{\mathbf{p}^2+1})^{1/3}\alpha = 0.86976680\alpha = \tfrac{1}{157.555} \tag{3}$$

A plane wave function $\Psi(r,t)$ and the Born probability density $|\Psi(r,t)|^2$ are

$$\begin{aligned} \Psi(\mathbf{r},t) &= Ae^{-\frac{i}{\hbar}(\mathbf{p}\cdot\mathbf{r}-Et)} = Ae^{-\frac{i}{\hbar}Et}\psi(\mathbf{r}) = f(t)\psi(\mathbf{r}) \\ |\Psi(\mathbf{r},t)|^2 &= \Psi\Psi^* = [Ae^{-\frac{i}{\hbar}Et}\psi(\mathbf{r})][Ae^{-\frac{i}{\hbar}Et}\psi(\mathbf{r})]^* \\ &= Ae^{-\frac{i}{\hbar}Et}\psi(\mathbf{r})\cdot Ae^{\frac{i}{\hbar}Et}\psi^*(\mathbf{r}) = A^2|\psi(\mathbf{r})|^2 \end{aligned} \tag{4}$$

(2) Schrödinger Equation Derivation from α

We notice that $\mathbf{e}^2/c$ in (2) has the same dimension with $\hbar$, Et and $\mathbf{p}\cdot\mathbf{r}$. Let's use $i = -1/i$, and $e^{-i\pi} = -1 = i^2 = \mathrm{ln}e^{-1}$ to rewrite (2)

$$\tfrac{\mathbf{e}^2}{c} = \underline{\mathbf{p}\cdot\mathbf{r} - Et} = \pm\alpha\hbar = \pm\underline{i^2}\cdot\alpha\hbar\cdot\underline{\mathrm{ln}e^{-1}} = \underline{i\hbar\ln e^{\mp i\alpha}} \tag{5}$$

Applying the Einstein/de Broglie wave-particle duality [8,9]

$$\begin{aligned} E &= \hbar\omega \\ \mathbf{p} &= \hbar\mathbf{k} \end{aligned} \tag{6}$$

with conservation of angular momentum in the reduced Planck (Dirac) constant $\hbar$, [10,11]

$$\mathrm{ln}e^{\mp i\alpha} = -\tfrac{i}{\hbar}(\mathbf{p}\cdot\mathbf{r}-Et) = -i(\mathbf{k}\cdot\mathbf{r}-\omega t) = \int_A^\Psi\tfrac{du}{u} = \ln\tfrac{\Psi}{A} \tag{7}$$

In Hilbert space ($e^{i\hat{\boldsymbol{H}}t/\hbar} = e^{iEt/\hbar}$), a *non*-localized plane wavefunction $\Psi(\mathbf{r},t) = Ae^{-\frac{i}{\hbar}Et}\psi(\mathbf{r})$, i.e., $\Psi(\mathbf{r},t) = Ae^{-i(\mathbf{p}\cdot\mathbf{r}-Et)/\hbar}$ is defined as the exponential (7), and ***the wavefunction can be locally quantized by the fine structure constant*** as (5)

$$\Psi(\mathbf{r},t) = Ae^{-\frac{i}{\hbar}(\mathbf{p}\cdot\mathbf{r}-Et)} = Ae^{\mp i\alpha} = A\cos(\mp\alpha) \mp iA\sin(\mp\alpha) = a \mp bi \tag{8}$$

In (8), obviously, $\Psi(\mathbf{r},t) = \sum a_n\psi_n(\mathbf{r}) = \sum a_n e^{-iEt/\hbar}\psi_n(\mathbf{r})$ and $\int|\Psi|^2 dV = 1$. It is a Hermitian function, where the real part is an *even* function and the imaginary part is an *odd* function. From $\Psi(\mathbf{r},t) = Ae^{-\frac{i}{\hbar}(\mathbf{p}\cdot\mathbf{r}-Et)} = Ae^{-\frac{i}{\hbar}(P_x\cdot x+P_y\cdot y+P_z\cdot z-Et)}$,

$$\begin{aligned} \tfrac{\partial}{\partial t}\Psi(\mathbf{r},t) &= -\tfrac{i}{\hbar}E\Psi(\mathbf{r},t) \\ \tfrac{\partial^2}{\partial x^2}\Psi(x,t) &= -\tfrac{1}{\hbar^2}P_x^2\Psi(x,t) \end{aligned} \tag{9}$$

i.e., ***the operators can also be derived mathematically***

$$\begin{aligned} i\hbar\tfrac{\partial}{\partial t}\Psi(\mathbf{r},t) &= E\Psi(\mathbf{r},t) \\ -\hbar^2\tfrac{\partial^2}{\partial x^2}\Psi(x,t) &= P_x^2\Psi(x,t) \end{aligned} \tag{10}$$

The Hamiltonian $\hat{\boldsymbol{H}} = \hat{\boldsymbol{T}} + \hat{\boldsymbol{V}} = \boldsymbol{E}$ is the sum of kinetic and potential energy, and $\boldsymbol{T} = \frac{1}{2\mu}\mathbf{P}^2 = \frac{1}{2\mu}(P_x^2 {+} P_y^2 {+} P_z^2)$. In this way, the time-independent ***Schrödinger equation can be derived from the fine structure constant*** (2)

$$[\tfrac{-\hbar}{2\mu}\nabla^2 - V(r)]\Psi(\mathbf{r},t) = E\Psi(\mathbf{r},t) \tag{11}$$

From (8), the $\Psi(\mathbf{r},t) = Ae^{-\frac{i}{\hbar}(\mathbf{p}\cdot\mathbf{r}-Et)} = Ae^{-i\alpha}$ is for a negatively charged electron orbiting a positively charged nucleus, which has the complex number conjugates $\Psi(\mathbf{r},t) = a - bi$ and $\Psi(\mathbf{r},t)^* = a + bi$, then $\Psi\Psi^*$ is a real number for the probability density

$$|\Psi|^2 = \Psi\Psi^* = (a - bi)\,(a + bi) = a^2 + b^2 \tag{12}$$

Specially, if $\Psi = a - ai$, then the normalization from (12) yields $a = 1/\sqrt{2}$.

After localization, the wavefunctions (variables) become simple fixed complex numbers (vectors in Hilbert space). The complex addition Ψ_{12} of $\Psi_1 = a + bi$ and $\Psi_2 = c + di$ is

$$\Psi_{12} = \Psi_1 + \Psi_2 = (a + bi) \pm (c + di) = (a \pm c) + (b \pm d)i \tag{13}$$

There is an interference term when calculating the probability density from (10)

$$\begin{aligned} \Psi_{12}\Psi_{12}^* &= [(a \pm c) + (b \pm d)i][(a \pm c) - (b \pm d)i] \\ &= (a \pm c)^2 + (b \pm d)^2 \\ &= a^2 + b^2 + c^2 + d^2 \pm \underbrace{2(ac + bd)} \end{aligned} \tag{14}$$

This enlightens the physical interpretation of wavefunction. ***There is a hidden fine-structure constant*** α ***in the localized wavefunction*** (8). It naturally yields the eigenvalues $E_n = -E_e\underline{(\frac{\alpha}{n})^2}$ from Schrödinger equation, where $E_e = \frac{1}{2}m_ec^2$; and the fine-structures $\triangle E_F = \pm E_e\underline{(\frac{\alpha}{n})^4}[\frac{3}{4} - \frac{n}{j+1/2}]$ from Dirac relativistic equation, where $j = l + m_s$ is the total angular momentum. [4] In the magnetic field, QED yields the Lamb shift $\triangle E_L = E_e\underline{\frac{\alpha^5}{2n^3}}[k(n,l) \pm \frac{1}{\pi(j+1/2)(l+1/2)}]$, and the hyperfine structures due to the “nuclear spin” $\triangle E_H = \pm E_e\underline{\frac{g_p}{\beta}\frac{\alpha^4}{n^3}}\left[\frac{\boldsymbol{F}(\boldsymbol{F}+1)-j(j+1)-\boldsymbol{I}(\boldsymbol{I}+1)}{j(j+1)(2l+1)}\right]$, where the total angular momentum $\boldsymbol{F}$ is equal to the total nuclear spin $\boldsymbol{I}$ plus the total orbital angular momentum $\boldsymbol{J}$; the proton g-factor $g_p = 5.585$. This gives experimental confirmation of a connection between $\alpha \cong 1/137$ and $\beta \cong 1836$. In fact, the reduced masses $\mu = \frac{m_e\cdot m_p}{m_e+m_p} = \frac{\beta\cdot m_e}{\beta+1} \approx m_e$ always involve β in the Schrödinger equation (11). Quantum theory not only describes the electron configurations around the nucleus, but also considers the effect of nuclear mass. Neglecting interaction between electrons, the above eigenvalues are

$$\begin{aligned} &E_n{=}{-}E_\mu\underline{(\tfrac{\alpha}{n})^2} \propto \underline{\boldsymbol{\alpha^2} = 5.3{\times}10^{\boldsymbol{-5}}} \\ &\triangle E_F{=}{\pm}E_\mu\underline{(\tfrac{\alpha}{n})^4}[\tfrac{3}{4}{-}\tfrac{n}{j+1/2}] \propto \underline{\boldsymbol{\alpha^4} = 2.8{\times}10^{\boldsymbol{-9}}} \\ &\triangle E_L{=}E_\mu\underline{\tfrac{\alpha^5}{2n^3}}[k(n{,}l){\pm}\tfrac{1}{\pi(j+1/2)(l+1/2)}] \propto \underline{\boldsymbol{\alpha^5} = 2.1 \times 10^{\boldsymbol{-11}}} \\ &\triangle E_H{=}{\pm}E_\mu\underline{\tfrac{g_p}{\beta}\tfrac{\alpha^4}{n^3}}\left[\tfrac{\boldsymbol{F}(\boldsymbol{F}+1)-j(j+1)-\boldsymbol{I}(\boldsymbol{I}+1)}{j(j+1)(2l+1)}\right] \propto \underline{\boldsymbol{\alpha^4}/\beta = 1.5{\times}10^{\boldsymbol{-12}}} \end{aligned} \tag{15}$$

where $E_\mu = \frac{1}{2}\mu c^2 = 0.25536[\mathrm{MeV}]$. In (15), we have $E_n > \triangle E_F > \triangle E_L > \triangle E_H$ since $\alpha^2 > \alpha^4 > \alpha^5 > \alpha^4\beta^{-1}$.

The interpretation of entropy $S = k\ln\Psi$ in the statistical mechanics is the measure of *uncertainty*, or *mixedupness* to paraphrase Gibbs. [12] The wave-function Ψ is the number of micro-states as the statistical information as the probability amplitude.

(3) Uncertainty Principle Derivation from α

Landau reviewed the uncertainty principle as $\triangle P\triangle t \gtrsim \frac{\hbar}{c}\alpha^{1/2}$. He also pointed out a weakness in the interpretation of the uncertainty principle, such as, $\triangle E\triangle t \gtrsim \frac{\hbar}{2}$ "does not mean that the energy can not be measured with arbitrary accuracy within a short time." [13] Bohr used similar derivation on the debate in 1927. [13] Here we show, ***the uncertainty principle can also be derived from the fine structure constant*** (2), i.e., from $\mathbf{e}^2/c$ which has the same dimension with $\hbar$, Et and $\mathbf{p}\cdot\mathbf{r}$

$$\begin{gathered}\underline{\frac{\hbar}{2}} = \underline{\frac{\mathbf{e}_1\mathbf{e}_2}{2\alpha c}} = \frac{\mathbf{e^2}}{2v_e} = \frac{1}{2}\frac{\mathbf{e^2}}{\triangle x}\triangle t = \frac{1}{2}\underline{\triangle E\triangle t} = \\ \tfrac{1}{2}\tfrac{\mathbf{e^2}}{\triangle x^2}\triangle t\triangle x = \tfrac{1}{2}m_e\mathbf{a}\triangle t\triangle x = \tfrac{1}{2}m_e\triangle v\triangle x = \tfrac{1}{2}\underline{\triangle P_x\triangle x}\end{gathered} \tag{16}$$

where $E = \mathbf{e^2}/r$, $F = \mathbf{e^2}/r^2 = m\mathbf{a}$, $v = \mathbf{a}t$ and $P = mv$ are simple relationships in classical physics. In (16), if $\triangle t$ decease, $\triangle x$ will decease and make $\triangle E$ increase; so does $\triangle x$ and $\triangle P_x$. The $\triangle x$ and others can be defined as the standard deviations, e.g.,

$$\triangle x = \sigma_x = \sqrt{\tfrac{1}{N}\sum_{i=1}^{N}(x_i - \overline{x})^2} \tag{17}$$

If $\triangle x \to 0$, then the symmetric standard deviation $\sigma_{P_x} \gtrsim \frac{1}{2}\triangle P_x$. Therefore,

$$\begin{aligned}\triangle E\triangle t &\gtrsim \tfrac{\hbar}{2} \\ \triangle P_x\triangle x &\gtrsim \tfrac{\hbar}{2}\end{aligned} \tag{18}$$

Due to the conservation of angular momentum, the property of $\mathbf{e}_1$ must change if $\mathbf{e}_2$ is altered. The electrons in a atom are high-speed (αc) charged particles. The distance between each is constantly changing, which causes the measurement uncertainty. The nucleon has a micro-motion and thus the electron orbit could not be a perfect circle. The orthogonal and anti-commute pair $[\mathbf{A}, \mathbf{B}] = \pm i\hbar$, then $\triangle\mathbf{A}\triangle\mathbf{B} \geq \hbar/2$ given by Heisenberg-Kennard. [14,15]

The Quantum Theory is based on some postulates. The interpretation of Quantum Theory has been of great debate among theoretical physicists, specially, Bohr and Einstein. [13] Einstein was quoted: *God does not play dice*! However, he not only was a pioneer on Brownian motion, but also supported the Born statistic explanation of *Psi*-function. He had argued the realism, completeness, determinism, EPR-paradox, etc. What he really looking for is ***A DEFINITE RULE***. In 1953, he said "***That the Lord should play dice, all right; but that He should gamble according to definite rules, that is beyond me.***" [16] Here we show, the fine structure constant α is the ruler of quantum theory. The wavefunction, Schrödinger equation and Uncertainty principle can all be derived from it. Just like Pauli and Feynman point out: Quantum Theory to be inconclusive without understand the fine structure constant. [2,3]

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References

[1] M. Born, The Mysterious Number 137. *Proc. Indian Acad. Sci.* 2 (1935)

[2] W. Pauli, Exclusion Principle and Quantum Mechanics. *Nobel Lecture* (1946); The Connection Between Spin and Statistics. *Phys. Rev.*, **58**, 716-722 (1940)

[3] R. P. Feynman, QED: The Strange Theory of Light and Matter. *Princeton, NJ: Princeton University Press*, 129 (1985)

[4] A. Sommerfeld, Zur Quantentheorie der Spektrallinien. *Ann. Phys.* **51**, 1-94 (1916); P. A. M. Dirac, The Principles of Quantum Mechanics (Fourth Edition). *The Clarendon Press, Oxford*, 272 (1958)

[5] E. Schrödinger, Quantisierung als Eigenwertproblem. *Ann. Physik.* (Leipzig) **79**, 361, 489 (1926); **80**, 437 (1926); **81**, 109 (1926)

[6] M. Born, Zur Quantenmechanik der Stoßvorgänge, *Zeitschrift für Physik* **37** 863-867 (1926); The Statistical Interpretations of Quantum Mechanics, *Nobel Lecture* (1954)

[7] K. Xiao, Dimensionless Constants and Blackbody Radiation Laws. *EJTP*, **8** 379-388 (2011)

[8] A. Einstein, Concerning an Heuristic Point of View Toward the Emission and Transformation of Light. *Ann. Phys.* **17** 132-148 (1905)

[9] L. de Broglie, Ondes et Quanta. *Compt. Ren.* **177** 507-510 (1923)

[10] M. Planck, Ueber das Gesetz der Energieverteilung im Normalspectrum. *Ann. Phys.* **309** 553–563 (1901)

[11] P. A. M. Dirac, Zur Quantentheorie des Elektrons, *Leipziger Vorträge* (*Quantentheorie und Chemie*), 85-94 (1928)

[12] J. W. Gibbs, Elementary Principles in Statistical Mechanics. *New York: Scribner* (1902)

[13] L. D. Landau, R. Peierls, Extension of The Uncertainty Principle to Relativistic Quantum Theory; N. Bohr, Discuss With Einstein. *Quantum Theory and Measurement*. Ed. J. A. Wheeler and W. H. Zurek, 465-476 (1983)

[14] W. Heisenberg, Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik, *Zeitschrift für Physik* **43**, 172-198 (1927)

[15] E. H. Kennard, Zur Quantenmechanik einfacher Bewegungstypen. *Zeitschrift für Physik* **44**, 326-352 (1927)

[16] A. Einstein, Letter to Max Born (4 December 1926); *The Born-Einstein Letters* (translated by Irene Born) Walker and Company, New York, (1971); quoted in J. Wheeler, W. Zurek, Quantum and Measurement, Princeton U. Oress, p8 (1983)