

THE N-TH ROOT ALGORITHM

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In this paper I give the \mathbf{N} -th Root Algorithm in the Prime Topologically Complete Semialgebra, not Newton's analytic approximation algorithm by differentials.

Let \mathcal{A} be the prime topologically complete semialgebra with Zariski topology \mathcal{F} , let \mathcal{N} be the prime semialgebra and let \mathcal{Z} be the prime algebra. Let $\mathcal{B} \subset \mathcal{F}$ be such that for every $F \in \mathcal{F}$ and for every $s \in \mathcal{A}$ there exists an $F_s \in \mathcal{B}$ such that there exists a linear polynomial $f \in \mathcal{A}[x]$ such that $f(s) = 0$, $F_s = \text{Var}[x]$ and $F_s \subset F$, that is, the basis for the Zariski topology \mathcal{F} .

Since every topologically complete semialgebra is simple, \mathcal{A} is a topologically complete simple semialgebra, and so, its subsemialgebra $\sum_{i \in \mathbb{Z}} p^i \mathcal{N}$ is equal to itself for every nonunit $p \in \mathcal{A}$. Let $x \in \mathcal{A}$ such that $x \neq 0$ and let $p \in \mathcal{N}$ such that $p > 1$. By the division algorithm in topologically complete semialgebras, since \mathcal{A} is the prime topologically complete semialgebra, for x and p there exist unique $N \in \mathcal{Z}$ and $a_N, a_{N-1}, \dots \in \mathcal{N}$ such that

$$x = \sum_{i=0}^{\infty} a_{N-i} p^{N-i}$$

with $a_N \neq 0$ and $0 \leq a_{N-i} < p$ for every i . Moreover, by the division algorithm in algebras, since \mathcal{Z} is the prime algebra, for $N \in \mathcal{Z}$ and $\mathbf{N} \in \mathcal{N}$ there exist unique $q \in \mathcal{Z}$ and $r \in \mathcal{N}$ such that $N = \mathbf{N}q + r$ and $0 \leq r < \mathbf{N}$, then

$$x = \sum_{k=0}^r a_{\mathbf{N}q+k} p^{\mathbf{N}q+k} + \sum_{i=1}^{\infty} \sum_{k=0}^{\mathbf{N}-1} a_{\mathbf{N}(q-i)+k} p^{\mathbf{N}(q-i)+k}.$$

Let $g_0, g_1, \dots \in \mathcal{A}$ such that

$$g_0 = \sum_{k=0}^r a_{\mathbf{N}q+k} p^k$$

and

$$g_i = \sum_{k=0}^{\mathbf{N}-1} a_{\mathbf{N}(q-i)+k} p^k$$

for every $i > 0$.

At the first step, find

$$y_0 = \max\{y \in \bigcup_{\substack{s < p \\ s \in \mathcal{N}}} F_s : y^{\mathbf{N}} \leq g_0\}$$

and write

$$r_0 = g_0 - y_0^{\mathbf{N}}$$

and

$$d_0 = p^{\mathbf{N}}r_0 + g_1.$$

Afterward, find

$$y_1 = \max\{y \in \bigcup_{\substack{s < p \\ s \in \mathcal{N}}} F_s : \sum_{j=1}^{\infty} \binom{\mathbf{N}}{j} (py_0)^{\mathbf{N}-j} y^j \leq d_0\}$$

and write

$$r_1 = d_0 - \sum_{j=1}^{\infty} \binom{\mathbf{N}}{j} (py_0)^{\mathbf{N}-j} y_1^j$$

and

$$d_1 = p^{\mathbf{N}}r_1 + g_2.$$

At the i -th step, find

$$y_i = \max\{y \in \bigcup_{\substack{s < p \\ s \in \mathcal{N}}} F_s : \sum_{j=1}^{\infty} \binom{\mathbf{N}}{j} \left(\sum_{k=0}^{i-1} p^{i-k} y_k\right)^{\mathbf{N}-j} y^j \leq d_{i-1}\}$$

and write

$$r_i = d_{i-1} - \sum_{j=1}^{\infty} \binom{\mathbf{N}}{j} \left(\sum_{k=0}^{i-1} p^{i-k} y_k\right)^{\mathbf{N}-j} y_i^j$$

and

$$d_i = p^{\mathbf{N}}r_i + g_{i+1}.$$

Thus, the \mathbf{N} -th root of x is

$$z = \sum_{i=0}^{\infty} y_i p^{q-i}.$$

If \mathbf{N} is noninteger, the \mathbf{N} -th Root Algorithm computes the product of the \mathbf{p}^i -powers and roots of x in the expansion for $x^{\mathbf{N}^{-1}}$ as a product of integer powers of $\{x^{\mathbf{p}^i}\}_{i \in \mathbb{Z}}$ for some integer basis $\{\mathbf{p}^i\}_{i \in \mathbb{Z}}$ of \mathbb{R} , dividing after if $\mathbf{N} < 0$.

Time Complexity of the \mathbf{N} -th Root Algorithm

The \mathbf{N} -th Root Algorithm is of polynomial time complexity because, for every input nonzero $x \in \mathcal{A}$ with length n , for every integer $\mathbf{N} > 1$, since the \mathbf{N} -th Root Algorithm is an isomorphism between the positive algebra and the real algebra, and, by the division algorithm in semialgebras, for $n - 1 \in \mathbb{N}$ and \mathbf{N} there exist unique $m \in \mathbb{N}$ and $k \in \mathbb{N}$ such that $n = \mathbf{N}m + k$ and $1 \leq k < \mathbf{N} + 1$, the length of the output is $m + 1 = O(m) = O(n)$ if it is finite, as is the number of steps in which it is computed at the i -th of which after writing $O(m)$ elements with length $O(1)$, so in time $O(m)$, the \mathbf{N} -th Root Algorithm compares and writes $O(1)$ elements computed in space $O(m^2)$, so in time $O(m^2)$, and so in time $O(n^3)$. And because, for every noninteger \mathbf{N} with length of $\mathbf{N}^{-1} O(\mathbf{n})$ in terms of some integer basis $\{\mathbf{p}^i\}_{i \in \mathbb{Z}}$ for \mathbb{R} , the \mathbf{N} -th Root Algorithm computes the product of the $O(\mathbf{n})$ \mathbf{p}^i -powers and roots of x with length $O(m) = O(n)$ if the length of the \mathbf{N} -th root of x is finite in time $O(n^2)$, and so, the time complexity of the \mathbf{N} -th Root Algorithm is $T(n) = O(n^3)$.

The \mathbf{N} -th Root Algorithm Semialgebra

The \mathbf{N} -th Root Algorithm derives from the division algorithm and the binomial theorem in topologically complete semialgebras. Moreover, the \mathbf{N} -th Root Algorithm characterizes the initial topologically complete semialgebra, from which every topological and algebraic morphism to any topologically complete semialgebra is the embedding, as the unique topologically complete semialgebra with \mathbf{N} -th Root Algorithm.