Triangle exact solution of 3-bodies problem.

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Here is presented a system of equations of 3-bodies problem in well-known *Lagrange's form* (*describing a relative motions of 3-bodies*). Analyzing of such a system, we obtain an exact solution in special case of *constant ratios* of relative distances between the bodies.

Above simplifying assumption reduces all equations of initial system *to a proper similar form*, which leads us to a final solution: initial triangle of bodies m_1, m_2, m_3 is moving *as entire construction*, simultaneously rotating over the common center of masses as well as increasing or decreasing of it's size proportionally.

Let us consider the system of an ordinary differential equations for 3-bodies problem, at given initial conditions [1-3]:

$$m_1 \boldsymbol{q}_1'' = -\gamma \left\{ \frac{m_1 m_2 (\boldsymbol{q}_1 - \boldsymbol{q}_2)}{|\boldsymbol{q}_1 - \boldsymbol{q}_2|^3} + \frac{m_1 m_3 (\boldsymbol{q}_1 - \boldsymbol{q}_3)}{|\boldsymbol{q}_1 - \boldsymbol{q}_3|^3} \right\},$$

$$m_2 \boldsymbol{q}_2'' = -\gamma \left\{ \frac{m_2 m_1 (\boldsymbol{q}_2 - \boldsymbol{q}_1)}{|\boldsymbol{q}_2 - \boldsymbol{q}_1|^3} + \frac{m_2 m_3 (\boldsymbol{q}_2 - \boldsymbol{q}_3)}{|\boldsymbol{q}_2 - \boldsymbol{q}_3|^3} \right\},$$

$$m_3 \boldsymbol{q}_3'' = -\gamma \left\{ \frac{m_3 m_1 (\boldsymbol{q}_3 - \boldsymbol{q}_1)}{|\boldsymbol{q}_3 - \boldsymbol{q}_1|^3} + \frac{m_3 m_2 (\boldsymbol{q}_3 - \boldsymbol{q}_2)}{|\boldsymbol{q}_3 - \boldsymbol{q}_2|^3} \right\} .$$

- here $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\boldsymbol{q}_3$ - means the radius-vector of bodies m_1, m_2, m_3, accordingly.

For the purposes of exploring a relative motions of 3-bodies one to each other, let's rewrite the system above as below (*by linear transformation of initial equations*):

$$(\boldsymbol{q}_1 - \boldsymbol{q}_2)'' + \gamma (m_1 + m_2) \frac{(\boldsymbol{q}_1 - \boldsymbol{q}_2)}{|\boldsymbol{q}_1 - \boldsymbol{q}_2|^3} = \gamma m_3 \left\{ \frac{(\boldsymbol{q}_3 - \boldsymbol{q}_1)}{|\boldsymbol{q}_3 - \boldsymbol{q}_1|^3} + \frac{(\boldsymbol{q}_2 - \boldsymbol{q}_3)}{|\boldsymbol{q}_2 - \boldsymbol{q}_3|^3} \right\},$$

$$(\boldsymbol{q}_2 - \boldsymbol{q}_3)'' + \gamma (m_2 + m_3) \frac{(\boldsymbol{q}_2 - \boldsymbol{q}_3)}{|\boldsymbol{q}_2 - \boldsymbol{q}_3|^3} = \gamma m_1 \left\{ \frac{(\boldsymbol{q}_3 - \boldsymbol{q}_1)}{|\boldsymbol{q}_3 - \boldsymbol{q}_1|^3} + \frac{(\boldsymbol{q}_1 - \boldsymbol{q}_2)}{|\boldsymbol{q}_1 - \boldsymbol{q}_2|^3} \right\},$$

$$(\boldsymbol{q}_3 - \boldsymbol{q}_1)'' + \gamma (m_1 + m_3) \frac{(\boldsymbol{q}_3 - \boldsymbol{q}_1)}{|\boldsymbol{q}_3 - \boldsymbol{q}_1|^3} = \gamma m_2 \left\{ \frac{(\boldsymbol{q}_1 - \boldsymbol{q}_2)}{|\boldsymbol{q}_1 - \boldsymbol{q}_2|^3} + \frac{(\boldsymbol{q}_2 - \boldsymbol{q}_3)}{|\boldsymbol{q}_2 - \boldsymbol{q}_3|^3} \right\} .$$

Let's designate as below:

$$\boldsymbol{R}_{1,2} = (\boldsymbol{q}_1 - \boldsymbol{q}_2), \quad \boldsymbol{R}_{2,3} = (\boldsymbol{q}_2 - \boldsymbol{q}_3), \quad \boldsymbol{R}_{3,1} = (\boldsymbol{q}_3 - \boldsymbol{q}_1) \qquad (*)$$

Above designating causes the transformation of a previous system to another form:

$$\boldsymbol{R}_{1,2}'' + \gamma\,(m_1 + m_2)\frac{\boldsymbol{R}_{1,2}}{\left|\boldsymbol{R}_{1,2}\right|^3} = \gamma\, m_3 \left\{\frac{\boldsymbol{R}_{3,1}}{\left|\boldsymbol{R}_{3,1}\right|^3} + \frac{\boldsymbol{R}_{2,3}}{\left|\boldsymbol{R}_{2,3}\right|^3}\right\},$$

$$\boldsymbol{R}_{2,3}'' + \gamma\,(m_2 + m_3)\frac{\boldsymbol{R}_{2,3}}{\left|\boldsymbol{R}_{2,3}\right|^3} = \gamma\, m_1 \left\{\frac{\boldsymbol{R}_{1,2}}{\left|\boldsymbol{R}_{1,2}\right|^3} + \frac{\boldsymbol{R}_{3,1}}{\left|\boldsymbol{R}_{3,1}\right|^3}\right\}, \qquad (1.1)$$

$$\boldsymbol{R}_{3,1}'' + \gamma\,(m_1 + m_3)\frac{\boldsymbol{R}_{3,1}}{\left|\boldsymbol{R}_{3,1}\right|^3} = \gamma\, m_2 \left\{\frac{\boldsymbol{R}_{2,3}}{\left|\boldsymbol{R}_{2,3}\right|^3} + \frac{\boldsymbol{R}_{1,2}}{\left|\boldsymbol{R}_{1,2}\right|^3}\right\} .$$

Analysing system (1.1) we should note that if we sum all the above equations one to each other it would lead us to the result below:

$$\boldsymbol{R}_{1,2}'' + \boldsymbol{R}_{2,3}'' + \boldsymbol{R}_{3,1}'' = 0 .$$

If we also sum all the equalities (*) one to each other, we should obtain

$$\boldsymbol{R}_{1,2} + \boldsymbol{R}_{2,3} + \boldsymbol{R}_{3,1} = 0 \qquad (**)$$

Besides, if we substitute an expression for $\boldsymbol{R}_{3,1}/|\boldsymbol{R}_{3,1}|^3$ - *from 2-nd to 1-st equation of* (1.1)*, then* $\boldsymbol{R}_{1,2}/|\boldsymbol{R}_{1,2}|^3$ *to the 3-d* - we should obtain below:

$$\left\{\boldsymbol{R}_{1,2}{}'' + \gamma\,(m_1 + m_2 + m_3)\frac{\boldsymbol{R}_{1,2}}{|\boldsymbol{R}_{1,2}|^3}\right\}\cdot\frac{1}{m_3} = \boldsymbol{F}(t),$$

$$\left\{\boldsymbol{R}_{2,3}{}'' + \gamma\,(m_1 + m_2 + m_3)\frac{\boldsymbol{R}_{2,3}}{|\boldsymbol{R}_{2,3}|^3}\right\}\cdot\frac{1}{m_1} = \boldsymbol{F}(t), \qquad (1.2)$$

$$\left\{\boldsymbol{R}_{3,1}{}'' + \gamma\,(m_1 + m_2 + m_3)\frac{\boldsymbol{R}_{3,1}}{|\boldsymbol{R}_{3,1}|^3}\right\}\cdot\frac{1}{m_2} = \boldsymbol{F}(t) \ .$$

So, the linear recombining of equations (1.1) let us define some vector function $\boldsymbol{F}(t)$ *which seems to be unique for all equations of* (1.2). Otherwise, taking into consideration (**), we also obtain

$$\boldsymbol{R}_{1,2}{}'' + \boldsymbol{R}_{2,3}{}'' + \boldsymbol{R}_{3,1}{}'' + \gamma\,(m_1 + m_2 + m_3)\left\{\frac{\boldsymbol{R}_{1,2}}{|\boldsymbol{R}_{1,2}|^3} + \frac{\boldsymbol{R}_{2,3}}{|\boldsymbol{R}_{2,3}|^3} + \frac{\boldsymbol{R}_{3,1}}{|\boldsymbol{R}_{3,1}|^3}\right\} =$$
$$= \boldsymbol{F}(t)\cdot(m_1 + m_2 + m_3)\ , \quad \Rightarrow$$

$$\Rightarrow \quad \boldsymbol{F}(t) = \gamma\left\{\frac{\boldsymbol{R}_{1,2}}{|\boldsymbol{R}_{1,2}|^3} + \frac{\boldsymbol{R}_{2,3}}{|\boldsymbol{R}_{2,3}|^3} + \frac{\boldsymbol{R}_{3,1}}{|\boldsymbol{R}_{3,1}|^3}\right\} \quad .$$

It is well-known fact [1-3] that there are existing only *5 cases* of exact (1.1) solutions (*below* $\boldsymbol{R}_i = (x_i,\ y_i,\ z_i)$, $i = 1{,}2;\ 2{,}3;\ 3{,}1$):

- *3 Lagrange's linear cases*, when $\boldsymbol{R}_{1,2} \sim \boldsymbol{R}_{2,3} \sim \boldsymbol{R}_{3,1}$

- *2 Euler's cases of equipotential triangle*, when

$$\left|\boldsymbol{R}_{1,2}\right| = \left|\boldsymbol{R}_{2,3}\right| = \left|\boldsymbol{R}_{3,1}\right| \quad \Leftrightarrow \quad (**) \quad \Rightarrow \quad \boldsymbol{F}(t) = 0 \ .$$

Let's consider a solutions of (1.2) for which is valid an assumption below

$$\frac{\left|\boldsymbol{R}_{2,3}\right|}{\left|\boldsymbol{R}_{1,2}\right|} = a = const, \ \frac{\left|\boldsymbol{R}_{2,3}\right|}{\left|\boldsymbol{R}_{3,1}\right|} = b = const \qquad (***)$$

It means that proportions of absolute meanings of relative distances $\boldsymbol{R}_{1,2}$, $\boldsymbol{R}_{2,3}$, $\boldsymbol{R}_{3,1}$ between the bodies *should be the same all the time & should be equal to the initial proportions* (*which are given by special initial conditions*).

It could be possible only if such a triangle of bodies m_1, m_2, m_3 is moving *as entire construction*, rotating over the center of masses as well as increasing or decreasing the lengths of sides of such a triangle proportionally.

Besides, we obtain:

$$\boldsymbol{F}(t) = \gamma \cdot \left(\frac{a^3 \cdot \boldsymbol{R}_{1,2} + \boldsymbol{R}_{2,3} + b^3 \cdot \boldsymbol{R}_{3,1}}{\left|\boldsymbol{R}_{2,3}\right|^3} \right), \quad \boldsymbol{R}_{3,1} = -\boldsymbol{R}_{2,3} - \boldsymbol{R}_{1,2} \ ,$$

$$\Rightarrow \ \boldsymbol{F}(t) = \gamma \cdot \left(\frac{(a^3 - b^3) \cdot \boldsymbol{R}_{1,2} \ + \ (1 - b^3) \cdot \boldsymbol{R}_{2,3}}{\left|\boldsymbol{R}_{2,3}\right|^3} \right) .$$

So, in case $a = b = 1$ we obtain *Euler's cases of equipotential triangle*, but in the case $\boldsymbol{R}_{1,2} \sim \boldsymbol{R}_{2,3} \sim \boldsymbol{R}_{3,1}$ all the equations of system (1.2) could be reduced to one of *Lagrange's linear cases* [1].

In according with assumption (***) above, such a solution of (*1.2*) should be factorized as below ($\boldsymbol{R}_0 = \boldsymbol{R}(t_0)$):

$$\boldsymbol{R} = \frac{\boldsymbol{R}_0}{\left|\boldsymbol{R}_0\right|} \cdot R(t) \cdot \sin(\omega t + \varphi_0) \ ,$$

- here $\boldsymbol{R} = \boldsymbol{R}_i$ – is a vector of general motion, which describes *an identical character* of evolution for each of 3-bodies relative distances $\boldsymbol{R}_{1,2}$, $\boldsymbol{R}_{2,3}$, $\boldsymbol{R}_{3,1}$ ($i = 1{,}2,\ 2{,}3,\ 3{,}1$), besides ($\varphi_0 = \varphi(t_0)$ – *the initial angle when triangle of bodies began to rotate*):

$$R(t) = |\boldsymbol{R}| \cdot \max\{\sin(\omega t + \varphi_0)\}$$

– is the scale factor or measure of appropriate relative distances between the bodies.

As for the integral of momentum, for such a case it describes *a harmonic character* of rotation of triangle m_1, m_2, m_3 over the center of masses (ω – *is the angle velocity of triangle rotation, I – is the proper moment of inertia of such a triangle*):

$$I \cdot \omega = const, \quad I \sim R^2, \quad \Rightarrow \quad \omega = \frac{C}{R^2}, \quad C = const \qquad (1.3)$$

Thus, if an assumption below is valid

$$\frac{|\boldsymbol{R}_{2,3}|}{m_1} = \frac{|\boldsymbol{R}_{1,2}|}{m_3}, \qquad \frac{|\boldsymbol{R}_{2,3}|}{m_1} = \frac{|\boldsymbol{R}_{3,1}|}{m_2}$$

$$\Rightarrow \quad \frac{m_1}{m_3} = a \ , \qquad \frac{m_1}{m_2} = b \ ,$$

- above (1.2) system of vector equations could be reduced *to only one* ODE below:

$$\left(R(t)\cdot\sin(\omega t+\varphi_0)\right)'' + \gamma\left\{m_1+m_2+m_3\cdot\left(\frac{b^3 + (b^3-1)\cdot\alpha_{xi}}{a^3}\right)\right\}\frac{sign\,[\sin(\omega t+\varphi_0)]}{\left(R(t)\cdot\sin(\omega t+\varphi_0)\right)^2} = 0,$$

- where α_{xi} – are the coefficients of proportionality between the initial coordinates of vectors $\boldsymbol{R}_{2,3}/|\boldsymbol{R}_{2,3}|$, $\boldsymbol{R}_{1,2}/|\boldsymbol{R}_{1,2}|$. We also should take into consideration the expression (1.3) for angle velocity: $\omega = C/R^2$, $C = const$.

Such an ordinary differential equation for finding the function $R(t)$ is very complicated to solve by analytical methods, so it should be solved by numerical math methods.

Besides, according to the *Bruns theorem* [4], we know that there is no other invariants except well-known 10 integrals for 3-bodies problem (*including integral of energy, momentum, etc.*).

Let's summarise:

First of all, we represent the equations of 3-bodies problem in appropriate *Lagrange form* (1.2), describing *a relative motions* of 3-bodies. Such a system is proved to describe a motions of *similar character* for evolution of each of 3-bodies relative distances $\boldsymbol{R}_{1,2}$, $\boldsymbol{R}_{2,3}$, $\boldsymbol{R}_{3,1}$.

Then we consider a solutions of (1.2) for which is valid an assumption below:

$$\frac{m_1}{m_3} = \frac{|\boldsymbol{R}_{2,3}|}{|\boldsymbol{R}_{1,2}|} = a\,, \quad \frac{m_1}{m_2} = \frac{|\boldsymbol{R}_{2,3}|}{|\boldsymbol{R}_{3,1}|} = b\,.$$

Besides, we assume that triangle of 3-bodies m_1, m_2, m_3 is rotating on circle orbit around the common center of masses as well as increasing or decreasing the size of above triangle proportionally. Size (*radius*) of such an orbit is determined by masses m_1, m_2, m_3 as well as by parameters a, b. It means that proportions of absolute meanings of the relative distances $\boldsymbol{R}_{1,2}$, $\boldsymbol{R}_{2,3}$, $\boldsymbol{R}_{3,1}$ should be *the same* all the time & should be equal to the initial proportions (*which are given by special initial conditions*).

Thus, such a solution of (*1.2*) should be factorized as below ($\boldsymbol{R}_0 = \boldsymbol{R}(t_0)$):

$$\boldsymbol{R} = \frac{\boldsymbol{R}_0}{|\boldsymbol{R}_0|} \cdot R(t) \cdot \sin(\omega t + \varphi_0)\,,$$

- here $\omega = C/R^2$, $\varphi_0 = \varphi(t_0)$ – the initial angle when triangle of bodies began to rotate, $\boldsymbol{R} = \boldsymbol{R}_i$ – is a vector of general motion, which describes *the identical character* of evolution for each of 3-bodies relative distances $\boldsymbol{R}_{1,2}$, $\boldsymbol{R}_{2,3}$, $\boldsymbol{R}_{3,1}$ ($i = 1{,}2,\ 2{,}3,\ 3{,}1$), where:

$$R(t) = |\boldsymbol{R}| \cdot \max\{\sin(\omega t + \varphi_0)\}$$

– is the scale factor or measure for appropriate relative distances between the bodies.

Finally, all vector equations of (1.2) could be reduced *only to one* ODE below:

$$\left(R(t)\cdot\sin(\omega t+\varphi_0)\right)'' + \gamma\left\{m_1+m_2+m_3\cdot\left(\frac{b^3 + (b^3-1)\cdot\alpha_{xi}}{a^3}\right)\right\}\frac{sign\,[\sin(\omega t+\varphi_0)]}{\left(R(t)\cdot\sin(\omega t+\varphi_0)\right)^2} = 0,$$

- where α_{xi} – are the coefficients of proportionality between the proper coordinates of initial vectors $\boldsymbol{R}_{2,3}/|\boldsymbol{R}_{2,3}|$, $\boldsymbol{R}_{1,2}/|\boldsymbol{R}_{1,2}|$.

The last ordinary differential equation - *in regard to the function R*(*t*) - is very complicated to solve by analytical methods, but it could be solved properly only by numerical math methods.

References:

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