New exact solution of 3-bodies problem.

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Here is presented a system of equations of 3-bodies problem in well-known *Lagrange's form* (*describing a relative motions of 3-bodies*). Analyzing of such a system, we obtain an exact solution in special case of *constant ratios* of relative distances between the bodies.

Above simplifying assumption reduces all equations of initial system *to a proper unique form*, which leads us to a final solution: initial triangle of bodies m_1, m_2, m_3 is moving *as entire construction*, simultaneously rotating over the common center of masses as well as increasing or decreasing of it's size proportionally.

Let us consider the system of an ordinary differential equations for 3-bodies problem, at given initial conditions [1-3]:

$$m_1 \boldsymbol{q}_1'' = -\gamma \left\{ \frac{m_1 m_2 (\boldsymbol{q}_1 - \boldsymbol{q}_2)}{|\boldsymbol{q}_1 - \boldsymbol{q}_2|^3} + \frac{m_1 m_3 (\boldsymbol{q}_1 - \boldsymbol{q}_3)}{|\boldsymbol{q}_1 - \boldsymbol{q}_3|^3} \right\},$$

$$m_2 \boldsymbol{q}_2'' = -\gamma \left\{ \frac{m_2 m_1 (\boldsymbol{q}_2 - \boldsymbol{q}_1)}{|\boldsymbol{q}_2 - \boldsymbol{q}_1|^3} + \frac{m_2 m_3 (\boldsymbol{q}_2 - \boldsymbol{q}_3)}{|\boldsymbol{q}_2 - \boldsymbol{q}_3|^3} \right\},$$

$$m_3 \boldsymbol{q}_3'' = -\gamma \left\{ \frac{m_3 m_1 (\boldsymbol{q}_3 - \boldsymbol{q}_1)}{|\boldsymbol{q}_3 - \boldsymbol{q}_1|^3} + \frac{m_3 m_2 (\boldsymbol{q}_3 - \boldsymbol{q}_2)}{|\boldsymbol{q}_3 - \boldsymbol{q}_2|^3} \right\}.$$

- here $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\boldsymbol{q}_3$ - means the radius-vector of bodies m_1, m_2, m_3, accordingly.

For the purposes of exploring a relative motions of 3-bodies one to each other, let's rewrite the system above as below (*by linear transformation of initial equations*):

$$(\boldsymbol{q}_1 - \boldsymbol{q}_2)'' + \gamma (m_1 + m_2) \frac{(\boldsymbol{q}_1 - \boldsymbol{q}_2)}{|\boldsymbol{q}_1 - \boldsymbol{q}_2|^3} = \gamma m_3 \left\{ \frac{(\boldsymbol{q}_3 - \boldsymbol{q}_1)}{|\boldsymbol{q}_3 - \boldsymbol{q}_1|^3} + \frac{(\boldsymbol{q}_2 - \boldsymbol{q}_3)}{|\boldsymbol{q}_2 - \boldsymbol{q}_3|^3} \right\},$$

$$(\boldsymbol{q}_2 - \boldsymbol{q}_3)'' + \gamma (m_2 + m_3) \frac{(\boldsymbol{q}_2 - \boldsymbol{q}_3)}{|\boldsymbol{q}_2 - \boldsymbol{q}_3|^3} = \gamma m_1 \left\{ \frac{(\boldsymbol{q}_3 - \boldsymbol{q}_1)}{|\boldsymbol{q}_3 - \boldsymbol{q}_1|^3} + \frac{(\boldsymbol{q}_1 - \boldsymbol{q}_2)}{|\boldsymbol{q}_1 - \boldsymbol{q}_2|^3} \right\},$$

$$(\boldsymbol{q}_3 - \boldsymbol{q}_1)'' + \gamma (m_1 + m_3) \frac{(\boldsymbol{q}_3 - \boldsymbol{q}_1)}{|\boldsymbol{q}_3 - \boldsymbol{q}_1|^3} = \gamma m_2 \left\{ \frac{(\boldsymbol{q}_1 - \boldsymbol{q}_2)}{|\boldsymbol{q}_1 - \boldsymbol{q}_2|^3} + \frac{(\boldsymbol{q}_2 - \boldsymbol{q}_3)}{|\boldsymbol{q}_2 - \boldsymbol{q}_3|^3} \right\}.$$

Let's designate as below:

$$\boldsymbol{R}_{1,2} = (\boldsymbol{q}_1 - \boldsymbol{q}_2), \quad \boldsymbol{R}_{2,3} = (\boldsymbol{q}_2 - \boldsymbol{q}_3), \quad \boldsymbol{R}_{3,1} = (\boldsymbol{q}_3 - \boldsymbol{q}_1) \qquad (*)$$

Above designating causes the transformation of a previous system to another form:

$$\boldsymbol{R}_{1,2}'' + \gamma(m_1 + m_2)\frac{\boldsymbol{R}_{1,2}}{|\boldsymbol{R}_{1,2}|^3} = \gamma m_3 \left\{\frac{\boldsymbol{R}_{3,1}}{|\boldsymbol{R}_{3,1}|^3} + \frac{\boldsymbol{R}_{2,3}}{|\boldsymbol{R}_{2,3}|^3}\right\},$$

$$\boldsymbol{R}_{2,3}'' + \gamma(m_2 + m_3)\frac{\boldsymbol{R}_{2,3}}{|\boldsymbol{R}_{2,3}|^3} = \gamma m_1 \left\{\frac{\boldsymbol{R}_{1,2}}{|\boldsymbol{R}_{1,2}|^3} + \frac{\boldsymbol{R}_{3,1}}{|\boldsymbol{R}_{3,1}|^3}\right\}, \qquad (1.1)$$

$$\boldsymbol{R}_{3,1}'' + \gamma(m_1 + m_3)\frac{\boldsymbol{R}_{3,1}}{|\boldsymbol{R}_{3,1}|^3} = \gamma m_2 \left\{\frac{\boldsymbol{R}_{2,3}}{|\boldsymbol{R}_{2,3}|^3} + \frac{\boldsymbol{R}_{1,2}}{|\boldsymbol{R}_{1,2}|^3}\right\}.$$

Analysing system (1.1) we should note that if we sum all the above equations one to each other it would lead us to the result below:

$$\boldsymbol{R}_{1,2}'' + \boldsymbol{R}_{2,3}'' + \boldsymbol{R}_{3,1}'' = 0 \ .$$

If we also sum all the equalities (*) one to each other, we should obtain

$$\boldsymbol{R}_{1,2} + \boldsymbol{R}_{2,3} + \boldsymbol{R}_{3,1} = 0 \qquad (**)$$

Besides, if we substitute an expression for $\boldsymbol{R}_{3,1}$ - *from 2-nd to 1-st equation of* (1.1)*, then to the 3-d* - we should obtain below:

$$\begin{cases} \left(\boldsymbol{R}_{1,2}'' + \gamma(m_1 + m_2 + m_3) \dfrac{\boldsymbol{R}_{1,2}}{\left|\boldsymbol{R}_{1,2}\right|^3} \right) \cdot \dfrac{\left|\boldsymbol{R}_{1,2}\right|^2}{m_3} = \boldsymbol{F}(t) \\ \left(\boldsymbol{R}_{2,3}'' + \gamma(m_1 + m_2 + m_3) \dfrac{\boldsymbol{R}_{2,3}}{\left|\boldsymbol{R}_{2,3}\right|^3} \right) \cdot \dfrac{\left|\boldsymbol{R}_{2,3}\right|^2}{m_1} = \boldsymbol{F}(t) \\ \left(\boldsymbol{R}_{3,1}'' + \gamma(m_1 + m_2 + m_3) \dfrac{\boldsymbol{R}_{3,1}}{\left|\boldsymbol{R}_{3,1}\right|^3} \right) \cdot \dfrac{\left|\boldsymbol{R}_{3,1}\right|^2}{m_2} = \boldsymbol{F}(t) \end{cases} \qquad (1.2)$$

So, the linear recombining of equations (1.1) let us define some vector function ***F** (t) which seems to be unique for all equations of* (1.2). Otherwise, taking into consideration (**), we also obtain

$$\boldsymbol{R}_{1,2}'' + \boldsymbol{R}_{2,3}'' + \boldsymbol{R}_{3,1}'' + \gamma(m_1 + m_2 + m_3) \left\{ \frac{\boldsymbol{R}_{1,2}}{\left|\boldsymbol{R}_{1,2}\right|^3} + \frac{\boldsymbol{R}_{2,3}}{\left|\boldsymbol{R}_{2,3}\right|^3} + \frac{\boldsymbol{R}_{3,1}}{\left|\boldsymbol{R}_{3,1}\right|^3} \right\} =$$

$$= \boldsymbol{F}(t) \left\{ \frac{m_3}{\left|\boldsymbol{R}_{1,2}\right|^2} + \frac{m_1}{\left|\boldsymbol{R}_{2,3}\right|^2} + \frac{m_2}{\left|\boldsymbol{R}_{3,1}\right|^2} \right\}, \quad \Rightarrow$$

$$\Rightarrow \quad \boldsymbol{F}(t) = \gamma(m_1 + m_2 + m_3) \frac{\left\{ \dfrac{\boldsymbol{R}_{1,2}}{\left|\boldsymbol{R}_{1,2}\right|^3} + \dfrac{\boldsymbol{R}_{2,3}}{\left|\boldsymbol{R}_{2,3}\right|^3} + \dfrac{\boldsymbol{R}_{3,1}}{\left|\boldsymbol{R}_{3,1}\right|^3} \right\}}{\left(\dfrac{m_3}{\left|\boldsymbol{R}_{1,2}\right|^2} + \dfrac{m_1}{\left|\boldsymbol{R}_{2,3}\right|^2} + \dfrac{m_2}{\left|\boldsymbol{R}_{3,1}\right|^2} \right)} \; .$$

It is well-known fact [1-3] that there are existing only *5 cases* of exact (1.1) solutions (*below* $\boldsymbol{R}_i = (x_i, y_i, z_i)$, $i = 1,2;\ 2,3;\ 3,1$):

- *3 Lagrange's linear cases*, when $\boldsymbol{R}_{1,2} \sim \boldsymbol{R}_{2,3} \sim \boldsymbol{R}_{3,1}$

- *2 Euler's cases of equipotential triangle*, when

$$\left|\boldsymbol{R}_{1,2}\right| = \left|\boldsymbol{R}_{2,3}\right| = \left|\boldsymbol{R}_{3,1}\right| ,$$

$$\Leftrightarrow \ (**) \ \Rightarrow \ \boldsymbol{F}(t) = 0 \ .$$

Let's consider a solutions of (1.2) for which is valid an assumption below

$$\frac{\left|\boldsymbol{R}_{2,3}\right|}{\left|\boldsymbol{R}_{1,2}\right|} = a \, , \ \frac{\left|\boldsymbol{R}_{2,3}\right|}{\left|\boldsymbol{R}_{3,1}\right|} = b \, , \qquad (***)$$

- then we obtain:

$$\boldsymbol{F}(t) = \frac{\gamma(m_1 + m_2 + m_3)}{\left|\boldsymbol{R}_{2,3}\right|} \cdot \left(\frac{a^3 \cdot \boldsymbol{R}_{1,2} + \boldsymbol{R}_{2,3} + b^3 \cdot \boldsymbol{R}_{3,1}}{a^2 \cdot m_3 + m_1 + b^2 \cdot m_2} \right), \quad \boldsymbol{R}_{3,1} = -\boldsymbol{R}_{2,3} - \boldsymbol{R}_{1,2} \ ,$$

$$\Rightarrow \ \boldsymbol{F}(t) = \frac{\gamma(m_1 + m_2 + m_3)}{\left(a^2 \cdot m_3 + m_1 + b^2 \cdot m_2\right)} \cdot \left(\frac{(a^3 - b^3)}{a} \cdot \frac{\boldsymbol{R}_{1,2}}{\left|\boldsymbol{R}_{1,2}\right|} + (1 - b^3) \cdot \frac{\boldsymbol{R}_{2,3}}{\left|\boldsymbol{R}_{2,3}\right|} \right).$$

From equality above we could conclude that in case $a = b = 1$, we obtain *Euler's cases of equipotential triangle*, but in case $\boldsymbol{R}_{1,2} \sim \boldsymbol{R}_{2,3} \sim \boldsymbol{R}_{3,1}$ all the equations of system (1.2) could be reduced to one of *Lagrange's linear cases* [1].

Besides, let's consider only solutions, for which the equalities below are valid:

$$\begin{cases} \dfrac{|R_{1,2}|^2}{m_3} = \dfrac{|R_{2,3}|^2}{m_1} \Rightarrow \dfrac{m_1}{m_3} = \dfrac{|R_{2,3}|^2}{|R_{1,2}|^2} = a^2 \\ \\ \dfrac{|R_{3,1}|^2}{m_2} = \dfrac{|R_{2,3}|^2}{m_1} \Rightarrow \dfrac{m_1}{m_2} = \dfrac{|R_{2,3}|^2}{|R_{3,1}|^2} = b^2 \end{cases} \qquad (1.3)$$

In such a case, system of equations (1.2) could be reduced as below:

$$\begin{cases} \left(\boldsymbol{R}_{1,2}'' + \gamma(m_1 + m_2 + m_3) \dfrac{\boldsymbol{R}_{1,2}}{|\boldsymbol{R}_{1,2}|^3} \right) = \widetilde{\boldsymbol{F}}(t) = \dfrac{m_3}{|\boldsymbol{R}_{1,2}|^2} \cdot \boldsymbol{F}(t) \\ \left(\boldsymbol{R}_{2,3}'' + \gamma(m_1 + m_2 + m_3) \dfrac{\boldsymbol{R}_{2,3}}{|\boldsymbol{R}_{2,3}|^3} \right) = \widetilde{\boldsymbol{F}}(t) = \dfrac{m_1}{|\boldsymbol{R}_{2,3}|^2} \cdot \boldsymbol{F}(t) \\ \left(\boldsymbol{R}_{3,1}'' + \gamma(m_1 + m_2 + m_3) \dfrac{\boldsymbol{R}_{3,1}}{|\boldsymbol{R}_{3,1}|^3} \right) = \widetilde{\boldsymbol{F}}(t) = \dfrac{m_2}{|\boldsymbol{R}_{3,1}|^2} \cdot \boldsymbol{F}(t) \end{cases} \qquad (1.4)$$

So, analysing above (*1.4*) we can observe *an identical character of equations* for evolution of each of 3-bodies relative distances $\boldsymbol{R}_{1,2}$, $\boldsymbol{R}_{2,3}$, $\boldsymbol{R}_{3,1}$.

It could be possible only if such a triangle of bodies m_1, m_2, m_3 is moving *as entire construction*, rotating over the center of masses as well as increasing or decreasing the lengths of sides of such a triangle proportionally.

In according with our previous assumption (***), it means that absolute meanings of proportions between the relative distances $\boldsymbol{R}_{1,2}$, $\boldsymbol{R}_{2,3}$, $\boldsymbol{R}_{3,1}$ *should be the same all the time & should be equal to the initial proportions* (*given by special initial conditions*).

For the reason that entire triangle m_1, m_2, m_3 are proved to be moving *in identical way*, we could make a conclusion from (*1.4*) that every set of points of such a triangle shall also be moving in the same way (expression for $\boldsymbol{F}$ is given above):

$$\boldsymbol{F}'' + \gamma(m_1+m_2+m_3)\frac{\boldsymbol{F}}{|\,\boldsymbol{F}\,|^3} = \frac{m_3}{|\boldsymbol{R}_{1,2}|^2}\cdot\boldsymbol{F}(t)\,, \qquad |\,\boldsymbol{F}\,| = const = C\,,$$

$$\Rightarrow \boldsymbol{F}'' - \left(\frac{m_3}{|\boldsymbol{R}_{1,2}|^2} - \frac{\gamma(m_1+m_2+m_3)}{C^3}\right)\boldsymbol{F} = \boldsymbol{0}\,,$$

- here $\boldsymbol{F}$ *- is a vector, which does not depend on relative distances* between masses m_1, m_2 *or* m_3. Besides, above equality describes *a harmonic character* of triangle m_1, m_2, m_3 rotation over the center of masses (ω – *is the angle velocity of triangle rotation*):

$$\Rightarrow \boldsymbol{F} = \boldsymbol{F_0}\cdot\sin(\omega t+\varphi_0)\,, \qquad \omega = \sqrt{\frac{m_3}{|\boldsymbol{R}_{1,2}|^2} - \frac{\gamma(m_1+m_2+m_3)}{C^3}}\,,$$

$$C = |\,\boldsymbol{F}(t)\,| = \frac{\gamma(m_1+m_2+m_3)}{\left(a^2\cdot m_3+m_1+b^2\cdot m_2\right)}\cdot\left|\frac{(a^3-b^3)}{a}\cdot\frac{\boldsymbol{R}_{1,2}}{|\boldsymbol{R}_{1,2}|} + (1-b^3)\cdot\frac{\boldsymbol{R}_{2,3}}{|\boldsymbol{R}_{2,3}|}\right| =$$

$$= \frac{\gamma}{3}(1+\frac{1}{b^2}+\frac{1}{a^2})\cdot\sqrt{\left(\frac{(a^3-b^3)}{a}\right)^2 + 2\frac{(a^3-b^3)}{a}(1-b^3)\cdot\cos\left\{\boldsymbol{R}_{1,2},\boldsymbol{R}_{2,3}\right\} + (1-b^3)^2}\,.$$

We could obtain a proper restriction for the regime of triangle m_1, m_2, m_3 rotation (*from expression above for angle velocity* ω):

$$\frac{m_3}{|\boldsymbol{R}_{1,2}|^2} - \frac{\gamma(m_1+m_2+m_3)}{C^3} \geq 0\,, \quad \Rightarrow \quad |\boldsymbol{R}_{1,2}|^2 \leq \frac{C^3}{\gamma\left(\frac{1}{a^2}+\frac{a^2}{b^2}+1\right)}\,.$$

Finally, the general solution of (*1.4*) should be factorized as below ($\boldsymbol{R}_0 = \boldsymbol{R}(t_0)$):

$$\boldsymbol{R} = \frac{\boldsymbol{R}_0}{|\boldsymbol{R}_0|} \cdot R(t) \cdot \sin(\omega t + \varphi_0) \ ,$$

- here $\boldsymbol{R} = \boldsymbol{R}_i$ – is a vector of general motion, which describes *an identical character* of evolution for each of 3-bodies relative distances $\boldsymbol{R}_{1,2}$, $\boldsymbol{R}_{2,3}$, $\boldsymbol{R}_{3,1}$ ($i = 1,2,\ 2,3,\ 3,1$), besides:

$$R(t) = \|\boldsymbol{R}| \cdot \max\{\sin(\omega t + \varphi_0)\}$$

– is the scale factor or measure of appropriate relative distances between the bodies.

Thus, we obtain:

$$\frac{\boldsymbol{R_0}}{|\boldsymbol{R}_0|} \cdot \left\{ (R \cdot \sin(\omega t + \varphi_0))'' + \frac{\gamma(m_1 + m_2 + m_3)}{(R \cdot \sin(\omega t + \varphi_0))^2} \right\} = \left(\frac{m_i}{R^2 \cdot \sin^2 \omega t} \right) \cdot \boldsymbol{F_0} \cdot \sin(\omega t + \varphi_0) \ ,$$

$$\Rightarrow \quad (R \cdot \sin(\omega t + \varphi_0))'' + \frac{\gamma(m_1 + m_2 + m_3)}{(R \cdot \sin(\omega t + \varphi_0))^2} = \left(\frac{\alpha_i \cdot m_i}{R^2 \cdot \sin^2 \omega t} \right) \cdot \sin(\omega t + \varphi_0) \ ,$$

$$\Rightarrow (R \cdot \sin(\omega t + \varphi_0))'' + \{\gamma(m_1 + m_2 + m_3) - \alpha_i \cdot m_i \cdot \sin(\omega t + \varphi_0)\} \cdot \frac{1}{(R \cdot \sin(\omega t + \varphi_0))^2} = 0 \ ,$$

- where α_i – are the coefficients of proportionality between the proper coordinates of vectors $\boldsymbol{R}_{1,2} / |\boldsymbol{R}_{1,2}|$, $\boldsymbol{R}_{2,3} / |\boldsymbol{R}_{2,3}|$. We also should take into consideration the above expression for angle velocity ω:

$$\omega = \sqrt{\frac{m_i}{R^2} - \frac{\gamma(m_1 + m_2 + m_3)}{C^3}} \ .$$

If we designate $y(t) = R(t)\cdot sin(\omega\cdot t+\varphi o)$, the last equation could be represented as below:

$$y'' + \{\gamma(m_1+m_2+m_3)-\alpha_i\cdot m_i\cdot \sin(\omega t+\varphi_0)\}\cdot\frac{1}{y^2} = 0 ,$$

- but such an ordinary differential equation for finding the function $R(t)$ is very complicated to solve by analytical methods, so it should be solved by numerical math methods.

Besides, according to the *Bruns theorem* [4], we know that there is no other invariants except well-known 10 integrals for 3-bodies problem (*including integral of energy, momentum, etc.*).

<u>Let's summarise</u>:

First of all, we represent the equations of 3-bodies problem in appropriate *Lagrange form* (1.2), describing *a relative motions* of 3-bodies. Then we consider a solutions of (1.2) for which is valid an assumption (1.3):

$$\frac{m_1}{m_3} = \frac{|\boldsymbol{R}_{2,3}|^2}{|\boldsymbol{R}_{1,2}|^2} = a^2, \quad \frac{m_1}{m_2} = \frac{|\boldsymbol{R}_{2,3}|^2}{|\boldsymbol{R}_{3,1}|^2} = b^2$$

For such a kind of solutions we obtain that equations (1.4) describe a motions of *identical character* for evolution each of 3-bodies relative distances $\boldsymbol{R}_{1,2}$, $\boldsymbol{R}_{2,3}$, $\boldsymbol{R}_{3,1}$.

Besides, we obtain that triangle of 3-bodies m_1, m_2, m_3 is proved to be rotating on circle orbit around the common center of masses as well as increasing or decreasing the size of above triangle proportionally. Size (*radius*) of such an orbit is determined by masses m_1, m_2, m_3 as well as by parameters *a, b*. It means that absolute meanings of proportions between the relative distances $\boldsymbol{R}_{1,2}$, $\boldsymbol{R}_{2,3}$, $\boldsymbol{R}_{3,1}$ should be the same all the time & should be equal to the initial proportions (*given by special initial conditions*).

Thus, the general solution of (*1.4*) should be factorized as below ($\boldsymbol{R}_0 = \boldsymbol{R}(t_0)$):

$$\boldsymbol{R} \;=\; \frac{\boldsymbol{R}_0}{\left|\boldsymbol{R}_0\right|} \cdot R(t) \cdot \sin\,(\omega t + \varphi_0) \;\;,$$

- here $\boldsymbol{R} = \boldsymbol{R}_i$ – is a vector of general motion, which describes *the identical character* of evolution for each of 3-bodies relative distances $\boldsymbol{R}_{1,2}$, $\boldsymbol{R}_{2,3}$, $\boldsymbol{R}_{3,1}$ (i = $1,2$, $2,3$, $3,1$), where:

$$R\,(t) = \|\boldsymbol{R}| \cdot \max\,\{\sin\,(\omega t + \varphi_0)\}$$

– is the scale factor or measure for appropriate relative distances between the bodies.

Besides, here:

$$\omega \;=\; \sqrt{\frac{m_i}{R^2} - \frac{\gamma(m_1 + m_2 + m_3)}{C^3}} \;\;,$$

- where expression for C is given above.

It means that *angle velocity* ω depends on the *radius* of circle orbit of 3-bodies triangle rotation: - the larger radius, the less *an angle velocity* ω; - the less radius, the larger *an angle velocity* ω, but such a regime of rotation is valid only up to the appropriate meaning of radius below:

$$R \;\le\; \sqrt{\frac{C^3}{\gamma\left(\frac{1}{a^2} + \frac{a^2}{b^2} + 1\right)}} \;\;.$$

Finally, all vector equations (1.4) could be reduced only to one ODE below:

$$(R\cdot\sin(\omega t+\varphi_0))'' + \frac{\{\,\gamma(m_1+m_2+m_3)-\alpha_i\cdot m_i\cdot\sin(\omega t+\varphi_0)\,\}}{(R\cdot\sin(\omega t+\varphi_0))^2} = 0 ,$$

- where α_i – are the coefficients of proportionality between the proper coordinates of initial vectors $\boldsymbol{R}_{1,2}/|\boldsymbol{R}_{1,2}|$, $\boldsymbol{R}_{2,3}/|\boldsymbol{R}_{2,3}|$.

The last ordinary differential equation - *in regard to the function* $R(t)$ - is very complicated to solve by analytical methods, but it could be solved properly by numerical math methods.

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